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ON THE EDGE SINGULARITIES IN COMPOSITE MEDIA. INFLUENCE OF ANISOTROPY

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1. Introduction

Let us consider an elliptic problem of the form

$$-\frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial U}{\partial x_i}\right) = f \tag{1.1}$$

in a domain Ω of $\boldsymbol{R}^2$, with appropriate boundary conditions. It is known that, under suitable smoothness hypotheses about the coefficients $a_{ij}(x)$ and the boundary of the domain (denoted by $\partial\Omega$), the classical regularity theory holds. In particular, if f belongs locally (i.e. on a domain D which may go up to the boundary) to the Sobolev space H^m (m real $\geqslant 0$) and the boundary conditions are homogeneous (i.e. equal to zero), the solution belongs to

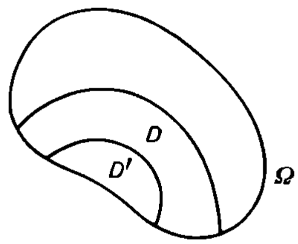

Fig. 1

H^{m+2} on any subdomain D' included in D. Moreover, if Ω is bounded, an inequality of the type

$$\|U\|_{H^{m+2}(\Omega)} \leqslant C(\|f\|_{H^m(\Omega)} + \|U\|_{H^m(\Omega)}) \tag{1.2}$$

holds for $m \geqslant 0$, with a constant C which only depends on Ω, m, and the coefficients of the equation.

Roughly speaking, let U be a solution obtained by variational techniques belonging to $H^1(\Omega)$. If the coefficients and $\partial\Omega$ are smooth, and f

belongs to, say, $L^2(\Omega)$, the solution actually belongs to $H^2(\Omega)$; moreover, if $f = 0$ in the vicinity of a point, the solution is actually of class C^∞ there. Of course, more complicated situations occur for nonhomogeneous boundary conditions: see Lions–Magenes [7] for these questions.

A different situation appears if the coefficients are smooth (constant, say) but the boundary $\partial\Omega$ is not, in particular if it exhibits angular points.

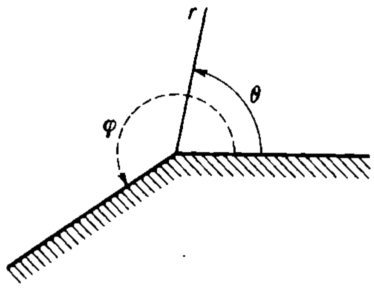

Fig. 2

It is easy to see that in such a situation, the local regularity depends on the angle φ of the domain. Indeed, let us take the example of the Laplace equation with Neumann boundary condition:

$$-\Delta U = 0, \tag{1.3}$$

$$\frac{\partial U}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{1.4}$$

(in fact the right-hand side is zero in a neighbourhood of the origin but may be different from zero elsewhere and consequently the solution U under consideration is not necessarily zero). Let us search for solutions of the form

$$U(x_1, x_2) = r^\alpha u(\theta) \tag{1.5}$$

where (r, θ) are polar coordinates with origin at the angular point. From (1.3), (1.4) we obtain for $u(\theta)$

$$\begin{gathered} -u'' - \alpha^2 u = 0, \qquad \theta \in (0, \varphi), \\ u' = 0 \quad \text{for} \quad \theta = 0,\ \theta = \varphi, \end{gathered} \tag{1.6}$$

and we have solutions of the form

$$u = A \cos \alpha\theta \quad \text{with} \quad \alpha = 0,\ \pm\frac{\pi}{\varphi},\ \pm 2\frac{\pi}{\varphi}, \ldots \tag{1.7}$$

Of course, the gradient of (1.5) behaves as $r^{\alpha-1}$, and we are interested in solutions exhibiting a singularity as $r \to 0$, i.e. with $\operatorname{Re} \alpha - 1 < 0$. On the other hand, if the solution exists according to some variational problem in H^1, then u and grad u are square-integrable, and this implies $\operatorname{Re} \alpha > 0$. We see from (1.7) that such solutions exist if $\varphi \in (\pi, 2\pi)$, i.e. if the domain is concave, but they do not exist if $\varphi \in (0, \pi)$, i.e. if the domain is convex. As a result, *the solution which exists according to a variational theory may exhibit a singular behavior (i.e. its gradient may tend to infinity at the origin) if the domain is concave.*

A picture of the flux lines (i.e. the lines tangent to grad U) furnishes some insight into the physical phenomenon: for a concave domain (resp. a convex domain) the flux lines push to each other (resp. spread out) as shown in Figs. 3 and 4.

