SEQUENTIAL METHODS IN STATISTICS
BANACH CENTER PUBLICATIONS, VOLUME 16
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1985

A MODIFICATION OF SUDAKOV'S LEMMA AND EFFICIENT SEQUENTIAL PLANS FOR SOME JUMP MARKOV PROCESSES

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1. A modification of Sudakov's lemma and the Cramér–Rao inequality

In 1968 S. Trybuła proved in [7] that for the Poisson process $\xi(t)$, $t \geqslant 0$, with the parameter λ the joint distribution m_λ of a random vector $(\tau, \xi(\tau))$, where τ is a Markov stopping time, is absolutely continuous with respect to m_{λ_0} for some λ_0 with the density $dm_\lambda/dm_{\lambda_0}$. Using this fact, the author obtained a characterization of efficient sequential plans for the Poisson process with an unknown parameter λ. In the same article analogous results for the negative-binomial, gamma and Wiener processes were obtained. Later, in 1969, V. N. Sudakov ([6]) proved the following lemma, generalizing the result obtained by Trybuła.

LEMMA ([6] or [2], 55–59). *Let $(\Omega, \mathscr{F}, P_\theta)$ be a probability space and let $\xi(t)$, $t \geqslant 0$, be a stochastic process with right-continuous realizations — P_ϑ almost surely for every $\vartheta \in \Theta$.*

We assume that $\xi(t)$ is a sufficient statistic for ϑ and the one-dimensional distribution $P_\vartheta(\xi(t) \in B)$ of $\xi(t)$, $t \geqslant 0$, is absolutely continuous with respect to the one-dimensional distribution $P_{\vartheta_0}(\xi(t) \in B)$ and the density function $h(t, x; \vartheta, \vartheta_0)$ is continuous.

Then for any Markov stopping time τ the joint distribution m_ϑ of $(\tau, \xi(\tau))$ is absolutely continuous with respect to the joint distribution m_{ϑ_0} and

$$\frac{dm_\vartheta}{dm_{\vartheta_0}}(u) = h\big(t(u), x(u); \vartheta, \vartheta_0\big),$$

$u = \big(t(u), x(u)\big) \in U = [0, \infty) \times R^l$.

Let us introduce the following notation:

$D([0, \infty))$ — the space of functions $x(\cdot)$: $[0, \infty) \to R^k$ which are right continuous and have left-side limits.

$\mathscr{D}$ — the least σ-algebra of subsets of $D([0, \infty))$ with respect to which the coordinate mappings $x(t)$: $D([0, \infty)) \to R^k$ are measurable.

$\mathscr{D}_t$ — the least σ-algebra of subsets of $D([0, \infty))$ with respect to which the coordinate mappings $x(s)$: $D([0, \infty)) \to R^k$, $s \in [0, t]$ are measurable.

μ_ϑ — a probability measure defined on $(D([0, \infty)), \mathscr{D})$, depending on the parameter $\vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_s) \in \Theta \subset R^s$.

DEFINITION 1. A *Markov stopping time* is a random variable τ: $D([0, \infty)) \to [0, \infty]$ for which

$$\{x(\cdot)\colon \tau(x(\cdot)) \leqslant t\} \in \mathscr{D}_t \qquad \text{for every} \quad t \geqslant 0.$$

Further we also assume that

$$\mu_\vartheta(\{x(\cdot)\colon \tau(x(\cdot)) < \infty\}) = 1 \qquad \text{for every} \quad \vartheta \in \Theta.$$

1°. For every t, let $Z(t, x(\cdot))$, be the mapping from $D([0, \infty))$ to R^l, measurable with respect to $\mathscr{D}_t$ and right-continuous with respect to t — μ_ϑ almost surely for every $\vartheta \in \Theta$.

2° By $\mu_{\vartheta,t}$ we denote the restriction of μ_ϑ to the σ-algebra $\mathscr{D}_t$. Let us assume that $\mu_{\vartheta,t}$ is absolutely continuous with respect to $\mu_{\vartheta_0,t}$ and

$$\frac{d\mu_{\vartheta,t}}{d\mu_{\vartheta_0,t}}(x(\cdot)) = h(t, Z(t, x(\cdot)); \vartheta, \vartheta_0),$$

where h is a continuous function and $Z(t) = Z(t, x(\cdot))$ is a mapping satisfying the previous condition.

The problem is what we can say about the joint distribution of the random vector $(\tau, Z(\tau))$.

3°. Let $U = [0, \infty) \times R^l = T \times R^l$,

$$U \in u = (t(u), z(u)).$$

$\mathscr{B}_U$ — the σ-algebra of Borel subsets of U.

