DYNAMICAL SYSTEMS AND ERGODIC THEORY
BANACH CENTER PUBLICATIONS, VOLUME 23
PWN-POLISH SCIENTIFIC PUBLISHERS
WARSZAWA 1989

BIFURCATION SEQUENCES IN THE DISSIPATIVE SYSTEMS WITH SADDLE EQUILIBRIA

M. A. ZAKS and D. V. LYUBIMOV

Institute of Continuous Media Mechanics, Urals Scientific Center,
Academy of Sciences of the U.S.S.R.
Perm, U.S.S.R.

The bifurcation sequences associated with formation of saddle-connections are studied with use of renormalization methods. In the systems with a single saddle equilibrium the sequence converges at the exponential rate whereas in the case of two symmetrical saddle-points the convergence is superexponential. In both cases the quantitative aspects of the scenario prove to be entirely determined by the ratio of two real eigenvalues of the vector field linearized near the saddle-point.

The initial elements of the universal scenarios of transition to chaos are commonly time-dependent states (periodic for period-doublings [1] and intermittency [2] or quasiperiodic [3, 4]). This work is focused on the bifurcation sequences in which the key role is given to even more simple objects — the non-stable equilibria represented in dynamical systems by saddle-points. The parameter variation may produce saddle connections (homoclinic orbits) — captures of unstable manifolds of such points by their stable manifolds. This leads to creation or destruction of the closed trajectories — the limit cycles [5]. In presence of certain symmetries more complicated structures may arise. The most known example of this is probably the formation of the chaotic attractor in the famous Lorenz model [6].

The aim of this note is to study with the help of some (intuitive rather than rigorous) arguments two bifurcation sequences associated with creation of homoclinic orbits. The first one occurs in the systems with the single saddle equilibrium. It bears some features of both the Lorenz [5, 6] Feigenbaum [1] scenarios and was described earlier [7, 8]. The numerical renormalization-group (RG) study shows the universality of this route for symmetrical systems. Then we shall follow the peculiar transformation of the bifurcation picture in case of a dynamical system with a couple of symmetrical equilibria. In this situation, the bifurcation sequence converges not at a geometric rate but much more rapidly. The simplified RG-analysis finds this unusual behaviour to be due to the fact that the fixed-point of the RG-transformation which governs the dynamics of such systems is singular.

At the end we use the continuous model – the set of three ordinary differential equations (ODE) to illustrate the preceding qualitative and quantitative analysis of discrete systems.

Bifurcations in the system with a single saddle-point

Consider the system of N ($N \geqslant 3$) autonomous ODE which satisfies the following conditions in some open set of parameter values:

(a) The system has a fixed point 0.

(b) In the spectrum of the vector field linearized near 0 two eigenvalues λ_1 and λ_2 with the biggest real parts are both real. One of them (λ_1) is positive and other is negative. This means that the point 0 is a saddle-point with 1-dimensional unstable manifold consisting of the point itself and two trajectories, which we shall call separatrices.

(c) There exists a transformation under which the system is invariant and each of the separatrices is transformed into the other one (this symmetry is typical of many dynamical systems arising in the problems of free thermal convection).

(d) In the parameter space there is a codimension-1 surface upon which the separatrices are captured by the stable manifold of the saddle-point; they return to 0 being tangent to each other and constitute a couple of structurally unstable homoclinic orbits.

Trajectories leaving the vicinity of the saddle-point return there again. Their behaviour is dominated by the dynamics upon the "slow manifold" – the invariant 2-dimensional surface which is tangent in the point 0 to the eigenvectors associated to λ_1 and λ_2. Hence the Poincaré mapping is reduced to one-dimensional recursion relation which in the lowest order can be written down as

$$x_{i+1} = f(x_i), \qquad f(x) = (a|x|^{\nu} - \mu)\cdot \operatorname{sign}(x). \tag{1}$$

Here the coordinate x is measured along the eigenvector associated to λ_1; $\nu = -\lambda_2/\lambda_2$ is the so-called saddle ratio, μ is the value of x for the first return on the secant of the separatrix leaving for $x < 0$. The homoclinic orbits appear when $\mu = 0$. The orbits are assumed to be orientable [10] which makes the factor a in (1) positive (then the proper rescaling of x for $\nu \neq 1$ makes $a = 1$).

If $\nu < 1$ then the Lorenz route to chaos [9] via the "preturbulent state" [11] is observed. We consider the case of $\nu > 1$. Here the Poincaré-mapping has zero derivative at $x = 0$ and the states arising from the destruction of homoclinic orbits are stable.

The growth of μ changes the dynamics of (1). The associated bifurcations in terms of the flow are as follows:

(a) At $\mu = (a\nu)^{1/1-\nu} \cdot (1-\nu)/\nu$ the double tangent bifurcation produces two stable and two non-stable closed trajectories (cycles).

