GEOMETRIC AND ALGEBRAIC TOPOLOGY
BANACH CENTER PUBLICATIONS, VOLUME 18
PWN – POLISH SCIENTIFIC PUBLISHERS
WARSZAWA 1986

THE CANCELLATION PROBLEM FOR HOMOTOPY EQUIVALENT REPRESENTATIONS OF FINITE GROUPS: A SURVEY

PAWEŁ TRACZYK

Warsaw, Poland

0. Introduction

The purpose of this paper is to give an up-to-date account of the author's efforts to prove the cancellation law for homotopy equivalence of representations. Although little progress has been made as compared with the author's paper [6], some development and simplification of the methods has been achieved.

Some results presented in the sequel are due to J. Tornehave or A. G. Wasserman, some other are probably well known.

Up to the author's knowledge no counterexample for the cancellation law is known.

1. Preliminaries

Here we give a brief survey of facts, and definitions which we need.

Two real representations V and W of a finite group G are said to be stably G-homotopy equivalent iff the spheres $S(V \oplus U)$ and $S(W \oplus U)$ are G-homotopy equivalent for some representation U of G. V and W are G-homotopy equivalent if and only if $S(V)$ and $S(W)$ are G-homotopy equivalent, without the need of introducing stabilizing summand U. We will usually omit "G" and write just that V and W are homotopy equivalent or stably homotopy equivalent.

We say that the cancellation law holds for homotopy equivalence of representations of G if stable G-homotopy equivalence implies G-homotopy equivalence.

Now assume that for every subgroup $H \leqslant G$ we have chosen an orientation of $S(V)^H$ and $S(W)^H$ (of course, we assume that if $S(V)^H = S(V)^K$

and $S(W)^H = S(W)^K$, then the same orientation is chosen for H and K). It is well known that an equivariant map $f\colon S(V) \to S(W)$ is a G-homotopy equivalence if and only if $\deg f^H = \pm 1$ for every $H \leqslant G$; in particular, a necessary condition for V and W to be G-homotopy equivalent is that $\dim V^H = \dim W^H$ for every $H \leqslant G$.

For a given choice of orientations, an equivariant map $f\colon S(V) \to S(W)$ defines a degree function $\operatorname{Deg} f$, $\operatorname{Deg} f(H) = \deg f^H$. If $S(V)^H = \emptyset$, we put $\deg f^H = 1$, so as to obtain the formula $\deg(f * g)^H = \deg f^H \cdot \deg g^H$ for the degree of the join of two equivariant maps.

If V and W satisfy the condition $\dim V^H = \dim W^H$ for every $H \leqslant G$, then it is known that there exists an equivariant map $f\colon S(V) \to S(W)$. Moreover, if H is an isotropy subgroup of the action of G on $S(V)$ (we denote the family of all isotropy subgroups by $ISO(S(V))$), then the degrees $\deg f^K$ taken over all $K > H$ determine $\deg f^H \bmod (|N(H)/H|)$, where $N(H)$ denotes the normalizer of H. Further, given an arbitrary integer k, f may be modified so as to obtain an equivariant map f_1 such that $\deg f_1^K = \deg f^K$ for every K which is not conjugate to a subgroup of H and $\deg f_1^H = \deg f^H + k\,|N(H)/H|$. (Of course, there is one natural exception to the freedom in changing f: if $S(V)^H = S^0$ then $\deg f^H$ may be equal only to 1, -1 or 0 and a modification may consist only in a choice from among the three possibilities and only in the case where $|N(H)/H| = 1$ or 2.) This result concerning the existence of equivariant maps will be the main tool used in this paper. For the proof see Tornehave [5], Theorem A.

2. Constructing G-homotopy equivalence when a suitable degree function is given

Let V and W satisfy the condition $\dim V^H = \dim W^H$ for every $H \leqslant G$. Assume that the orientations of fixed point sets are chosen as described in Tornehave [5], § 1. Let D be a function assigning to every subgroup H of the group G an integer. In order to be the degree function of an equivariant map, D must satisfy the following obvious preconditions:

1) if H and H' are conjugate then $D(H) = D(H')$,
2) if $V^H = V^{H'}$ then $D(H) = D(H')$,
3) if $V^H = 0$ then $D(H) = 1$.