On $(U, \mathscr{B}_U)$ we define, for every $A \in \mathscr{B}_U$, the measure m_ϑ generated by the statistics Z and the Markov time τ:

$$m_\vartheta(A) = \mu_\vartheta\Big(\big\{x(\cdot)\colon \big(\tau(x(\cdot)), Z(\tau(x(\cdot)), x(\cdot))\big) \in A\big\}\Big).$$

LEMMA 1 ([4]). *Under assumptions* 1°–3° *the measure* m_ϑ *is absolutely continuous with respect to the measure* m_{ϑ_0} *and the density function takes the form*:

$$\frac{dm_\vartheta}{dm_{\vartheta_0}}(u) = h(t(u), z(u); \vartheta, \vartheta_0).$$

DEFINITION 2. By a *sequential plan* we mean the pair $(\tau, f(\tau, Z(\tau)))$, where τ is a Markov stopping time and f: $U \to R$ is a $\mathscr{B}_U$-measurable function.

The function f will be called an *estimator* of a given function $g(\vartheta)$. We also assume that:

(i) $E_{m_\vartheta} f = E_\vartheta f(\tau, Z(\tau)) = g(\vartheta)$,

(ii) $E_{m_\vartheta} f^2 = \int\limits_U f^2(u)\, h(u;\, \vartheta,\, \vartheta_0)\, dm_{\vartheta_0}(u) < \infty$ for every $\vartheta \in \Theta$,

(iii) the function $g(\vartheta)$ is differentiable and $g(\vartheta) \not\equiv \text{const}$. Let us denote:

$$V_\vartheta(u) = \left(\frac{\partial}{\partial \vartheta_1} \log h(u;\, \vartheta,\, \vartheta_0), \ldots, \frac{\partial}{\partial \vartheta_s} \log h(u;\, \vartheta,\, \vartheta_0)\right),$$

$$\nabla_\vartheta g = \left(\frac{\partial}{\partial \vartheta_1} g(\vartheta), \ldots, \frac{\partial}{\partial \vartheta_s} g(\vartheta)\right), \qquad J(\vartheta) = E_{m_\vartheta} V_\vartheta^T V_\vartheta, \; .$$

$D_\vartheta(\cdot)$ – the variance evaluated at ϑ.

We can formulate the following theorem:

THEOREM 1. *If a sequential plan (τ, f) satisfies the regularity conditions which guarantee that*

$$\frac{\partial}{\partial \vartheta_i} \int\limits_U h(u;\, \vartheta,\, \vartheta_0)\, dm_{\vartheta_0}(u) = \int\limits_U \frac{\partial}{\partial \vartheta_i} h(u;\, \vartheta,\, \vartheta_0)\, dm_{\vartheta_0}(u),$$

$$\frac{\partial}{\partial \vartheta_i} \int\limits_U f(u)\, h(u;\, \vartheta,\, \vartheta_0)\, dm_{\vartheta_0}(u) = \int\limits_U f(u) \frac{\partial}{\partial \vartheta_i} \log h(u;\, \vartheta,\, \vartheta_0)\, dm_{\vartheta_0}(u),$$

then

$$D_\vartheta f(\tau, Z(\tau)) \geqslant (\nabla_\vartheta g) J^{-1}(\vartheta) (\nabla_\vartheta g)^T. \tag{1}$$

The equality holds at a particular value of ϑ if and only if

$$f(u) = (\nabla_\vartheta g) J^{-1}(\vartheta) V_\vartheta^T + g(\vartheta) \qquad m_{\vartheta_0}\text{-almost surely.}$$

DEFINITION 3. A sequential estimation plan (τ, f) for $g(\vartheta)$ is said to be *efficient at* ϑ if (1) becomes an equality at ϑ.

The estimator f is then called *efficient at the value* ϑ and the function $g(\vartheta)$ is *efficiently estimable at* ϑ.

DEFINITION 4. A sequential estimation plan (τ, f) for $g(\vartheta)$ is said to be *efficient* if it is efficient at each $\vartheta \in \Theta$.

The estimator f is then called *efficient* and the function $g(\vartheta)$ is *efficiently estimable*.

2. The characterization of efficient sequential plans for an m-state Markov homogeneous process ([8]).

Let $\xi = \xi(t)$, $t \geqslant 0$, be a homogeneous m-state Markov process with intensity matrix Λ and with intensities λ_{ij} for $i, j = 1, 2, \ldots, m$ $(i \neq j)$.