(b) The stable cycles approach the saddle and at $\mu = 0$ coincide with the separatrices forming the couple of homoclinic orbits. The destruction of this couple produces stable 2-looped cycle (trajectory twice returning to the neighbourhood of the saddle-point) which consists of two symmetrical halves.

(c) At $\mu = (a\nu)^{1/1-\nu} \cdot (1+1/\nu)$ this solution loses stability and the couple of symmetrical to each other 2-looped cycles bifurcates from it (the symmetry-breaking bifurcation).

(d) At $\mu = a^{1/1-\nu}$ these cycles coincide with separatrices. Here $f^2(0) = 0$ and we observe the couple of 2-looped homoclinic orbits out of which a 4-looped stable cycle is born.

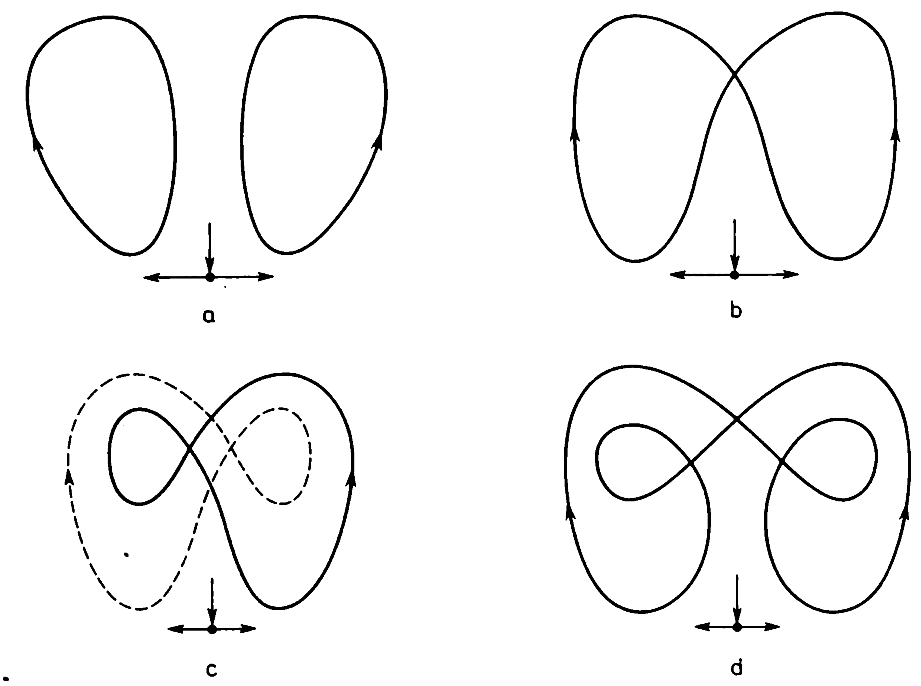

Fig. 1. Transformation of attracting states in systems with single saddle-point

The first steps of the process are sketched in Fig. 1. As the scenario goes on, two kinds of bifurcation alternate infinitely many times. The first one is the formation of a couple of homoclinic orbits through which out of two stable cycles a new one (twice as long) is created, and the second kind is the symmetrybreaking bifurcation. Both are structurally unstable but become generic in dynamical systems of considered symmetry.

The computations have shown that if v is fixed then the differences between the successive same-kind-bifurcation values of μ decrease showing asymptotically the exponential law. The explanation of this is in the similarity of high iterates of (1).