If G-homotopy equivalencies $f\colon S(V \oplus U) \to S(W \oplus U)$ and $h\colon S(U) \to S(U)$ are given, it is not clear whether $\operatorname{Deg} f/\operatorname{Deg} h$ satisfies conditions 2) and 3) (counterexamples are easily provided). However, if this is assumed, we are able to obtain a G-homotopy equivalence denoted by f/h, $f/h\colon S(V) \to S(W)$, such that $\operatorname{Deg} f/h = \operatorname{Deg} f/\operatorname{Deg} h$, and thus cancel U.

THEOREM 2.1. *Assume that* $f\colon S(V \oplus U) \to S(W \oplus U)$ *and* $h\colon S(U) \to S(U)$

are G-homotopy equivalences. Assume that Deg f/Deg h *satisfies the conditions* 2) *and* 3), *i.e.* $\deg f^H/\deg h^H = \deg f^{H'}/\deg h^{H'}$ *whenever* $S(V)^H = S(V)^{H'}$ *and* $\deg f^H/\deg h^H = 1$ *if* $V^H = 0$.

Then there exists a G-homotopy equivalence f/h: $S(V) \to S(W)$ *such that*

$$\deg(f/g)^H = \deg f^H/\deg h^H$$

for every $H \leqslant G$.

Proof. It is known that there exists an equivariant map m: $S(V) \to S(W)$ and that it can be modified in the way described in § 1. Let H_0 be a maximal element in the set of those subgroups of G for which the equality

$$\deg m^H = \deg f^H/\deg h^H$$

is not satisfied. We will be done if we show that we can modify m so as to obtain an equivariant map m' such that

$$\deg m'^{H_0} = \deg f^{H_0}/\deg h^{H_0}$$

and $\deg m'^K = \deg m^K$ for every subgroup K of G which is not conjugate to any subgroup of H_0 (because then we can repeat the procedure). To this end, let us first note that H_0 is an isotropy subgroup of the action of G on $S(V)$. This is an immediate consequence of the fact there exists a uniquely determined isotropy subgroup H such that $S(V)^H = S(V)^{H_0}$ and the assumption that Deg f/Deg h satisfies the condition 2). Thus if H were greater than H_0, H_0 would not be a maximal element, as assumed. To modify m as required, it is now enough to prove that

$$\deg m^{H_0} \equiv \deg f^{H_0}/\deg h^{H_0} (\mathrm{mod}\ |N(H_0)/H_0|).$$

Consider the equivariant map $m * h$: $S(V \oplus U) \to S(W \oplus U)$. Evidently, for every subgroup $H \leqslant G$ which contains H_0 as a proper subgroup we have

$$\deg(m*h)^H = \deg m^H \cdot \deg h^H = (\deg f^H/\deg h^H) \cdot \deg h^H = \deg f^H.$$

It follows that $\deg(m*h)^{H_0} \equiv \deg f^{H_0} (\mathrm{mod}\ |N(H_0)/H_0|)$, because the degrees $\deg(m*h)^H$ determine $\deg(m*h)^{H_0}$ modulo $N(H_0)/H_0$ and they are the same as for the map f. But $\deg(m*h)^{H_0} = \deg m^{H_0} \cdot \deg h^{H_0}$, and so we have

$$\deg m^{H_0} \equiv \deg f^{H_0}/\deg h^{H_0}\ (\mathrm{mod}\ |N(H_0)/H_0|),$$

as required. This completes the proof of Theorem 2.1.

We will now list some immediate consequences of Theorem 2.1.