Let $P(X_0 = k) = p_k$, $0 < p_k < 1$ for $k = 1, 2, \ldots, m$; $\sum_{k=1}^{m} p_k = 1$. Denote $p = (p_1, \ldots, p_m)$, $\lambda_i = \sum_{\substack{j=1\\ j\neq i}}^{m} \lambda_{ij}$. Let $N_{ij}(t)$, $i, j = 1, 2, \ldots, m$ $(i \neq j)$, be the number of jumps of the process ξ from state i to state j in the interval $[0, t]$, let $T_i(t)$, $i = 1, 2, \ldots, m$, be total time during which the process ξ remains in state i in $[0, t]$, and let V_k, $k = 1, 2, \ldots, m$, be a random variable defined in the following way:

$$V_k = \begin{cases} 1 & \text{if } X_0 = k, \\ 0 & \text{otherwise.} \end{cases}$$

Denote

$$N = \begin{bmatrix} 0 & N_{12} & N_{22} & \ldots & N_{1m} \\ N_{21} & 0 & & \ldots & N_{2m} \\ \ldots & \ldots & & \ldots & \ldots \\ N_{m1} & N_{m2} & \ldots & \ldots & 0 \end{bmatrix},$$

$T = (T_1, T_2, \ldots, T_m)$, $V = (V_1, V_2, \ldots, V_m)$. Let $\vartheta = (p, \Lambda)$.

The process ξ generates the measure μ_ϑ in the space $D_m([0, \infty)) \subset D([0, \infty))$ of right continuous functions with values $1, 2, \ldots, m$. Let $\mathscr{F}$ be the least σ-algebra of subsets of D_m with respect to which the functions $x(t)$, $t \geqslant 0$ are measurable, and let $\mathscr{F}_t$ be the least σ-algebra of subsets of D_m with respect to which $x(s)$, $s \in [0, t]$ are measurable. If $\mu_{\vartheta,t}$ denotes the restriction of μ_ϑ on $\mathscr{F}_t$, then we have $\mu_{\vartheta,t} \ll \mu_{\vartheta_0,t}$ and

$$\frac{d\mu_{\vartheta,t}}{d\mu_{\vartheta_0,t}} = \prod_{k=1}^{m} \left(\frac{p_k}{p_k^0}\right)^{v_k} \prod_{\substack{i,j=1\\ i\neq j}}^{m} \left(\frac{\lambda_{ij}}{\lambda_{ij}^0}\right)^{n_{ij}(t)} \exp\Big[\sum_{\substack{r,s=1\\ r\neq s}}^{m} (\lambda_{rs}^0 - \lambda_{rs}) t_r(t)\Big]$$

$$= c\big(v(t), n(t), t(t); \vartheta_0\big) \prod_{k=1}^{m} p_k^{v_k} \prod_{\substack{i,j=1\\ i\neq j}}^{m} \lambda_{ij}^{n_{ij}(t)} \exp\Big[-\sum_{\substack{r,s=1\\ r\neq s}}^{m} \lambda_{rs} t_r(t)\Big],$$

where v_k, $n_{ij}(t)$, $t_i(t)$, v, $n(t)$, $\boldsymbol{t}(t)$ are the values of the random variables V_k, $N_{ij}(t)$, $T_i(t)$, V, $N(t)$, $T(t)$, respectively.

Let τ be a Markov stopping time with respect to $\mathscr{F}_t$. By the modification of Sudakov's lemma it follows that the joint distribution m_ϑ of

$(\tau, V, N(\tau), T(\tau))$ is absolutely continuous with respect to the distribution m_{ϑ_0} for some ϑ_0 and

$$\frac{dm_\vartheta}{dm_{\vartheta_0}} = c(v, n, t; \vartheta_0) \prod_{k=1}^{m} p_k^{v_k} \prod_{\substack{i,j=1\\ i\neq j}}^{m} \lambda_{ij}^{n_{ij}} \exp\Big(-\sum_{\substack{r,s=1\\ r\neq s}}^{m} \lambda_{rs} t_r\Big),$$

where v_k, n_{ij}, t_r, v, n, t are the values of the random variables V_k, $N_{ij}(\tau)$, $T_r(\tau)$, V, $N(\tau)$, $T(\tau)$, respectively.

Denote $g'_k = \dfrac{\partial g}{\partial p_k}$, $g'_{ij} = \dfrac{\partial g}{\partial \lambda_{ij}}$.