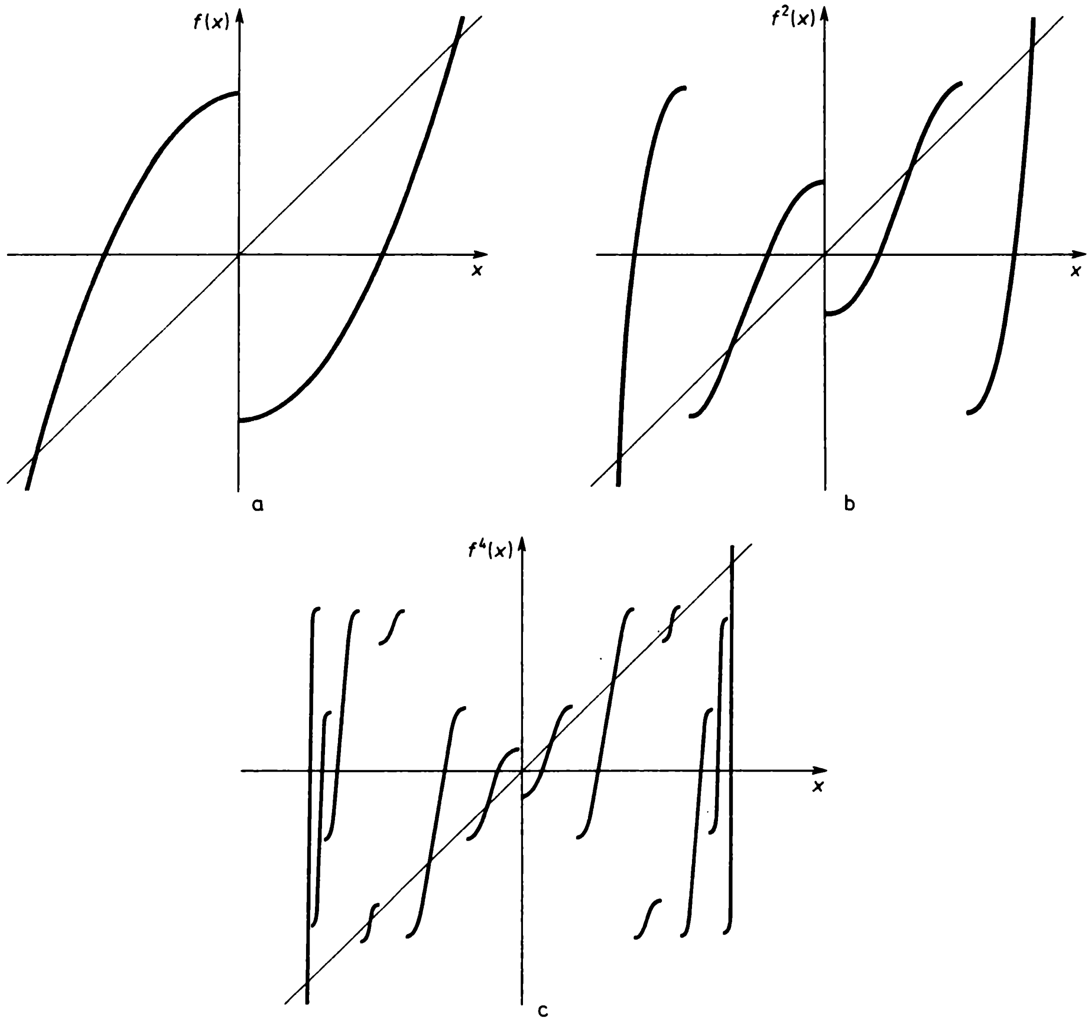

Fig. 2. The return mapping

The Fig. 2 shows the graphs of $f(x)$, $f^2(x)$ and $f^4(x)$ for $\mu \geqslant a^{1/1-v}$. The central parts of the graphs are all alike and it is only the scale that differs. This permits to apply the renormalization methods similar to the case of period-doublings [1]. (Note that there are no doublings of time period among the considered bifurcations – period tends to infinity at the bifurcations of the 1st kind and does not change at all at those of the 2nd kind.)

Consider the "doubling operator" $I(f(x)) = \alpha ff(x/\alpha)$ where α is the factor determined from the scale invariance. Let us search for the fixed point of this operator – the solution of the equation $f_*(x) = \alpha_* f_* f_*(x/\alpha_*)$ in the class of functions

$$f(x) = (-1 + a_1 |x|^\nu + a_2 |x|^{2\nu} + \ldots)\,\mathrm{sign}(x). \tag{2}$$

The equation differs from the well-known equation of Feigenbaum–Cvitanovic [1] by the sign of the righthand side only. This is quite natural because they have been dealing with continuous unimodal functions whereas in our problem $f(x)$ is piecewise-monotonic and discontinuous due to the passing of the trajectories near the saddle-point. The equation was solved numerically for various values of the saddle ratio ν by the application of the Newton method to the truncated series (2). Then the operator was linearized near the fixed point and its spectrum was examined. The calculations permit to conclude that for all $\nu \geqslant 1$ the operator has a fixed point in the spectrum of which only one eigenvalue lies outside the unit disc. It is precisely this eigenvalue that determines the convergence rate of the bifurcation sequence, while the value of α_* associated to this point describes the scaling of trajectories on the attractor near the saddle-point.

The fixed point depends on ν. Hence there is a 1-parameter family of universality classes. The parameter is the value of the saddle ratio ν, which entirely determines all the asymptotic characteristics of the bifurcation sequence as well as the properties (scaling, dimension etc.) of the chaotic attractor. The transition to chaos is associated to the intersection of the stable manifold of the fixed point which occurs in the universal way.