COROLLARY 2.2. *Assume that* f: $S(V \oplus U) \to S(W \oplus U)$ *is a G-homotopy equivalence and that* $ISO(S(V \oplus U)) = ISO(S(V))$. *Then there exists a G-homotopy equivalence* g: $S(V) \to S(W)$ *such that* $\deg g^H = \deg f^H$ *for every* $H \leqslant G$.

Proof. First observe that $S(V \oplus U)^H = S(V \oplus U)^{H'}$ if and only if $S(V)^H$

$= S(V)^{H'}$. It is clear that $S(V)^H = \emptyset$ implies $S(V\oplus U)^H = \emptyset$, otherwise the isotropy subgroups of the points belonging to $S(V\oplus U)^H$ would belong to $ISO(S(V\oplus U))$, without belonging to $ISO(S(V))$. If $S(V)^H \neq \emptyset$ then there exists a uniquely determined smallest subgroup $H_0 \in ISO(S(V))$ containing H and it is known that $S(V)^H = S(V)^{H_0} \cdot H_0$ is then the greatest subgroup of G having the same fixed points as H. Of course, H_0 is also the smallest isotropy subgroup containing H'. Since $ISO(S(V\oplus U)) = ISO(S(V))$, H_0 is also the smallest subgroup in $ISO(S(V\oplus U))$ containing H and H'. It follows that $S(V\oplus U)^H = S(V\oplus U)^{H_0} = S(V\oplus U)^{H'}$. Thus we have proved $S(V)^H = S(V)^{H'}$ yields $S(V\oplus U)^H = S(V\oplus U)^{H'}$; the opposite implication is trivial.

Let $h := \mathrm{id}\colon S(U) \to S(U)$. Clearly, $\mathrm{Deg} f/\mathrm{Deg} h$ satisfies the condition 3), since $S(V)^H = \emptyset$ implies $S(V\oplus U)^H = \emptyset$, whence $\deg f^H = 1$ and $\deg f^H/\deg h^H = 1$.

The condition 2) is obtained as an immediate consequence of the fact that $V^H = V^{H'}$ implies $(V\oplus U)^H = (V\oplus U)^{H'}$. By Theorem 2.1 there exists a G-homotopy equivalence $g\colon S(V) \to S(W)$ with degree function $\mathrm{Deg} f/\mathrm{Deg} h$; in other words, $\deg g^H = \deg f^H$ for every $H \leqslant G$.

Corollary 2.3 (due to A. G. Wasserman). *If V and W are stably homotopy equivalent, then $2V$ and $2W$ are homotopy equivalent.*

Proof. Let $f_0\colon S(V\oplus U) \to S(W\oplus U)$ be a G-homotopy equivalence. Then

$$f := f_0 * f_0\colon S(V\oplus U) * S(V\oplus U) = S(2(V\oplus U))$$
$$\to S(W\oplus U) * S(W\oplus U) = S(2(W\oplus U))$$

is a G-homotopy equivalence with a constant degree function $\mathrm{Deg} f = 1$. Now we apply Theorem 2.1 to $2V$, $2W$ and $2U$ taking

$$h := \mathrm{id}\colon S(2U) \to S(2U).$$

Corollary 2.4. *Assume that V and W are stably homotopy equivalent and $(2k+1)V$ and $(2k+1)W$ are homotopy equivalent. Then V and W are homotopy equivalent.*

Proof. By Corollary 2.3 $2kV$ and $2kW$ are homotopy equivalent. Let $U := 2k(V\oplus W)$. Then $V\oplus U$ and $W\oplus U$ are homotopy equivalent (because $V\oplus U = (2k+1)V\oplus 2kW$, $W\oplus U = (2k+1)W\oplus 2kV$ and the summands on the right are pairwise homotopy equivalent). Since $ISO(S(V)) = ISO(S(V\oplus U))$, we may apply Corollary 2.2.