LEMMA 2. *If $f = f(V, N, T)$ is an estimator of $Ef = g(p, \Lambda)$ fulfilling regularity conditions then*

$$E[(p_m V_k - p_k V_m) f] = p_k p_m g'_k,$$
$$E[(N_{ij} - \lambda_{ij} T_i) f] = \lambda_{ij} g'_{ij},$$
$$E(p_m V_k - p_k V_m)^2 = p_k p_m (p_k + p_m),$$
$$E[(p_m V_k - p_k V_m)(p_m V_{k'} - p_{k'} V_m)] = p_k p_{k'} p_m,$$
$$E[(N_{ij} - \lambda_{ij} T_i)(N_{i'j'} - \lambda_{i'j'} T_{i'})] = 0, \qquad (i, j) \neq (i', j'),$$
$$E[(p_m V_k - p_k V_m)(N_{ij} - \lambda_{ij} T_i)] = 0,$$
$$E(N_{ij} - \lambda_{ij} T_i)^2 = \lambda_{ij} E T_i.$$

THEOREM 2. *For a sequential plan $(\tau, f(\tau, V, N(\tau), T(\tau)))$ satisfying the regularity conditions we have*

$$D_\vartheta f \geqslant \sum_{k=1}^{m-1} p_k (g'_k)^2 - \Big(\sum_{k=1}^{m-1} p_k g'_k\Big)^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{m} \frac{\lambda_{ij}(g'_{ij})^2}{E T_i}.$$

Equality holds at a particular value of (p, Λ) if and only if f is a linear function of w_k and $w_{i,j}$, m_{ϑ_0} almost surely, where

$$w_k = \frac{p_m v_k - p_k v_m}{p_k p_m}, \qquad w_{ij} = \frac{n_{ij} - \lambda_{ij} t_i}{\lambda_{ij}}.$$

So we can formulate the following corollary.

COROLLARY. *If τ is a Markov stopping time, then a nonconstant estimator $f = f(v, n, t)$ is efficient for $Ef = g(p, \Lambda)$ at (p, Λ) if and only if there exist constants a_k, b_{ij}, $i, j = 1, 2, \ldots, m$; $k = 1, 2, \ldots, m$, not all equal to zero, and a constant c such that*

$$f(v, n, t) = \sum_{k=1}^{m-1} a_k (p_m v_k - p_k v_m) + \sum_{\substack{i,j=1\\ i\neq j}}^{m} b_{ij} (n_{ij} - \lambda_{ij} t_i) + c$$

for almost all (with respect to m_{ϑ_0}) points $u \in U$.

Let

$$\Lambda_i = (\lambda_{i1}, \ldots, \lambda_{i,i-1}, \lambda_{i,i+1}, \ldots, \lambda_{im}).$$

In the sequel we assume that g is a function of only Λ_i and $m > 2$. Without loss of generality let $i = 1$.

Two distinct values, $\Lambda_1^{(1)} = (\lambda_{11}^{(1)}, \ldots, \lambda_{1m}^{(1)})$ and $\Lambda_1^{(2)} = (\lambda_{11}^{(2)}, \ldots, \lambda_{1m}^{(2)})$, are equivalent with respect to $g(\Lambda_1)$ if $g(\Lambda_1^{(1)}) = g(\Lambda_1^{(2)})$.

THEOREM 3. *If for a Markov stopping time τ there exists a nonconstant estimator f which is efficient for a function $g(\Lambda_1)$ for two values of Λ_1 which are not equivalent with respect to $g(\Lambda_1)$, then there exist constants $\alpha_1, \ldots, \alpha_m$, β, not all zero, and $\gamma \neq 0$ such that*

$$\sum_{j=2}^{m} \alpha_j n_{1j} + \beta t_1 + \gamma = 0 \qquad m_{\vartheta_0}\text{-almost surely.}$$

THEOREM 4. *Let $P(T_1 > 0) = 1$. Let us assume that for a Markov stopping time τ there exist constants $\alpha_1, \ldots, \alpha_m$, β, not all zero, $\gamma \neq 0$, such that*

$$\sum_{j=1}^{m} \alpha_j n_{ij} + \beta t_1 + \gamma = 0 \qquad m_{\vartheta_0}\text{-almost surely,}$$

then for almost all $u \in U$ either

$$n_{1\sigma(2)} + n_{1\,\sigma(3)} + \ldots + n_{1\,\sigma(k)} = l \tag{2}$$

for some positive integer l, where $(\sigma(2), \ldots, \sigma(m))$ is a permutation of $(2, 3, \ldots, m)$ and k is an integer, $2 \leqslant k \leqslant m$, or

$$t_1 = \alpha \qquad \text{for some} \qquad \alpha > 0.$$

Let τ be the time at which line (2) is first attained for some $k = 1, 2, \ldots, m$ and some positive integer l. Such a plan we shall call an *inverse* plan.

Let τ be the time at which the line $t_1 = \alpha$ is first attained for some $\alpha > 0$. Such a plan we shall call a *simple* plan.