It was noted in [7] that one may put in correspondence the families of continuous unimodal mappings exhibiting the period-doubling scenario and the mappings similar to (1) with its sequence of homoclinic bifurcations. Formally, the previous also have the family of universality classes with the parameter being the value analogous to ν – the order of extremum. But the natural smoothness condition immediately singles out the integer even extrema and of them the simplest – the quadratic one. Hence the corresponding $\delta = 4.6692$ and $\alpha = 2.5029$ are the same for all typical systems in which the period-doublings take place. On the contrary the discontinuous mappings like (1) do not specify any value of the saddle ratio. In the usual cases the structure of the spectrum of the equilibrium (and hence the saddle ratio) depends on external "physical" parameters (e.g. the Rayleigh and the Prandtl numbers in thermal convection). On the plane of two such parameters the quantitative characteristics of the bifurcation sequence may well vary along the line corresponding to the boundary of chaos.

Systems with two saddle equilibria

In systems of symmetry considered the saddle-point may appear as a result of direct (supercritical) or inverted (subcritical) bifurcation of the stable node. In the last case the stable point in the subcritical zone coexists with two symmetrical saddle-points, which coalesce with it on the stability border and transfer to it their instability.

The property of having biasymptotical trajectories may be also transferred. We shall see that near the segment of the border to which the line of formation of homoclinic orbits comes from the supercritical zone one can find recombinations of invariant manifolds of the couple of saddle-points in subcritical region.

How can one obtain the return mapping? The usual technique of linearization of equations in the neighbourhood of the saddle-point cannot be employed here because the eigenvalue associated to the unstable direction vanishes at the border. Hence the corresponding equation on the slow manifold must include the nonlinear terms; for symmetry reasons this will be the cubic one. Then the slow manifold equations become

$$\dot{x} = \lambda_1 x + x^3, \qquad \dot{z} = \lambda_2 z \tag{3}$$

and the linearization of global segments of the trajectories near the separatrices produces the expression

$$x_{i+1} = f(x_i), \quad f(x) = \left(a \left(\frac{x^2+\lambda_1}{|x|} \right)^{-\lambda_2/\lambda_1} - \mu \right) \mathrm{sign}(x) \tag{4}$$

where μ is the first return of the separatrix leaving to the left from the saddle-point (or from the left saddle if there are two), the eigenvalue λ_1 is the parameter along with μ and the stability border is given by $\lambda_1 = 0$. The relation (4) describes the dynamics near the single saddle-point for $\lambda_1 > 0$ or near the set of the stable node $x = 0$ and two saddles $x = \pm\sqrt{-\lambda_1}$ for $\lambda_1 < 0$.

When $\mu = -\sqrt{-\lambda_1}$ the outer separatrices return back producing homoclinic orbits; when $\mu = \sqrt{-\lambda_1}$ there are saddle-connections of another kind: $f^2(\pm\sqrt{-\lambda_1}) = \mp\sqrt{-\lambda_1}$ – each separatrix goes to the other saddle-point and together they form the heteroclinic contour.

This part of the parameter plane is sketched in Fig. 3. The subcritical loss of stability of the equilibrium takes place on the ordinate axis. The lines $\mu = 0$ $(\lambda_1 > 0)$ and $\mu = -\sqrt{-\lambda_1}$ mark the formation of pairs of homoclinic orbits and the line $\mu = \sqrt{-\lambda_1}$ – of the heteroclinic contour. For $\lambda_1 < 0$ besides the separatrix going outwards each saddle-point has another one which goes to the node. Therefore at one side of each of the bifurcation lines (up from the lower and down from the upper) the destruction of saddle-connections is accompanied by the death of the periodic state – all trajectories in the saddle neighbourhood (except its stable manifold) belong to the basin of the stable node.

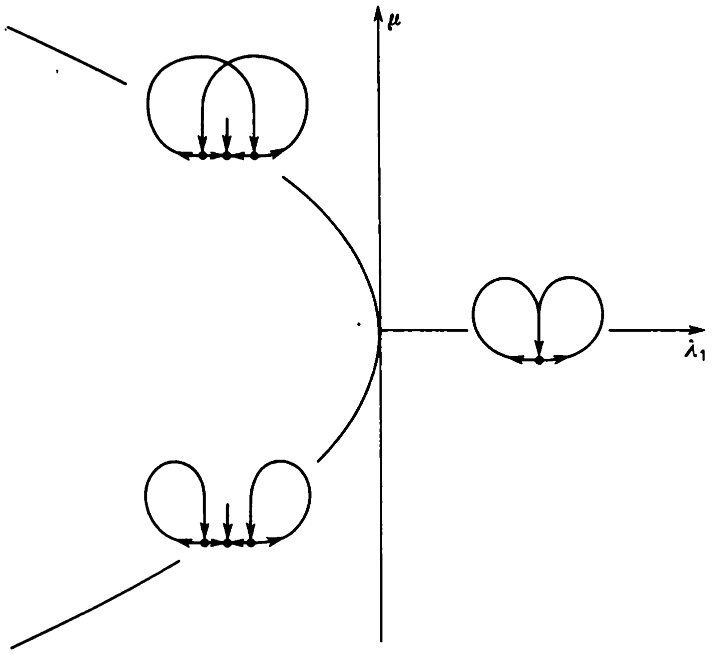

Fig. 3. Formation of saddle-connections near the line of neutral stability

As we move farther into the subcritical zone, we may omit the nonlinear term in the equation on the slow manifold and we get the more simple expression for the return mapping

