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A GENERAL ACYCLICITY LEMMA AND ITS USES

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1. Acyclicity statements

Let A be a commutative noetherian local ring and

$$L_{\bullet} = 0 \to L_s \to \ldots \to L_1 \to L_0 \to 0$$

be a complex of finitely generated A-modules. Then we have the

(1.1) Acyclicity Lemma. *Suppose that for all i, $0 < i \leqslant s$,* (i) *depth of $L_i \geqslant i$ and* (ii) *either $H_i(L_{\bullet}) = 0$ or depth of $H_i(L_{\bullet}) = 0$. Then $H_i(L_{\bullet}) = 0$ for $i > 0$.*

In this form the Acyclicity Lemma made its appearance [9, Lemme 1.8] in the joint thesis of Peskine and Szpiro, where they used it to good avail in their successful attack on several homological conjectures in local algebra. Its usefulness was soon realized, and several authors provided variations, generalizations or sharpenings, e.g. [5], [6], [10]. It remained however for Anne-Marie Simon (Université Libre de Bruxelles) to notice, early this year, that the lemma has nothing in particular to do with depth at all!

A sequence of covariant additive functors F^n, $n \in \mathbf{Z}$, between abelian categories $\boldsymbol{A}$ and $\boldsymbol{B}$ is called an *exact connected right sequence* if $F^n = 0$ for $n < 0$ and if every short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\boldsymbol{A}$ gives rise to a long exact sequence $\ldots \to F^n A' \to F^n A \to F^n A'' \to F^{n+1} A' \to \ldots$ in $\boldsymbol{B}$ which is functorial. Notice that in particular F^0 is left exact. For each object A in $\boldsymbol{A}$ we define $f^-(A) = \inf\{i \mid F^n A \neq 0\}$; then $f^-(A) \in \mathrm{N}$ or $f^-(A) = \infty$.

(1.2) Proposition (A.-M. Simon). *Let F^n be an exact connected right sequence of functors. If $A_{\bullet} = 0 \to A_s \to A_{s-1} \to \ldots \to A_0$ is a complex in $\boldsymbol{A}$ such that* (i) *$f^-(A_i) \geqslant i$ for $0 < i \leqslant s$ and* (ii) *either $H_i(A_{\bullet}) = 0$ or $f^-(H_i(A_{\bullet})) = 0$, then $A_{\bullet}$ is exact.*

This paper is in final form and no version of it will be submitted for publication elsewhere.

The proof, which is an exact transcription of the proof of (1.1), will be omitted; it can be found in [12].

The advantage of working in abelian categories is that we right away have a dual

(1.2)° PROPOSITION. *Let G_n be an exact connected left sequence of functors. If $A^\bullet = A^0 \to \ldots \to A^{s-1} \to \ldots \to A^s \to 0$ is a complex in A such that* (i) $g_-(A^i) \geqslant i$ *for* $0 < i \leqslant s$ *and* (ii) *either* $H^i(A^\bullet) = 0$ *or* $g_-(H^i(A^\bullet)) = 0$, *then* $A^\bullet$ *is exact.*

Of course there are contravariant versions as well.

In the next section we shall follow Simon and apply these general acyclicity lemmas to questions involving depth. The results are rather modest in that they generalize and simplify certain known theorems. Even so, the versatility of the method will become apparent. It seems likely that applications in different contexts will be forthcoming, which is why I decided to speak about this topic. At the conference I was told by P. Pragacz that he has applied a very particular acyclicity statement, most probably subsumed by (1.2), to a base change problem for vector bundles.

2. A few statements about depth

In the following, A will be a commutative ring, $x = x_1, \ldots, x_r$ elements of A which generate a proper ideal $\mathfrak{a}$ and M an A-module. The functors $\mathrm{Ext}^n_A(A/\mathfrak{a}, -)$ form an exact connected right sequence of functors F^n as in (1.2) and f^- shall be called E-dp$(\mathfrak{a}, -)$ as in [3]. For any A-module M, E-dp$(\mathfrak{a}, M)$ is the grade of M in $\mathfrak{a}$ as defined by Rees, and it is equal to the depth of M in $\mathfrak{a}$ as measured by the maximal length of M-sequences in $\mathfrak{a}$ when M is finitely generated, $M \neq \mathfrak{a}M$, and A is noetherian.