THEOREM 5. *The following functions are efficiently estimable ones*:

a) *for an inverse plan with some k, l and σ*

$$g(\Lambda_1) = \frac{\alpha_2 \lambda_{1\,\sigma(2)} + \ldots + \alpha_k \lambda_{1\,\sigma(k)} + \beta}{\lambda_{1\,\sigma(2)} + \ldots + \lambda_{1\,\sigma(k)}};$$

b) *for a simple plan*

$$g(\Lambda_1) = \alpha_2 \lambda_{12} + \ldots + \alpha_m \lambda_{1m} + \beta.$$

3. Sequential plans for the birth and death process ([5])

Let $(\Omega, \mathscr{F}, P_\vartheta)$ be a probability space and $\xi(t)$, $t \geqslant 0$, the birth and death process satisfying the following conditions:

$$P_\vartheta(\xi(t+\Delta t) = i+1|\xi(t) = i) = \lambda a_i \Delta t + o(\Delta t) > 0,$$

$$P_\vartheta(\xi(t+\Delta t) = i-1|\xi(t) = i) = \mu b_i \Delta t + o(\Delta t) > 0,$$

$$P_\vartheta(\xi(t+\Delta t) = i \pm k|\xi(t) = i) = o(\Delta t) > 0, \qquad k \geqslant 2,$$

$$P_\vartheta(\xi(t+\Delta t) = i|\xi(t) = i) = 1-(\lambda a_i + \mu b_i)\Delta t + o(\Delta t) > 0,$$

$$P_\vartheta(\xi(0) = n) = 1,$$

$$P_\vartheta(\xi(t+\Delta t) < 0|\xi(0) = 0) = 0$$

for every $\vartheta = (\lambda, \mu) \in \Theta \subset (0, \infty) \times (0, \infty)$, where $a_i > 0$ for every $i \geqslant 0$, and $b_0 = 0$, $b_i > 0$ for every $i \geqslant 1$. In the case $a_0 = 0$ we assume that

$$P_\vartheta(\xi(t+\Delta t) > 0|\xi(t) = 0) = 0 \qquad \text{for every} \quad \vartheta \in \Theta.$$

We also assume that almost all realizations of the process have only a finite number of jumps in the interval $[0, t]$, $t \geqslant 0$. The process $\xi(t)$, $t \geqslant 0$, generates the measure $\mu_{\vartheta,t}$ in the space $(D_1([0, \infty)), \mathscr{D}_{1,t})$, where $D_1([0, \infty))$ is the space of right continuous, integer-valued functions with unit jumps. The measure $\mu_{\vartheta,t}$ is absolutely continuous with respect to $\mu_{\vartheta_0,t}$ and

$$\frac{d\mu_{\vartheta,t}}{d\mu_{\vartheta_0,t}} = c(B(t), D(t), P_1(t), P_2(t); \vartheta_0) \cdot \lambda^{B(t)} \mu^{D(t)} \exp[-(\lambda P_1(t) + \mu P_2(t))],$$

where $B(t)$ – the number of births in the interval $[0, t]$, $D(t)$ – the number of deaths in the interval $[0, t]$,

$$P_1(t) = \sum_i a_i T_i(t), \qquad P_2(t) = \sum_i b_i T_i(t)$$

with $T_i(t)$ – the total time during which the process remains in the state i in the interval $[0, t]$.

Let us denote

$$Z(t) = (B(t), D(t), P_1(t), P_2(t)).$$

By the modification of Sudakov's lemma it follows that the joint distribution m_ϑ of $(\tau, Z(t))$ is absolutely continuous with respect to m_{ϑ_0} and

$$\frac{dm_\vartheta}{dm_{\vartheta_0}}(u) = c(b(u), d(u), p_1(u), p_2(u); \vartheta_0) \lambda^{b(u)} \mu^{d(u)} \exp[-(\lambda p_1(u) + \mu p_2(u))]$$
$$= h(u; \vartheta, \vartheta_0).$$

THEOREM 6. *If τ is a Markov stopping time, then under the regularity conditions we have*

$$E_\vartheta\left[\frac{B(\tau)}{\lambda}-P_1(\tau)\right]=0,\qquad E_\vartheta\left[\frac{D(\tau)}{\mu}-P_2(\tau)\right]=0,$$

$$E_\vartheta\left[\frac{B(\tau)}{\lambda}-P_1(\tau)\right]^2=\frac{1}{\lambda^2}E_\vartheta B(\tau),$$

$$E_\vartheta\left[\left(\frac{B(\tau)}{\lambda}-P_1(\tau)\right)\left(\frac{D(\tau)}{\mu}-P_2(\tau)\right)\right]=0$$

and

$$D_\vartheta f(\tau, Z(\tau))\geqslant\frac{\lambda^2}{E_\vartheta B(\tau)}\left(\frac{\partial}{\partial\lambda}g(\vartheta)\right)^2+\frac{\mu^2}{E_\vartheta D(\tau)}\left(\frac{\partial}{\partial\mu}g(\vartheta)\right)^2.$$