$$x_{i+1} = f(x_i), \quad f(x) = \left(a(|x| - d)^{\nu} - \mu\right)\operatorname{sign}(x) \tag{5}$$

where ν is the saddle ratio of symmetrical saddle-points. This mapping is defined for $|x| \geqslant d$, where d is the half-distance between the saddle-points. The interval $(-d, d)$ together with its preimages under (5) corresponds to the basin of the node (the trivial attractor). The trajectories never return to the saddles from this interval and hence the return mapping is not defined here.

On the stability border the saddle ratio is infinite and in the nearby strip it is greater than 1. The analysis of (5) showed that for $\nu > 1$ the following bifurcations are observed with the growth of μ;

– for $\mu = \mu_c < -d$ the double tangent bifurcation occurs producing two couples (for $x < 0$ and $x > 0$) of stable and non-stable fixed points of (5) associated with 1-looped cycles in the phase space:

– for $\mu = -d$ the stable cycles disappear in the homoclinic bifurcations and their basins are adjoined to the basin of the node.

The further growth of μ leads to the infinite sequence of 3 kinds of alternating bifurcations. The 1st one is the creation of a stable cycle from the heteroclinic contour. Two stable cycles bifurcate from this cycle in the 2-d kind of bifurcation (the same symmetry-breaking as in systems with a single saddle-point). At the bifurcation of the 3-d kind these cycles disappear through the homoclinic bifurcation. The first transformations of attracting trajectories are sketched in Fig. 4.

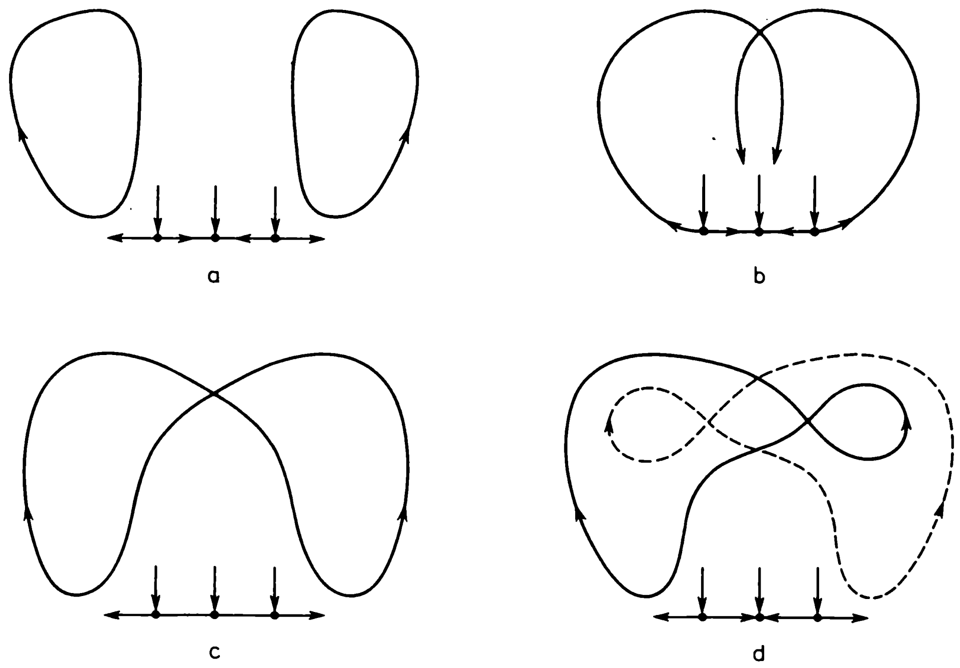

Fig. 4. Transformations of attractors in systems with the node and couple of saddle-points

In the parameter intervals between bifurcations of the 1st and the 2-d kinds there are two attracting states (the node and the stable cycle). Between the values of the 2-d and the 3-d kinds one sees three attractors (the node and the couple of stable cycles) with basins separated by the stable manifold of the non-stable cycle. Between a bifurcation of the 3-d kind and the next bifurcation of the 1st kind only the trivial state – the node – is attracting.