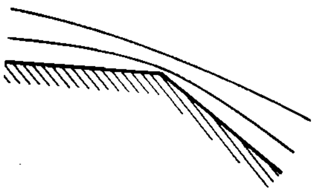

Fig. 3

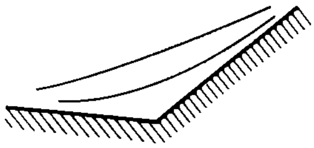

Fig. 4

The general reference for these questions (some examples of which are well known in hydrodynamics) are Kondrat'ev [6] and Grisvard [4]. The principal result of the corresponding theory is that, roughly speaking, solutions of the form (1.5) with $0 < \operatorname{Re} \alpha < 1$ are the only singularities of the problem, and regularity theory holds with these exceptions. Of course, more involved situations occur for nonhomogeneous boundary conditions.

The same general theory holds for problems with nonsmooth coefficients, in particular for transmission problems with piecewise constant coefficients, in particular for situations as in Fig. 5, where the coefficients a_{ij} of (1.1) are constant on each of the regions Ω_1, Ω_2.

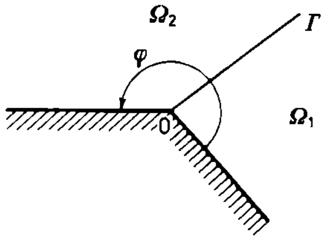

Fig. 5

A very particular case of this problem is considered in Sect. 2 and 3 where it is shown that for a smooth boundary of the domain, i.e. $\varphi = \pi$ (Fig. 5) with Neumann boundary conditions (corresponding to a free boundary), *the anisotropy of the medium* in the regions Ω_1 or Ω_2 *may generate singularities* even with a smooth boundary. A conjecture is stated in Sect. 4 about the existence of singularities in the general case: this constitutes a general criterion for homogeneous Neumann problems in second order equations, but it is not proved. An application to elasticity is given in Sect. 5. A deeper study of the elasticity system in domains with edges is given in Sect. 6, 7, 8, where it is proved that, in the situation of Fig. 5 with $\varphi = \pi$ and elasticity coefficients constant in Ω_1 and Ω_2 there exists an open domain in the space of pairs of elastic coefficients for which a singularity appears.

2. Setting the problem and derivation of the equation for $u(\theta)$

We consider the half-plane $x_2 > 0$ of the plane $\boldsymbol{R}^2$ with Cartesian (resp. polar) coordinates x_1, x_2 (resp. r, θ).

We consider equation (1.1) with $f = 0$ and coefficients a_{ij} taking constant values in the regions Ω_1 (i.e. $x_1 > 0$) and Ω_2 (i.e. $x_1 < 0$), namely

$$a_{11} = 1, \quad a_{22} = 1, \qquad (2.1)$$
$$a_{12} = a_{21} = \begin{cases} \varepsilon & \text{for } x_1 > 0, \\ 0 & \text{for } x_1 < 0, \end{cases}$$

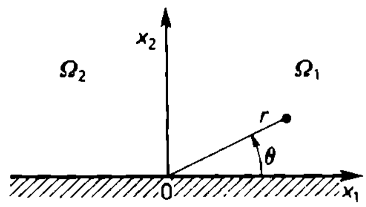

Fig. 6

where ε is a real parameter (which, for mathematical convenience, will also be taken complex in the sequel). It is clear that we have the Laplace equation for $\varepsilon = 0$; for $\varepsilon \neq 0$, the part $x_1 > 0$ is anisotropic.