This inequality becomes an equality if and only if

$$f(u)=\left(\frac{\partial}{\partial\lambda}g(\vartheta)\right)\frac{\lambda^2}{E_\vartheta B(\tau)}\frac{\partial}{\partial\lambda}\log h(u;\,\vartheta,\,\vartheta_0)+$$
$$+\left(\frac{\partial}{\partial\mu}g(\vartheta)\right)\frac{\mu^2}{E_\vartheta D(\tau)}\frac{\partial}{\partial\mu}\log h(u;\,\vartheta,\,\vartheta_0)+g(\vartheta)\qquad m_{\vartheta_0}\text{-almost surely.}$$

THEOREM 7. *If $(\tau, f(\tau, Z(\tau)))$ is an efficient plan for $g(\vartheta)$, then there exist constants α, β, γ_1, γ_2, γ_3 for which*

$$\alpha b(u)+\beta d(u)+\gamma_1 p_1(u)+\gamma_2 p_2(u)+\gamma_3=0\qquad m_{\vartheta_0}\text{-almost surely,}$$

with $\alpha^2+\gamma_1^2\neq 0$, $\beta^2+\gamma_2^2\neq 0$.

We also conclude that only

$$g(\vartheta)=\frac{k_0+k_1\lambda\mu+k_2\lambda+k_3\mu}{l_0+l_1\lambda\mu+l_2\lambda+l_3\mu}$$

is an efficiently estimable function.

EXAMPLES. a) Let $b_0=0$, $a_i>0$,

$$\tau_{t_0}(x(\cdot))=\inf\{t\colon \sum_i a_i T_i(t, x(\cdot))=t_0\}.$$

Then

$$E_\vartheta P_1(\tau_{t_0})=t_0\qquad\text{and}\qquad D_\vartheta P_1(\tau_{t_0})=0.$$

Let $g(\vartheta)=g(\lambda)$. We have

$$E_\vartheta B(\tau_{t_0})=\lambda t_0\qquad\text{and}\qquad D_\vartheta B(\tau_{t_0})=\lambda t_0.$$

If the estimator $f(\tau_{t_0}, Z(\tau_{t_0}))$ is efficient, then it takes the form

$$f(\tau_{t_0}, Z(\tau_{t_0})) = c_1 B(\tau_{t_0}) + c_2$$

and

$$g(\lambda) = c_1 \lambda t_0 + c_2$$

is an efficiently estimable function.

b) Let

$$\tau_{x_0}(x(\cdot)) = \inf\{t:\ B(t, x(\cdot)) = x_0\}.$$

Then

$$E_\vartheta B(\tau_{x_0}) = x_0 \quad \text{and} \quad D_\vartheta B(\tau_{x_0}) = 0.$$

Let $g(\vartheta) = g(\lambda)$. We have

$$E_\vartheta P_1(\tau_{x_0}) = \frac{x_0}{\lambda} \quad \text{and} \quad D_\vartheta P_1(\tau_{x_0}) = \frac{x_0}{\lambda^2}.$$

So, if the estimator $f(\tau_{x_0}, Z(\tau_{x_0}))$ is efficient, then there exist constants c_1, c_2 such that

$$f(\tau_{x_0}, Z(\tau_{x_0})) = c_1 P_1(\tau_{x_0}) + c_2$$

and

$$g(\lambda) = c_1 \frac{x_0}{\lambda} + c_2$$

is an efficiently estimable function.

c) Let us take $a_i = b_i$, $a_0 = b_0 = 0$,

$$\tau^1_{x_0}(x(\cdot)) = \inf\{t:\ B(t, x(\cdot)) + D(t, x(\cdot)) = x_0\},$$

where $x_0 < n$. Then we have

$$E_\vartheta [B(\tau^1_{x_0}) + D(\tau^1_{x_0})] = x_0, \qquad D_\vartheta [B(\tau^1_{x_0}) + D(\tau^1_{x_0})] = 0,$$

$$P_1(\tau^1_{x_0}) = P_2(\tau^1_{x_0}),$$

$$E_\vartheta P_1(\tau^1_{x_0}) = \frac{1}{\lambda+\mu} x_0, \qquad E_\vartheta B(\tau^1_{x_0}) = \frac{\lambda}{\lambda+\mu} x_0, \qquad E_\vartheta D(\tau^1_{x_0}) = \frac{\mu}{\lambda+\mu} x_0,$$

$$E_\vartheta [B(\tau^1_{x_0}) D(\tau^1_{x_0})] = \frac{\lambda\mu}{(\lambda+\mu)^2} x_0(x_0 - 1),$$