On this way the length of attracting closed trajectories grows: each new one arises from a heteroclinic contour, which consists of twice as many turns as any of the homoclinic orbits ending the previous 3 steps of the sequence. It should be noted that each 3 steps add a new non-stable cycle to the space.

The convergence law of this sequence differs from the exponential one (observed in the systems with single saddle equilibrium). Computations have shown that the progression is formed not by the differences $(\mu_{n+1}-\mu_n)$ of consecutive bifurcation values of the same kind but the logarithms of these differences:

$$\lim_{n\to\infty}\frac{\ln(\mu_{n+1}-\mu_n)}{\ln(\mu_n-\mu_{n-1})}=\varkappa(\nu)>1 \tag{6}$$

(We must note that an analogous relation was obtained in [12] from numerical analysis of the mapping with two extrema.)

The explicit formulae in case of $\nu = 1$ (piecewise-linear mapping) provide $\varkappa(1) = 2$ that is $(\mu_{n+1} - \mu_n) \sim (\mu_n - \mu_{n-1})^2$ and the sequence of bifurcation values converges much more rapidly than any geometric progression.

This fast convergence hampers the numerical studying of (5). After few steps the interval between the bifurcation parameter values becomes too small to be resolved within the computer precision. The asymptotical convergence law is not established yet and the estimates of the dependence $\varkappa(\nu)$ are rather crude.

To explain this feature of systems with two saddle-points we consider the dynamics of RG-transformation. This is the very same transformation $I(f(x)) = \alpha ff(x/\alpha)$ which was used above in the case of single-equilibrium systems but now it acts on the functions of another kind

$$f(x) = (-1 - d + a_1(|x| - d)^\nu + a_2(|x| - d)^{2\nu} + \ldots)\,\mathrm{sign}(x) \tag{7}$$

defined for $|x| \geqslant d$. The scaling factor α is now determined from

$$\alpha = \frac{f(d) + d}{ff(d) + d} \tag{8}$$

This recursion law transforms the function into another one with the "undefinedness interval" multiplied by α. The fixed-point associated with the single-equilibrium systems refers to the case $d = 0$. The spectrum analysis in this situation yields another relevant eigenvalue (in addition to δ). It is equal to $\alpha_*(\nu)$ – the value of the scaling factor (8) in the fixed-point – and is associated to the perturbations which generate the "undefinedness interval". The dynamics proves to be two-parametrous with the distance between the saddles being the second relevant parameter.

Because of this crossover the fixed-point shows the asymptotic dynamics only of the systems with zero distance between the saddles (that is with one saddle). The systems with small but finite distance behave similarly to them only at first stages of the scenario, but further on the convergence rate exhibits unlimited growth.

To evaluate the asymptotical dynamics we shall employ the truncated RG-analysis, retaining in the expansion (7) only the lowest order terms

$$f(x) = (-1 - d + q(|x| - d)^\nu)\,\mathrm{sign}(x). \tag{9}$$

(This truncation becomes asymptotically correct for $\nu \to 1$.)

The RG-transformation generates certain dynamics on the (q, d) plane: it transforms the function with some initial values of q and d into another function with (generally) altered q and d. The dynamics is governed by recurrent relations

$$d_{i+1} = \frac{d_i}{q_i - 2d_i - 1}, \tag{10}$$
$$q_{i+1} = \nu q_i^2 (q_i - 2d_i - 1)^{\nu - 1}.$$

If the initial values q_0 and $d_0 > 0$ are chosen on the accumulation line of the bifurcation curves family then, upon iterating (10) the values of q_i and d_i will grow on. In terms of the variables $b = q^{-1}$ and $u = d^{-1}$ the transformation (10) becomes

$$u_{i+1} = \frac{u_i - 2b_i - u_i b_i}{b_i}, \tag{11}$$
$$b_{i+1} = \frac{b_i^{1+\nu}}{\nu} \left(\frac{u_i}{u_i - 2b_i - u_i b_i} \right)^{\nu - 1}.$$

The point Q ($u = b = 0$) is the singular fixed point of the transformation (11). Consider the dynamics in its neighbourhood Q_0. For most of the small perturbations the application of (11) kicks the system out of Q_0: the next value of u will not be small. The systems close to the stable separatrix W_s of Q (i.e. those for which (11) can be performed in Q_0 many times) must at any moment satisfy the condition $u \approx 2b$. Then it is easy to verify that $u_{i+1} \sim u_i^{1+1/\nu}$ and the systems which belong to W_s approach the point Q superexponentially: $u_i \approx u_0^{\varkappa^i}$ where

$$\varkappa = 1 + 1/\nu. \tag{12}$$

The systems close to W_s move away from it and the law of departure yields the convergence law of the bifurcation sequences. Let $u_i = 2b_i + \varphi_i$ where φ_i is small. Having performed (11) one sees that $\varphi_{i+1} = \varphi_i / b_i$ – the decline from the separatrix grows superexponentially;

$$\varphi_n \approx \varphi_0 \cdot C^{\varkappa^n}/C, \quad C > 1 \tag{3}$$

which gives the relation (6) for the differences between the successive bifurcation values of the parameter.