Corollary 2.5. *Assume that $C \leqslant Z(G)$, where C is a cyclic group of prime order $\geqslant 3$. Let $f\colon S(V\oplus U) \to S(W\oplus U)$ be a G-homotopy equivalence. Assume that $V^C = U^C = 0$.*

Then there exists a G-homotopy equivalence $g\colon S(V) \to S(W)$ such that $\deg f^H = \deg g^H$ for every $H \leqslant G$.

For the proof we will need the following fact.

LEMMA 2.6. *Assume that $C \leqslant Z(G)$, where C is a cyclic group of prime order $p \geqslant 3$. Let $f\colon S(V) \to S(W)$ be a G-homotopy equivalence. Assume that $V^C = 0$. Then the degree function* $\operatorname{Deg} f$ *is uniquely determined.*

Proof. This fact is well-known; it is a simple consequence of the result mentioned in § 1. Obviously, for every subgroup $H \leqslant G$, V^H and W^H are representations of C and f^H is a C-homotopy equivalence. Thus $\deg f^H$ is determined modulo $|C|$ by $\deg f^{\langle H \cup C\rangle}$ and this is equal to 1, because $V^{\langle H \cup C\rangle} = 0$. But $\deg f^H$ may only equal 1 or -1, so it is in fact uniquely determined, because the difference between these two values is smaller than $|C|$.

Proof of Corollary 2.5. We will use induction on the order of G. Assume that there exists any G-homotopy equivalence $g\colon S(V) \to S(W)$. Then $g * \mathrm{id}\colon S(V \oplus U) \to S(W \oplus U)$ is also a G-homotopy equivalence. By the above lemma, $\operatorname{Deg} f = \operatorname{Deg} g * \mathrm{id}$, whence $\operatorname{Deg} f = \operatorname{Deg} g$. Let $h := \mathrm{id}\colon S(U) \to S(U)$. We want to show that $\operatorname{Deg} f/\operatorname{Deg} h = \operatorname{Deg} f$ satisfies the conditions 2) and 3) of § 2. The implication $V^H = 0 \Rightarrow \deg f^H = 1$ is clear: f^H is a self-C-homotopy equivalence of U^H and we apply Lemma 2.6. If $G = C$, the condition 3) is trivially satisfied, and by Theorem 2.1 there exists g with required properties. Assume that the corollary is true for groups of order less than $|G|$. Let H and H' be subgroups of G such that $V^H = W^{H'}$. We have to show that $\deg f^H = \deg f^{H'}$. It is enough to consider the case $H \leqslant H'$, because there exists an isotropy subgroup H_0 of the action of G on $S(V)$ containing both H and H' and such that $V^H = V^{H_0} = V^{H'}$. If $\langle H' \cup C\rangle$ is a proper subgroup of G, then by the inductive assumption there exists a $\langle H' \cup C\rangle$-homotopy equivalence and, so noted at the beginning of the proof, its degree function must be equal to $\operatorname{Deg} f$ restricted to subgroups of $\langle H' \cup C\rangle$; in particular, we have $\deg f^H = \deg f^{H'}$.

Now assume that $\langle H' \cup C\rangle = G$. We may assume that V and W is a minimal homotopy equivalent pair, i.e. V and W cannot be written as $V_1 \oplus V_2$, $W_1 \oplus W_2$, where V_i and W_i are homotopy equivalent (if this were the case, we might decompose V and W into sums $V_1 \oplus \ldots \oplus V_n$ and $W_1 \oplus \ldots \oplus W_n$, obtain homotopy equivalences $g_1, \ldots, g_n$ and take $g := g_1 * \ldots * g_n$). Then V and W factorize into representations of G/H' and hence they are homotopy equivalent by the inductive assumption. It remains to apply Theorem 2.1.