The boundary conditions are the Neumann ones associated with the operator (1.1) on $x_2 = 0$:

$$a_{2i} \frac{\partial U}{\partial x_i} = 0 \quad \text{for } x_2 = 0, \text{ with } x_1 > 0 \text{ and } x_1 < 0. \qquad (2.2)$$

Of course on the line $x_1 = 0$ of discontinuity of the coefficients we have the transmission conditions associated with (1.1) in the distribution sense and U of class H^1 locally:

$$[U] = 0, \quad \left[a_{1i} \frac{\partial U}{\partial x_i} \right] = 0 \quad \text{on } x_1 = 0, \qquad (2.3)$$

where the symbol [ ] denotes the jump across $x_1 = 0$.

Now, we search for solutions of (1.1), (2.2), (2.3) with the coefficients (2.1), of the form

$$U(x_1, x_2) = r^\alpha u(\theta) \qquad (2.4)$$

with $0 < \operatorname{Re} \alpha < 1$. This gives us an equation for $u(\theta)$ and boundary and transmission conditions for $\theta = 0, \pi/2, \pi$. In order to obtain the sesquilinear form associated with this boundary value problem for u, we recall that the classical sesquilinear form for U including the Neumann and transmission conditions is

$$\int_{\Omega} a_{ij} \frac{\partial U}{\partial x_j} \frac{\partial \bar{V}}{\partial x_i} dx \qquad \forall V \in H^1(\Omega) \tag{2.5}$$

where the bar denotes the complex conjugate. Then we take

$$\Omega = \{r, \theta;\ r \in]0, \infty[, \quad \theta \in]0, \pi[\}, \tag{2.6}$$

$$U = r^\alpha u(\theta), \qquad V = \varphi(r) v(\theta), \qquad \varphi \in \mathcal{D}(0, \infty), \tag{2.7}$$

and the problem of searching for solutions of the form (2.4) amounts to finding α such that there exists a nonzero $u(\theta) \in H^1(0, \pi)$ satisfying

$$\begin{aligned} 0 = \int_0^\infty r\, dr \int_0^\pi d\theta \Big\{ & a_{11} \frac{\partial r^\alpha u}{\partial x_1} \frac{\partial \varphi \bar{v}}{\partial x_1} + a_{22} \frac{\partial r^\alpha u}{\partial x_2} \frac{\partial \varphi \bar{v}}{\partial x_2} \\ & + a_{12} \left(\frac{\partial r^\alpha u}{\partial x_1} \frac{\partial \varphi \bar{v}}{\partial x_2} + \frac{\partial r^\alpha u}{\partial x_2} \frac{\partial \varphi \bar{v}}{\partial x_1} \right) \Big\} \end{aligned} \tag{2.8}$$

for any $\varphi \in \mathcal{D}(0, \infty)$, $v \in H^1(0, \pi)$. Performing the change of variables

$$\frac{\partial}{\partial x_1} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta}, \qquad \frac{\partial}{\partial x_2} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta}, \tag{2.9}$$

(2.8) becomes

$$0 = \int_0^\infty r^\alpha [F \varphi'(r) + \Phi r^{-1} \varphi(r)]\, dr \tag{2.10}$$

with

$$\begin{aligned} F \equiv \int_0^\pi \{ & a_{11} (\alpha \cos\theta u - \sin\theta u') \cos\theta \bar{v} \\ & + a_{22} (\alpha \sin\theta u + \cos\theta u') \sin\theta \bar{v} \\ & + a_{12} [\alpha \sin 2\theta u \bar{v} + \cos 2\theta u' \bar{v}] \}\, d\theta \end{aligned} \tag{2.11}$$

$$\begin{aligned} \Phi \equiv \int_0^\pi \{ & a_{11} (-\alpha \cos\theta u + \sin\theta u') \sin\theta \bar{v}' \\ & + a_{22} (\alpha \sin\theta u + \cos\theta u') \cos\theta \bar{v}' \\ & + a_{12} [\alpha \cos 2\theta u \bar{v}' - \sin 2\theta u' \bar{v}'] \}\, d\theta \end{aligned} \tag{2.12}$$

which of course holds for arbitrary coefficients a_{ij}. We integrate by parts (2.10) (or equivalently we consider it as a distribution product) and it becomes

$$\int_0^\infty (\Phi-\alpha F) r^{\alpha-1} \varphi(r)\, dr = 0 \qquad \forall \varphi \in \mathscr{D}(0, \infty),$$