The same asymptotic law describes the scaling on the attractor. The characteristic length is here the distance from the saddle $\xi_n = f^{2^n}(d) + d$. For large n we have $\xi_{n+1} \sim \xi_n^{\varkappa}$ that is

$$\xi_n \approx (B\xi_0)^{\varkappa^n}/B. \tag{14}$$

Numerical investigation of bifurcations

Here we present an example – the system of ODE's which exhibits the described bifurcations. The equations

$$\begin{aligned} \dot{X} &= -\sigma X + \sigma Y + \sigma D Y(Z-R), \\ \dot{Y} &= RX - Y - XZ, \\ \dot{Z} &= XY - bZ \end{aligned} \tag{15}$$

were derived in [13] for the description of thermal convection in the horizontal layer of fluid which is put into the oscillating vertical gravitational field. Here D ($D \geqslant 0$) is the intensity of vibrational modulation, the parameters σ, R and b have the same meaning as in the Lorenz model (which is Eq. (15) for $D = 0$).

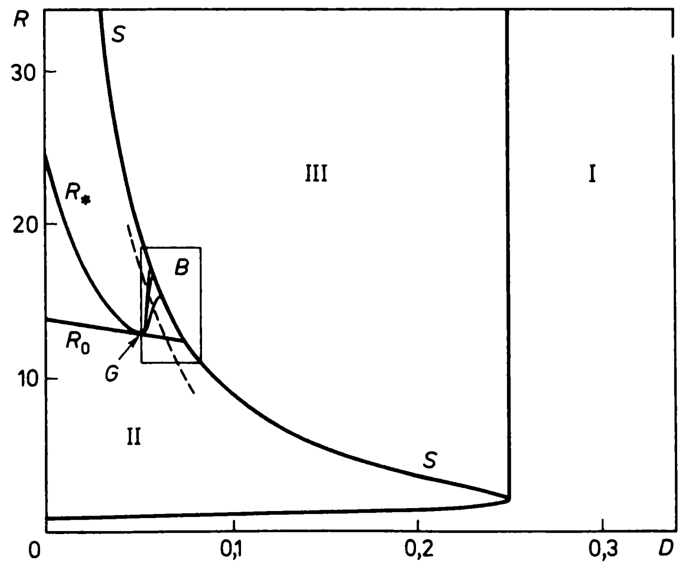

Fig. 5a

The parameter plane D-R consists of three zones (Fig. 5a). In the first of them the trivial solution $X = Y = Z = 0$ is globally stable, in the second it becomes the saddle-point with 1-dimensional unstable manifold and in the third it is stable with respect to infinitesimal perturbations. The border between the second and the third zones is the line S of inverted bifurcation where the couple of saddle-points bifurcates from the trivial solution.

Let us fix the traditional values $\sigma = 10$ and $b = 8/3$ and integrate equations (15) numerically for various R and D. The line $R_0(D)$ corresponds to the formation of the pair of homoclinic orbits of the trivial solution (it starts from the known point $R_0(0) = 13.926\ldots$). For D small the saddle ratio is less than 1 on $R_0(D)$ (just as in the Lorenz system) and the Lorenz scenario evolves (the lower boundary of the region of attracting chaotic states is shown by $R_*(D)$). The value of ν on $R_0(D)$ grows with D monotonically. To the right from the point G $\nu > 1$ and we observe the bifurcation sequence described in the first part of this paper. If one moves along the line corresponding to some constant value of ν then the sequence $\{R_n(\nu)\}$ of intersections with the curves of formation of 2^n-looped homoclinic orbits converges asymptotically at a geometric rate. This rate coincides within numerical accuracy with the value of $\delta(\nu)$ obtained for the same ν by RG-analysis. (For more details concerning equations (15) and this bifurcation sequence see [8] and [13].)

The above analysis predicts homoclinic and heteroclinic bifurcations of the couple of saddle-points near the intersection of $R_0(D)$ and the instability border S of the trivial solution. This region (the rectangle B from Fig. 5a) is shown in Fig. 5b more distinctly.