COROLLARY 2.7. *Assume that G contains a normal cyclic subgroup C of odd order such that $C_G(C)$ is contained in a certain normal subgroup $H \lhd G$ of odd index in G. Assume that the cancellation law holds for H. If V and W are stably homotopy equivalent and satisfy the condition $V^C = W^C = 0$, then V and W are homotopy equivalent.*

Proof. An easy application of the Mackey irreducibility criterion shows that V and W are induced from H. Consequently $\operatorname{ind}_H^G \operatorname{res}_H V = (G:H)\, V$,

$\mathrm{ind}_H^G \mathrm{res}_H W = (G:H) W$. By the assumption on H, $\mathrm{res}_H V$ and $\mathrm{res}_H W$ are H-homotopy equivalent. But it is known that the representations induced from homotopy equivalent representations are homotopy equivalent; thus $(G:H) V$ and $(G:H) W$ are G-homotopy equivalent and $(G:H)$ is an odd number, and we can apply Corollary 2.4.

3. The case of groups containing a normal p-torus

It is obvious that if V and W are stably G-homotopy equivalent then, for every subgroup $H \leqslant G$, V^H and W^H are stably $N(H)$-homotopy equivalent representations of $N(H)$. It follows that $\mathrm{ind}_{N(H)}^G V^H$ and $\mathrm{ind}_{N(H)}^G W^H$ are stably homotopy equivalent (representations induced from homotopy equivalent representations are homotopy equivalent, see tom Dieck [1] p. 251). If, moreover, the induced representations are irreducible then they are subrepresentations of V and W, as follows from the Frobenius reciprocity law. This works particularly neatly in the situation described in the following theorem.

THEOREM 3.1 *Let V and W be stably homotopy equivalent representations of a group G such that $V^H = W^H = 0$ for $H \trianglelefteq G$, $H \neq 1$. Assume that G contains a non-trivial normal p-torus $P = (Z_p)^n$, $n \geqslant 2$. Then there exists a subgroup P' of index p in P such that $V^{P'} \neq 0$ and for every such subgroup $\mathrm{ind}_{N(P')}^G V^{P'}$ and $\mathrm{ind}_{N(P')}^G W^{P'}$ are stably homotopy equivalent subrepresentations of V and W. If, moreover, the cancellation law holds for $N(P')$, then $\mathrm{ind}_{N(P')}^G V^{P'}$ and $\mathrm{ind}_{N(P')}^G W^{P'}$ are homotopy equivalent subrepresentations of V and W.*

The proof amounts to checking that the representation induced from any irreducible subrepresentation of the representation $V^{P'}$ of the group $N(P')$ is an irreducible representation of C. For details see the author's paper [7].

4. On changing the stabilizing summand

In an attempt to prove the cancellation law inductively for some group classes closed under the operations of taking subgroups and quotient groups it has appeared convenient to reduce the inductive step to the case of stably homotopy equivalent representations V and W such that $V^H = W^H = 0$ for every $H \trianglelefteq G$, $H \neq 1$ ($V, W \in RO(G, f)$ in the notation used by tom Dieck [1]; the character f indicates that V and W are the sums of faithful irreducible representations). The reduction is easy and well known (c.f. Traczyk [7]): If V^H, $W^H \neq 0$ for $H \trianglelefteq G$, $H \neq 1$, then V^H and W^H are stably homotopy equivalent representations of the quotient group G/H and the inductive assumption (induction is on the order of the group) guarantees the cancellation law for G/H. It is clear that V_H and W_H (the orthogonal complements of V^H and W^H) are also stably homotopy equivalent; for if $V\oplus U$ and $W\oplus U$ are homotopy equivalent then $V\oplus U\oplus W^H$ and $W\oplus U\oplus V^H$ are homotopy equivalent. But $V\oplus U\oplus W^H = V_H\oplus(V^H\oplus W^H\oplus U)$ and $W\oplus U\oplus V^H = W_H\oplus(V^H\oplus W^H\oplus U)$, the summands in brackets being equal. We may repeat the procedure until fixed point subspaces of all normal subgroups are split off.