$$D_\vartheta P_1(\tau^1_{x_0}) = \frac{1}{(\lambda+\mu)^2} x_0.$$

Let

$$f(\tau_{x_0}^1, Z(\tau_{x_0}^1)) = \frac{1}{x_0} P_1(\tau_{x_0}^1).$$

This estimator is efficient for the function

$$g(\lambda, \mu) = \frac{1}{\lambda + \mu}.$$

4. Sequential plans for a Markov process with migration ([3])

Let us assume that there is a flow of homogeneous objects entering a certain system A and each of the objects in the system may emigrate in one of the n directions $B_1, \ldots, B_n$. We also assume that the incoming objects form a Poisson flow with intensity α. Further, if an object is in the system A at a time $t > 0$, then it can emigrate during time $(t, t+\Delta t)$, independently of its entrance time, in the direction B_j, $j = 1, \ldots, n$, with the probability $\beta_j \Delta t + o(\Delta t)$. By $V(t)$ let us denote the number of objects which entered the system during the time interval $[0, t)$. Let $W_j(t)$ be the number of objects which emigrated during this time in the direction B_j, $j = 1, 2, \ldots, n$, and let k_0 be the number of objects present in A at the time $t = 0$. Let $I = \{0, 1, 2, \ldots\}$ and $T = [0, \infty)$. Next, let us denote $W_0(t) = k_0 + V(t)$ and let ϑ be the vector $(\alpha, \beta_1, \ldots, \beta_n) \in \Theta \subset (0, \infty)^{n+1}$

Let $(\Omega, \mathcal{F}, P_\vartheta)$ be a probability space. Let us consider a homogeneous Markov process

$$\xi(t) = (W_0(t), W_1(t), \ldots, W_n(t)), \qquad t \in T,$$

defined on $(\Omega, \mathcal{F}, P_\vartheta)$. The values of this process we shall denote by $x = (w_0, w_1, \ldots, w_n)$.

We assume that the process $\xi(t)$, $t \in T$, satisfies the following conditions:

a) $P_\vartheta(\xi(0) = (k_0, 0, \ldots, 0)) = 1$,

b) the transition probabilities are of the form

$$P_\vartheta(\xi(t+\Delta t) = y | \xi(t) = x)$$

$$= \begin{cases} \alpha\Delta t + o(\Delta t) & \text{if } x = (w_0, w_1, \ldots, w_n) \text{ and} \\ & \phantom{\text{if }} y = (w_0+1, w_1, \ldots, w_n), \\ \sum_{j=1}^{n} k\beta_j \Delta t + o(\Delta t) & \text{if } x = (w_0, w_1, \ldots, w_n) \text{ and} \\ & \phantom{\text{if }} y = (w_0, w_1, \ldots, w_{j-1}, w_j+1, w_{j+1}, \ldots, w_n), \\ & \phantom{\text{if }} j = 1, \ldots, n, \\ 1 - (\alpha + \sum_{j=1}^{n} k\beta_j)\Delta t + o(\Delta t) & \text{if } x = y, \\ o(\Delta t) & \text{otherwise,} \end{cases}$$

where $k = w_0 - \sum_{j=1}^{n} w_j$ denotes the value of the random variable,

$$K(t) = W_0(t) - \sum_{j=1}^{n} W_j(t)$$

determining the number of objects present in the system at time t,

c) $P_\vartheta(K(t) \geqslant 0) = 1$ for every $t > 0$.

Our problem is to estimate the intensities $\alpha, \beta_1, \ldots, \beta_n$ or their functions, using the observation of the process $\xi(t)$, $t \in T$. From the Skorokhod theorems ([1]) we have $\mu_{\vartheta,t} \ll \mu_{\vartheta_0,t}$ and

$$\frac{d\mu_{\vartheta,t}}{d\mu_{\vartheta_0,t}}(x(\cdot)) = \left(\frac{\alpha}{\alpha_0}\right)^{v(t)} \exp\Big[-(\alpha-\alpha_0)t - \sum_{j=1}^{n} (\beta_j - \beta_{0j}) \times \Big(k(t)t + \sum_{j=1}^{n} \sum_{l=1}^{w_j(t)} \sigma_{jl} - \sum_{r=1}^{v(t)} v_r\Big)\Big] \prod_{j=1}^{n} \left(\frac{\beta_j}{\beta_{0j}}\right)^{w_j(t)},$$

where v_r's are the arrival times,

$$0 < v_1 < \ldots < v_{v(t)} < t,$$

and σ_{jl}'s are the exit times in direction B_j,

$$0 < \sigma_{j1} < \ldots < \sigma_{j w_j(t)} < t, \qquad j = 1, \ldots, n.$$

Let us denote

$$w(t) = (w_1(t), \ldots, w_n(t)),$$

$$S(t, x(\cdot)) = k(t)t + \sum_{j=1}^{n} \sum_{l=1}^{w_j(t)} \sigma_{jl} - \sum_{r=1}^{v(t)} v_r,$$