which amounts to

$$\Phi - \alpha F = 0. \tag{2.13}$$

We remark that F and Φ defined by (2.11), (2.12) are sesquilinear forms for $u, v \in H^1(0, \pi)$. Our problem is to find α such that there exists a nonzero $u \in H^1(0, \pi)$ satisfying (2.13) for any $v \in H^1(0, \pi)$. We now take the specific coefficients (2.1), where the regions $x_1 > 0$, $x_1 < 0$ become $\theta \in (0, \pi/2)$, $\theta \in (\pi/2, \pi)$ respectively. The form $\Phi - \alpha F$ is written in the form

$$\begin{aligned} a(\varepsilon, \alpha; u, v) \equiv\; & a_1(u, v) + \alpha^2 a_2(u, v) + \varepsilon a_3(u, v) \\ & + \varepsilon\alpha a_4(u, v) + \varepsilon\alpha^2 a_5(u, v), \quad \text{with} \end{aligned} \tag{2.14}$$

$$a_1(u, v) \equiv \int_0^\pi u' \bar{v}'\, d\theta,$$

$$a_2(u, v) \equiv -\int_0^\pi u \bar{v}\, d\theta,$$

$$a_3(u, v) \equiv -\int_0^{\pi/2} \sin 2\theta u' \bar{v}'\, d\theta,$$

$$a_4(u, v) \equiv \int_0^{\pi/2} \cos 2\theta (u\bar{v}' - u'\bar{v})\, d\theta,$$

$$a_5(u, v) \equiv -\int_0^{\pi/2} \sin 2\theta u \bar{v}\, d\theta.$$

Of course, the problem of searching for the singularities of the boundary value problem stated at the beginning of this section is equivalent to the following one:

Problem P(ε). We consider ε as a parameter (taking small values, for instance). For fixed ε, we search for the values $\alpha(\varepsilon)$ with $0 < \operatorname{Re} \alpha < 1$ such that there exists a nonzero $u \in H^1(0, \pi)$ satisfying

$$a(\varepsilon, \alpha; u, v) = 0 \qquad \forall v \in H^1(0, \pi), \tag{2.15}$$

where the form a is defined in (2.14).

3. Asymptotic study for small ε. Existence of singularities for real, small positive ε

Before going on, we consider problem P(ε) for $\varepsilon = 0$. This amounts to the Laplace equation in the half-plane $x_2 > 0$ with Neumann boundary conditions. We know that there are no singularities. In fact, from (2.15) with $\varepsilon = 0$, we have

PROBLEM P(0). Find α such that there exists $u \in H^1(0, \pi)$ satisfying

$$\int_0^\pi (u'\bar{v}' - \alpha^2 u\bar{v})\, d\theta = 0 \qquad \forall v \in H^1(0, \pi), \tag{3.1}$$

which amounts to

$$\begin{aligned} -u'' &= \alpha^2 u \quad \text{for } \theta \in (0, \pi), \\ u'(0) &= u'(\pi) = 0, \end{aligned} \tag{3.2}$$

which for $\alpha = 1$ has the solution (normalized in $L^2(0, \pi)$):

$$\alpha = 1, \qquad u = u_0(\theta) \equiv (2/\pi)^{-1} \cos\theta. \tag{3.3}$$

We note that (3.2) also has solutions for other values of α^2 (the eigenvalues of the Neumann problem), but $\alpha = 1$ is the only one which is at the boundary of the singular region $0 < \operatorname{Re} \alpha < 1$ (the value $\alpha = 0$ has only trivial solutions without physical meaning). Our aim is to introduce perturbation for $\varepsilon \neq 0$ in order to obtain corresponding singular $\alpha(\varepsilon)$. Note on the other hand that (3.3) is associated with the solution

$$U(x_1, x_2) = (2/\pi)^{-1} x_1 \tag{3.4}$$

which is a constant flux parallel to the boundary $x_2 = 0$. Before studying this perturbation, let us state the result of this section:

PROPOSITION 3.1. *For real, positive, sufficiently small ε, problem* P(ε) *has singular solutions. In fact, there exists a holomorphic branch $\alpha(\varepsilon)$ for complex ε with $|\varepsilon|$ sufficiently small of solutions with $\alpha(0) = 1$, $(d\alpha/d\varepsilon)(0) < 0$.*

In order to prove this proposition, we write problem P(ε) in another form. Let us consider $H^1(0, \pi)$ contained in $L^2(0, \pi)$ which we identify with its dual. For small $|\varepsilon|$, $|\alpha - 1|$, the form $a(\varepsilon, \alpha; u, v)$ is sesquilinear continuous on H^1, and (2.15) amounts to

$$A(\varepsilon, \alpha)u = 0 \tag{3.5}$$

where $A(\varepsilon, \alpha)$ is the operator associated with $a(\varepsilon, \alpha; u, v)$; it may be considered either as a continuous operator from H^1 into its dual or as an unbounded operator with compact resolvent on L^2. If we consider the eigenvalue problem

$$A(\varepsilon, \alpha)u = \lambda u \tag{3.6}$$

problem P(ε) amounts to finding $\alpha(\varepsilon)$ such that $\lambda = 0$ is an eigenvalue of $A(\varepsilon, \alpha(\varepsilon))$. But, for $\varepsilon = 0$, $\alpha = 1$, the value $\lambda = 0$ is a simple eigenvalue (see (3.2), (3.3)). Moreover, the classical analytic perturbation theory applies (note that the operators $A(\varepsilon, \alpha)$ form a holomorphic family of class **B** of Kato [5] with respect to ε and α), there exists an eigenvalue $\lambda(\varepsilon, \alpha)$ which is simple and holomorphic in ε and α (and consequently in ε, α together) for $|\varepsilon|$, $|\alpha-1|$ sufficiently small, taking the value 0 for $\varepsilon=0$, $\alpha=1$. Our problem amounts to proving that the equation

$$\lambda(\varepsilon, \alpha)=0 \tag{3.7}$$

defines $\alpha(\varepsilon)$ for small ε and that $(d\alpha/d\varepsilon)(0)<0$. According to the implicit function theorem, Proposition 3.1 will be proved if we prove that

$$\frac{\partial\lambda}{\partial\alpha}(0,1)<0, \quad \frac{d\lambda}{d\varepsilon}(0,1)<0. \tag{3.8}$$

Let us prove the first of (3.8). It only concerns the operator for $\varepsilon=0$. From (2.14) we see that the eigenvalue problem

$$A(0,\alpha)u=\lambda u$$

amounts to

$$-u''-\alpha^2u=\lambda u, \quad u'(0)=u'(\pi)=0,$$

and the eigenvalue $\lambda(0,\alpha)$ taking the value 0 for $\alpha=1$ is $\lambda(0,\alpha)\equiv 1-\alpha^2$; consequently

$$\frac{d\lambda}{d\alpha}(0,1)=-2. \tag{3.9}$$

Let us now prove the second of (3.8). It is a little bit more complicated, and we shall use the standard perturbation theory (which also furnishes (3.9), of course). The second of (3.8) only concerns $\alpha\equiv 1$. The eigenvalue problem for $\lambda(\varepsilon, 1)$ then reads

$$A(\varepsilon,1)u(\varepsilon)=\lambda(\varepsilon)u(\varepsilon) \tag{3.10}$$

where $\lambda(\varepsilon)$, $u(\varepsilon)$ are the corresponding eigenvalue and eigenvector. This is of course equivalent to finding $u(\varepsilon)\in H^1$ such that, $\forall v\in H^1$,

$$a_1(u,v)+a_2(u,v)+\varepsilon\big(a_3(u,v)+a_4(u,v)+a_5(u,v)\big)=\lambda(\varepsilon)(u,v)_{L^2}. \tag{3.11}$$