It is not so clear if the adherence to $RO(G,f)$ may be also imposed on the stabilizing summand U. If $U = U^H$, $H \trianglelefteq G$, $H \neq 1$, one can try to use Theorem 2.1 to cancel U. It seems to be a natural choice for h: $S(U) \to S(U)$ to take $h := f^H$ (where f: $S(V\oplus U) \to S(W\oplus U)$ is a G-homotopy equivalence). But it is not clear if the function $\operatorname{Deg} f/\operatorname{Deg} h$ satisfies the necessary conditions. To avoid this obstacle we will prove the following

THEOREM 4.1. *Let V and W be stably homotopy equivalent representations of a finite solvable group G. Assume that $V^H = W^H = 0$ for every $H \trianglelefteq G$, $H \neq 1$. Then there exists a representation Y such that $V\oplus Y$ and $W\oplus Y$ are homotopy equivalent and $Y^H = 0$ for every $H \trianglelefteq G$, $H \neq 1$.*

Proof. Let $H_1, \ldots, H_n$ be the set of all minimal normal subgroups of G. It is enough to find a stabilizing summand Y such that $Y^{H_i} = 0$ for each i.

We will use induction on the order of G. It is clear that the theorem is true for abelian groups, because the cancellation law holds for abelian groups (Kawakubo [3], Theorem 2.5). The group $\langle H_1, \ldots, H_n \rangle$ is abelian. It follows that either there is a p-torus $p = (Z_p)^N$, $n \geqslant 2$, contained in it, or it is cyclic. Assume the first possibility. We will make an inductive step, using a slightly different but obviously equivalent version of the theorem:

Suppose that V and W are stably homotopy equivalent representations of G. Then there exists a representation Y such that $V\oplus Y$ and $W\oplus Y$ are homotopy equivalent and that $Y^H \neq 0$ only for those normal subgroups of G (possibly for which $V^H \neq 0$.

Now assume that the theorem is true for all groups of order less than $|G|$. We may restrict ourselves to the case where V and W are a minimal pair of stably homotopy equivalent representations of G; we may also assume that $V^H = W^H = 0$ for every $H \trianglelefteq G$, $H \neq 1$. Then by Theorem 3.1 there exists a subgroup $P' < P$ of index p in P such that $V = \operatorname{ind}_{N(P')}^G V^{P'}$, $W = \operatorname{ind}_{N(P')}^G W^{P'}$ and $V^{P'}$, $W^{P'}$ are stably homotopy equivalent representations of $N(P')$. Of course, $N(P') \neq G$, because V was assumed to have no fixed points with respect to non-trivial normal subgroups of G and P' is a non-trivial subgroup.

By the inductvie assumption applied to the group $N(P')$ there exists a representation Y_0 of $N(P')$ such that $V^{P'}\oplus Y_0$ and $W^{P'}\oplus Y_0$ are $N(P')$-homotopy equivalent and that $Y_0^H \neq 0$ only for those normal subgroups of $N(P')$ (possibly) for which $(V^{P'})^H \neq 0$. In particular, $Y_0^{H_i} = 0$, (of course, H_i is a normal subgroup of $N(P')$; this is normal in the whole G and it is contained in $N(P')$, because is commutes with P'). It follows that $\mathrm{ind}^G_{N(P')}(V^{P'} \oplus Y_0)$ and $\mathrm{ind}^G_{N(P')}(W^{P'} \oplus Y_0)$ are G-homotopy equivalent; hence $V \oplus Y$ and $W \oplus Y$ are homotopy equivalent, where $Y := \mathrm{ind}^G_{N(P')} Y_0$. But it is clear that $Y^{H_i} = 0$ for $i = 1, \ldots, n$, and this completes the inductive step.

5. The cancellation law for supersolvable groups and for products of a 2-group and an odd order group

We now show some applications of the methods described in the preceding sections. First we give a proof of the cancellation law for supersolvable groups. This result (due to J. Tornehave), although known to specialists, has not been published yet.