We shall also consider the Koszul complex $K_\bullet(x, A)$ and standardly define the homological complex $K_\bullet(x, M) = K_\bullet(x, A) \otimes_A M$ and its cohomological counterpart $K^\bullet(x, M) = \mathrm{Hom}_A(K_\bullet(x, A), M)$. Their properties are well known and can for instance be found in the references [11] or [4]. We mention that their chain modules $K_n(x, M)$ and $K^n(x, M)$ only can live between 0 and r and similarly can their homologies $H_n(x, M)$ and $H^n(x, M)$. Furthermore the $H_n(x, -)$ resp. $H^n(x, -)$ form an exact connected left resp. right sequence of functors and we recall the notation $h_-(x, M)$ and $h^-(x, M)$ introduced in Section 1. Notice that these invariants are either ∞ or an integer between 0 and r.

(2.1) THEOREM. $h^-(x, M) =$ E-dp$(\mathfrak{a}, M)$ *for every module* M.

Proof. E-dp$(\mathfrak{a}, M) \leqslant h^-(x, M)$: We may assume that $h^-(x, M) = s < \infty$. Suppose E-dp$(\mathfrak{a}, M) \geqslant s+1$, and consider the truncated Koszul conplex

$$C^\bullet = 0 \to K^0(x, M) \to \ldots \to K^s(x, M) \to K^{s+1}(x, M).$$

Observe that the chain modules are all finite direct sums of copies of M, so $\text{E-dp}(\mathfrak{a}, C^i) \geqslant s+1$ for all the chain modules C^i in the complex $C^{\cdot}$, so (i) of Proposition (1.2) is satisfied. As for (ii), either $H^i(x, M) = 0$ or it is annihilated by the ideal $\mathfrak{a}$. In this case $\operatorname{Hom}_A(A/\mathfrak{a}, H^i(C^{\cdot})) \simeq H^i(C^{\cdot}) \neq 0$, so $\text{E-dp}(\mathfrak{a}, H^i(C^{\cdot})) = 0$. Proposition (1.2) now shows that the complex $C^{\cdot}$ is exact. But $h^-(x, M) = s$ means that $C^{\cdot}$ has homology at the sth place, yielding a contradiction.

$h^-(x, M) \leqslant \text{E-dp}(\mathfrak{a}, M)$: Now we may suppose that $\text{E-dp}(\mathfrak{a}, M) = s < \infty$. Take a free resolution of the A-module $A/\mathfrak{a}$ and call $L_{\bullet}$ the associated complex with $H_0(L_{\bullet}) = A/\mathfrak{a}$. The chain modules of the complex $C^{\cdot} = \operatorname{Hom}_A(L_{\bullet}, M)$ are direct products of copies of M, and since $K^{\cdot}(x, \Pi M) = \Pi K^{\cdot}(x, M)$, we find that $h^-(x, C^i) = h^-(x, M)$ for all i.

Suppose now that $h^-(x, M) \geqslant s+1$ and consider the complex

$$C^{\cdot} = 0 \to C^0 \to \ldots \to C^s \to C^{s+1}$$

with respect to the connected sequence of functors $H^n(x, -)$. Since $H^i(C^{\cdot}) = \operatorname{Ext}^i_A(A/\mathfrak{a}, M)$, its first nonvanishing homology is at spot s. On the other hand, condition (i) of (1.2) is satisfied, while $H^0(x, H^s(C^{\cdot})) \simeq \operatorname{Hom}(A/\mathfrak{a}, H^s(C^{\cdot})) \simeq H^s(C^{\cdot}) \neq 0$ clinches (ii). Therefore $C^{\cdot}$ is exact by (1.2) and the theorem has been established.

Theorem (2.1) is often referred to as "grade-sensitivity" of the Koszul complex. It not only generalizes the version in the literature, [1, Prop. 5] but to my mind its proof is considerably more straightforward than certain arguments in that paper and in [8].