$$\beta = \sum_{j=1}^{n} \beta_j, \qquad \beta_0 = \sum_{j=1}^{n} \beta_{0j}$$

and

$$Z(t, x(\cdot)) = (v(t), w(t), S(t, x(\cdot))).$$

The function S determines the overall time spent in the system by the objects which arrived during the time $[0, t)$ or were in the system at the time $t = 0$. So we can write

$$\frac{d\mu_{\vartheta,t}}{d\mu_{\vartheta_0,t}}(x(\cdot)) = \left(\frac{\alpha}{\alpha_0}\right)^{v(t)} \exp[-(\alpha-\alpha_0)t - (\beta-\beta_0)S(t, x(\cdot))] \prod_{j=1}^{n} \left(\frac{\beta_j}{\beta_{0j}}\right)^{w_j(t)}$$

$$= c(t, Z(t, x(\cdot)); \vartheta_0) \cdot \alpha^{v(t)} \exp[-\alpha t - \beta S(t, x(\cdot))] \prod_{j=1}^{n} \beta_j^{w_j(t)}.$$

From the modification of Sudakov's lemma we have

$$\frac{dm_{\vartheta}}{dm_{\vartheta_0}}(u) = c(u;\, \vartheta_0)\, \alpha^{v(u)} \exp[-\alpha t(u) - \beta s(u)] \prod_{j=1}^{n} \beta_j^{w_j(u)}.$$

The Cramér–Rao inequality takes the form

$$D_{\vartheta} f(\tau,\, Z(\tau)) \geqslant \frac{\alpha}{E_{\vartheta}\tau}[g'_{\alpha}(\vartheta)]^2 + \frac{1}{E_{\vartheta} S(\tau)} \sum_{j=1}^{n} \beta_j [g'_j(\vartheta)]^2.$$

The equality holds at a particular value of ϑ iff

$$f(\tau,\, Z(\tau)) = \frac{g'_{\alpha}(\vartheta)}{E_{\vartheta}\tau}[V(\tau) - \alpha\tau] + \frac{1}{E_{\vartheta} S(\tau)} \sum_{j=1}^{n} g'_j(\vartheta)[W_j(\tau) - \beta_j S(\tau)] + g(\vartheta).$$

EXAMPLES. a) Let

$$\tau^{(1)}(x(\cdot)) = T_0 > 0.$$

For this stopping time

$$f^{(1)} = \frac{c_1}{T_0} V(T_0) + c_2$$

is efficient for $g(\vartheta) = c_1\alpha + c_2$.

b)

$$\tau^{(2)}(x(\cdot)) = \inf\{t:\ v(t) = v_0\},$$

where v_0 is a positive integer number.

For this stopping time

$$f^{(2)} = \frac{c_1}{v_0}\tau^{(2)} + c_2$$

is efficient for $g(\vartheta) = c_1/\alpha + c_2$.

c)

$$\tau^{(3)}(x(\cdot)) = \inf\{t:\ S(t,\, x(\cdot)) = s_0\}, \qquad s_0 > 0.$$

For this stopping time

$$f^{(3)} = \frac{1}{s_0} \sum_{j=1}^{n} c_j W_j(\tau^{(3)}) + d$$

is efficient for $g(\vartheta) = \sum_{j=1}^{n} c_j \beta_j + d$.

d)

$$\tau^{(4)}(x(\cdot)) = \inf\{t:\ \sum_{i=1}^{k} w_{\sigma(i)}(t) = m_0\},$$

where m_0 is a positive integer number, $2 \leqslant k \leqslant n$, σ is a permutation of the set $\{1, 2, \ldots, n\}$. For this stopping time

$$f^{(4)} = \frac{1}{m_0} \sum_{i=1}^{k} c_i W_{\sigma(i)}(\tau^{(4)}) + d$$

is efficient for

$$g(\vartheta) = \Big(\sum_{i=1}^{k} c_i \beta_{\sigma(i)}\Big)\Big(\sum_{i=1}^{k} \beta_{\sigma(i)}\Big)^{-1} + d.$$

e)

$$\tau^{(5)}(x(\cdot)) = \inf\{t\colon\ w_j(t) = l_0\},$$

where t_0 is positive integer.
For this stopping time

$$f^{(5)} = \frac{c_1}{l_0} S(\tau^{(5)}) + c_2$$

is efficient for $g(\vartheta) = c_1/\beta_j + c_2$.

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Presented to the semester
Sequential Methods in Statistics
September 7–December 11, 1981