THEOREM 5.1. *The cancellation law holds for homotopy equivalence of representations of supersolvable groups.*

Proof. We use induction on the order of G. To begin with, recall that the cancellation law holds for abelian groups. According to § 4, it is enough to consider stably homotopy equivalent representations V and W of a non-abelian group G such that $V^H = W^H = 0$ for every $H \trianglelefteq G$, $H \neq 1$. We assume that the cancellation law holds for all proper subgroups of G. It is well known that there exists a normal abelian subgroup $A \trianglelefteq G$ such that $C_G(A) = A$. If A is not cyclic then G contains a non-trivial normal p-torus, and V and W are G-homotopy equivalent by Theorem 3.1.

If A is cyclic then G is an extension of the form $1 \to A \to G \to G/A \to 1$, where G/A acts effectively on A (here we use the condition $C_G(A) = A$). Let U be a representation of G such that $V \oplus U$ and $W \oplus U$ are homotopy equivalent. By Theorem 4.1 U may be chosen so that $U^H = 0$ for every $H \trianglelefteq G$, $H \neq 1$. We shall show that if K is a subgroup of G such that no normal non-trivial subgroup of G is contained in K, then $K \in ISO(S(V))$. An easy consequence of the Mackey irreducibility criterion is that every faithful irreducible complex representation of G is induced from A. It follows that if X is an irreducible subrepresentation of V then either X or $2X$ is induced from A. In both cases it is clear that $K \in ISO(S(X))$, whence $K \in ISO(S(V))$. Consequently $ISO(S(V)) = ISO(S(V \oplus U))$ and U can be cancelled, by Corollary 2.2.

Remark. Let us consider the class of such solvable groups G for which there exists a normal subgroup N with abelian Sylow subgroups such that G/N is supersolvable. Of course, all subgroups and quotient groups of G also belong to this class. It can be shown that every group G in this class has the property that was crucial for our proof for supersolvable groups, namely, there is a normal abelian subgroup $A \trianglelefteq G$ such that $C_G(A) = A$. It follows that the cancellation law holds for all such groups G (this result is also due to J. Tornehave).

The following theorem is an improvement of the result of the author's paper [6] in which it was shown that the cancellation law holds for odd order groups.

THEOREM 5.2. *Let G be a product of a 2-group and an odd order group. Then the cancellation law holds for homotopy equivalence representations of G.*

Proof. As usual we restrict ourselves to the case where V and W are a minimal pair of stably homotopy equivalent representations of G and $V^H = W^H = 0$ for every $H \trianglelefteq G$, $H \neq 1$. We already know that the cancellation law is true for abelian groups and 2-groups and we assume it is true for groups of order less than $|G|$.

Let us first consider the case when G contains a cyclic normal subgroup C of odd prime order.

If C is central in G, we may apply Corollary 2.5 and infer that V and W are homotopy equivalent.

If C is not central in G, then $C_G(C)$ is a normal subgroup of odd index in G (because C commutes with $Syl_2(G)$), and we may apply Corollary 2.7.

Now assume that G contains a non-trivial normal p-torus $P = (Z_p)^n$, $n \geqslant 2$. Then we may apply Theorem 4.1. This completes the proof.

References

[1] T. tom Dieck, *Transformation Groups and Representation Theory*, Lecture Notes in Math. 766, Berlin–Heidelberg New York 1979.

[2] I. M. Isaacs, *Character theory of finite groups*, New York–San Francisco–London.

[3] K. Kawakubo, *Cancellation law for G-homotopy equivalent representations*, Japan. J. Math. 6, (1980), 259–266.

[4] J.-P. Serre, *Linear Representations of Finite Groups*, New York-Heidelberg–Berlin 1977.

[5] J. Tornehave, *Equivalent maps of spheres with conjugate orthogonal actions*, in *CMS conference proceedings*, vol. 2, part 2, 275–302, 1982.

[6] P. Traczyk, *Cancellation law for homotopy equivalent representations of groups of odd order*, Manuscripta Math. 40 (1982), 135–154.

[7] —, *A splitting principle for homotopy equivalent representations of supersolvable groups*, to appear in Demonstratio Math.

Presented to the Topology Semester
April 3 – June 29, 1984