Following [3] and [2], we next introduce the notion of Tor-codepth. The functors $\operatorname{Tor}^A_n(A/\mathfrak{a}, -)$ form an exact connected left sequence and we define $\text{T-codp}\, M = \inf\{n \mid \operatorname{Tor}^A_n(A/\mathfrak{a}, M)\} \neq 0$. Using Proposition $(1.2)^{\circ}$ rather than (1.2), the reader is invited to prove a companion theorem to (2.1):

(2.2) THEOREM. $h_-(x, M) = \text{T-codp}(\mathfrak{a}, M)$ *for every module* M.

In order to connect these two identities we recall the autoduality of the Koszul complex [4, Remarque 2), p. 149], which entails that $H^n(x, M) \simeq H_{r-n}(x, M)$ for all n. Thus, if we put $h^+(x, M) = \sup\{n \mid H^n(x, M) \neq 0\}$ and similarly for h_+, we find that

$$h^+(x, M) + h_-(x, M) = h_+(x, M) + h^-(x, M) = r, \tag{2.3}$$

provided that these invariants are finite for M; otherwise they are all four ∞.

(2.4) COROLLARY. $\text{E-dp}(\mathfrak{a}, M) < \infty$ *if and only if* $\text{T-codp}(\mathfrak{a}, M) < \infty$. *In this case,* $\text{E-dp}(\mathfrak{a}, M) + \text{T-codp}(\mathfrak{a}, M) \leqslant r$ *and equality is achieved precisely when* $K^{\cdot}(x, M)$ *has its homology concentrated in* $h^-(x, M) = h^+(x, M)$, *or equivalently for* $K_{\bullet}(x, M)$.

Theorems (2.1) and (2.2) show that $h^-(x, M)$ and $h_-(x, M)$ do not depend on the number of generators $x_1, \ldots, x_r$ which we have chosen for the ideal $\mathfrak{a}$, but according to (2.3), the invariants $h^+(x, M)$ and $h_-(x, M)$ of course do. Actually, more is true. It is well known that E-dp is also measured by local cohomology. Let $H^n_{\mathfrak{a}}$ be Grothendieck's local cohomology functors with respect to the ideal $\mathfrak{a}$ [7]. Then $\text{E-dp}(\mathfrak{a}, M) = \inf\{n \mid H^n_{\mathfrak{a}}(M) \neq 0\}$ [6]. Since local cohomology only depends on the topology induced by the power of the ideal, so does E-dp.

A more direct way of showing this is to prove a refined version of (2.1) as in [12]; a similar proof is given for (2.2), so that also $\text{T-codp}(\mathfrak{a}, -)$ only depends on the topology defined by $\mathfrak{a}$.

Let us now turn to the important case where A is a noetherian local ring and $\mathfrak{m}$ its maximal ideal. It is customary to just write E-dp for $\text{E-dp}(\mathfrak{m}, -)$ and the same for T-codp. The above remarks show that Simon's Corollary (2.4) implies the first part of the following result of Bartijn [2, Cor. 4.4 and Th. 4.15] which was proved by a different method. In fact, her search for such a generalization led to the discovery of the general Acyclicity Lemmas of Section 1.

(2.5) THEOREM. *Let A be a d-dimensional noetherian local ring. For any A-module M we have* $\text{E-dp}\, M < \infty$ *if and only if* $\text{T-codp}\, M < \infty$.

In this case,

(i) $\text{E-dp}\, M + \text{T-codp}\, M \leqslant d$;

(ii) *Equality is achieved for some M precisely when A possesses a balanced Big Cohen–Macaulay module.*

We shall not discuss (ii) further, only consider the case of a finitely generated module $M \neq 0$. Then $\text{T-codp}\, M = 0$ because $M \neq \mathfrak{m}M$ by Nakayama. Also $\text{E-dp}\, M = d$ means that there exists a regular sequence of length d on M, and it is well known that M is then even a small (= finitely generated) balanced Cohen–Macaulay module. For a proof of (ii) and unexplained terminology, see [2].

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