We then consider the analytic expansions

$$\lambda(\varepsilon)=0+\lambda_1\varepsilon+\lambda_2\varepsilon^2+\dots, \qquad u(\varepsilon)=u_0+u_1\varepsilon+u_2\varepsilon^2+\dots, \tag{3.12}$$

where u_0 is the eigenfunction of the unperturbed problem given by (3.3) and

$$\lambda_1=\frac{\partial\lambda}{\partial\varepsilon}(0,1). \tag{3.13}$$

In order to obtain an eigenvector $u(\varepsilon)$ holomorphic in ε, we must prescribe some normalization condition, for instance (see Friedrichs [3] for these expansions):

$$\big(u(\varepsilon),u_0\big)_{L^2}=1. \tag{3.14}$$

We substitute (3.12) into (3.11) and we expand it in the powers of ε. At order 1 we have an identity (which corresponds to the unperturbed problem), and at order ε

$$a_1(u_1, v)+a_2(u_1, v) = \lambda_1(u_0, v)_{L^2}-\big(a_3(u_0, v)+a_4(u_0, v)+a_5(u_0, v)\big) \quad \forall v\in H^1. \tag{3.15}$$

But the form a_1+a_2 on the left-hand side is the form associated with the unperturbed operator $A(0, 1)$; consequently, (3.15) is equivalent to

$$A(0, 1)u_1 = F \tag{3.16}$$

where $F\in(H^1)'$ is the element of the dual space associated with the right-hand side of (3.15). But $A(0, 1)$ has zero as (simple) eigenvalue, and consequently there is a compatibility condition to be satisfied by F in order for u_1 to exist, namely the duality product of F with the eigenfunction u_0 must be zero (note that $A(0, 1)$ is selfadjoint; this is a form of the Fredholm alternative; see Sanchez [8], p. 23, if necessary). This amounts to the fact that the right-hand side of (3.15) must be zero for $v = u_0$. This gives

$$\|u_0\|_{L^2}\lambda_1 = a_3(u_0, u_0)+a_4(u_0, u_0)+a_5(u_0, u_0)$$

which suffices (the asymptotic expansion may be continued, but this is not useful for our purpose) to obtain, by (3.3),

$$\frac{\partial\lambda}{\partial\varepsilon}(0, 1) = \lambda_1 = -\frac{2}{\pi},$$

which proves the second relation (3.8) and thus Proposition 3.1 is proved.

4. A conjecture on the general case

The (rigorously proved) result of the preceding sections shows that the problem (1.1) with the coefficients (2.1) and homogeneous Neumann boundary conditions has a singular behavior for small $\varepsilon > 0$ but not for $\varepsilon < 0$ (in fact, we only know that there are no singularities in the vicinity of $\alpha = 1$). In fact, it is not difficult to obtain an interpretation of this result in terms of convexity or concavity of the boundary with respect to the lines formed by the refracted rays. We now give such an interpretation which furnishes a general criterion for the existence of singularities in the geometric situation of Fig. 5, with Neumann conditions on $\partial\Omega$. This criterion is not proved, but we give it as a plausible conjecture.

Let us consider equation (1.1) with $f \equiv 0$ in the case where the coefficients $a_{ij}(x)$ take constant values in two regions Ω_1, Ω_2 of the plane with a straight interface Γ (Fig. 7); there is no boundary for the time being. Let us define, for any solution U, the vectors $\boldsymbol{q}$ and $\boldsymbol{p}$ with components

$$q_i = \frac{\partial U}{\partial x_i}, \tag{4.1}$$

$$p_i = a_{ij} \frac{\partial U}{\partial x_j} = a_{ij} q_j; \tag{4.2}$$

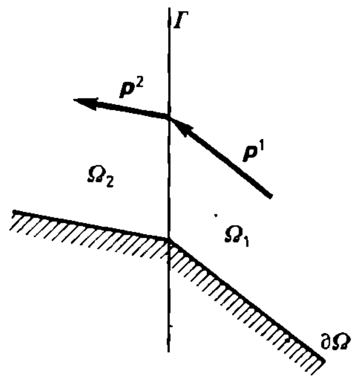

Fig. 7

in fact, $\boldsymbol{q}$ is the gradient of U and $\boldsymbol{p}$ is the associated flux; note that equation (1.1) with $f \equiv 0$ is

$$\operatorname{div} \boldsymbol{p} = 0. \tag{4.3}$$

Of course, the transmission conditions through the interface Γ (analogous to (2.3)) are

$$[q_t] = 0, \quad [p_n] = 0, \tag{4.4}$$

where the indexes t, n denote the tangential and normal components to Γ, and the brackets denote the jump across Γ.