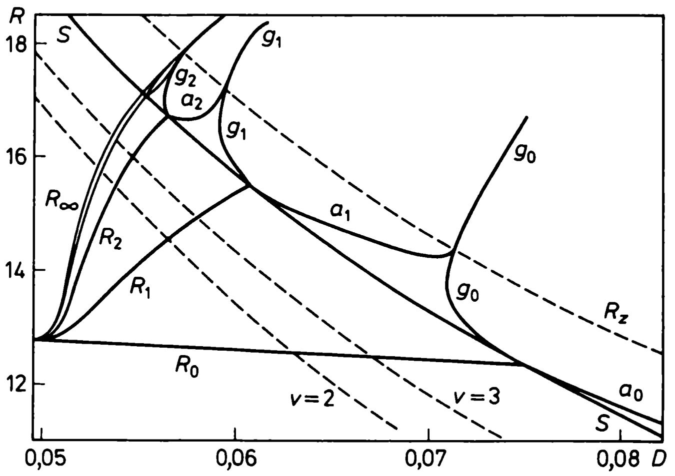

Fig. 5b

To the left from S one sees the curves of existence of 2^n-looped homoclinic orbits of the single saddle-point – the lines R_n which accumulate to the line R_∞. From the points where these lines stick to S couples of bifurcation curves a_n and g_n come out to the right marking the formation of 2^n-looped homoclinic orbits and 2^{n+1}-looped heteroclinic contour respectively. Moving further to the right we approach the line $R_z(D)$ on which the saddle ratio again becomes equal to 1. Here the curves of 2^{n+1}-looped homoclinic orbits and 2^{n+1}-looped heteroclinic contours merge and intervals of existence of non-trivial stable states shrink. To the right of this line homoclinic orbits never appear whereas the destruction of heteroclinic contours leads to the creation of non-stable closed trajectories. Computations have shown that in this parameter region trajectories with any initial conditions are eventually attracted to the stable trivial solution.

To evaluate the convergence of the bifurcation sequences for fixed ν one can use either the differences between the bifurcation parameter values (relation (6)) or the spatial characteristics of trajectories (relation (14)). The results of integration show that these estimates are in accordance with each other as well as with the data obtained from the numerical examination of the mapping (5). The following table gives the values

$$\varkappa_n = \frac{\ln \dfrac{a_{n-1} - a_{n-2}}{a_n - a_{n-1}}}{\ln \dfrac{a_{n-2} - a_{n-3}}{a_{n-1} - a_{n-2}}} \tag{16}$$

calculated for various ν; we see that they are not too far from those predicted by the formula (12).

Table

ν	$\varkappa_3$	$\varkappa_4$	$\varkappa_5$	$\varkappa_6$	$1+\frac{1}{\nu}$
1.000	1.3983	1.9302	1.9445	2.0057	2.00000
1.250	1.6598	1.8740	1.7683	1.7587	1.80000
1.333	1.6691	1.8663	1.7374	1.7036	1.75000
1.500	1.6702	1.8331	1.6942	1.6203	1.66667
1.750	1.6648	1.7746	1.6407	1.5404	1.57143
2.000	1.6581	1.7276	1.5930	1.4858	1.50000

All this demonstrates the important role the saddle equilibria can play in transition to complicated dynamical states. The universal quantitative aspects of global dynamics appear to be entirely determined by the local value which is comparatively easy in computation — the ratio of two eigenvalues from the linearized problem of the stability of equilibrium.

References

[1] M. J. Feigenbaum, Journ. Stat. Phys. 19 (1978), p. 25.
[2] Y. Pomeau, P. Manneville, Commun. Math. Phys. 74 (1980), p. 189.
[3] S. Ostlund, D. Rand, S. Sethna, E. Siggia, Physica 8D (1983), p. 303.
[4] J. P. Sethna, E. Siggia, Physica 11D (1984), p. 193.
[5] L. P. Shilnikov, Math. Sbornik 77 (119) (1968), p. 461.
[6] E. N. Lorenz, Journ. Atmos. Sci. 20 (1963), p. 120.
[7] A. Arneodo, P. Coullet, C. Tresser, Phys. Lett. 81A (1981), p. 197.
[8] D. V. Lyubimov, M. A. Zaks, Physica 9D (1983), 52.
[9] V. S. Afraimovich, V. V. Bykov, L. P. Shilnikov, Trans. Moscow Math. Soc. 44 (1982), p. 150.

[10] L. P. Shilnikov, *Appendix to the Russian edition of J. Marsden and M. McCracken, The Hopf Bifurcation and its applications*, Nauka, Moscow 1980.
[11] E. Yorke, J. A. Yorke, Journ. Stat. Phys. 21 (1979), p. 263.
[12] J. Coste, N. Peyraud, Physica 5D (1982), p. 188.
[13] M. A. Zaks, D. V. Lyubimov, V. I. Tschernatinskiĭ, Izv. Acad. Sci. USSR, Fiz. Atmos. Okean. 19 (1983), p. 312.