Let us study solutions with $\boldsymbol{q}$, $\boldsymbol{p}$ constant in each of the regions Ω_1, Ω_2 (we do not consider the boundary $\partial\Omega$ for the time being); the corresponding vectors will be denoted $\boldsymbol{q}^1$, $\boldsymbol{q}^2$, $\boldsymbol{p}^1$, $\boldsymbol{p}^2$. We may for instance take as $\boldsymbol{p}^1$ any vector; from (4.2) we obtain the corresponding $\boldsymbol{q}^1$ (note that the operator is elliptic and thus the matrix a_{ij} is invertible). Then from (4.4) we obtain q_t^2 and p_n^2 and (4.2) easily furnishes the other components of $\boldsymbol{q}^2$ and $\boldsymbol{p}^2$. Consequently, for any given $\boldsymbol{p}^1$ we can construct the corresponding $\boldsymbol{p}^2$ as in Fig. 7.

Let us now consider a boundary $\partial\Omega$ parallel in Ω_1 and Ω_2 to $\boldsymbol{p}^1$ and $\boldsymbol{p}^2$ respectively (Fig. 7). It is clear that the solution corresponding to the given constant vectors $\boldsymbol{p}^1$, $\boldsymbol{p}^2$ also holds in the presence of the boundary $\partial\Omega$ for Neumann boundary conditions, i.e.

$$a_{ij} \frac{\partial U}{\partial x_i} n_i = 0 \quad \Leftrightarrow \quad p_n = 0 \quad \text{on } \partial\Omega. \tag{4.5}$$

Now, this solution is, for the general problem with an interface Γ, analogous to the solution of constant flux tangent to a straight boundary for the Laplace equation (see (3.3) if necessary). The boundary $\partial\Omega$ of Fig. 7 plays in this problem a role analogous to that of a straight boundary for the Laplace equation. We then may guess (this is not proved!) the existence or nonexistence of singularities in terms of convexity or concavity of the boundary with respect to the $\partial\Omega$ of Fig. 7 (see also Figs. 3, 4). In fact, as $\operatorname{div} \boldsymbol{p} = 0$, if the field lines of $\boldsymbol{p}$ concentrate (resp. spread) at a point, the value of $|\boldsymbol{p}|$ grows (resp. decreases) there. Summing up we may state the following criterion for the existence of singularities which contains the known results for the Laplace equation and the asymptotic ones of Sect. 3:

CRITERION (unproved conjecture). We consider (see Figs. 8, 9) two regions Ω_1, Ω_2 with interface Γ and the boundary of the domain formed by Ω_1, Ω_2. We consider solutions of equation (1.1) with constant coefficients on Ω_1 and Ω_2, subject to the transmission conditions

$$[U] = 0, \quad \left[n_i a_{ij} \frac{\partial U}{\partial x_j} \right] = 0 \quad \text{on } \Gamma, \tag{4.6}$$

and the Neumann conditions

$$n_i a_{ij} \frac{\partial U}{\partial x_j} = 0 \quad \text{on } \Sigma_1 \text{ and } \Sigma_2. \tag{4.7}$$

In order to study the existence of singularities at the origin, we choose that of the two domains Ω_1, Ω_2 which has opening $\leqslant \pi$; let it be Ω_1. Then we construct a vector $\boldsymbol{p}^1$ parallel to Σ_1 (Figs. 8, 9). Using (4.4) and (4.2) we construct the vectors $\boldsymbol{q}^1$, $\boldsymbol{q}^2$, $\boldsymbol{p}^2$ and the line formed by the vectors $\boldsymbol{p}^1$, $\boldsymbol{p}^2$ (Figs. 8, 9). Then there is a (resp. there is no) singularity if the angle of the boundaries Σ_1, Σ_2 is more (resp. less) open than the line formed by $\boldsymbol{p}^1$, $\boldsymbol{p}^2$.

It is to be noticed that in the case where the two domains Ω_1, Ω_2 have openings $\leqslant \pi$, we may start in two different ways; but it is easily seen that the result is the same.

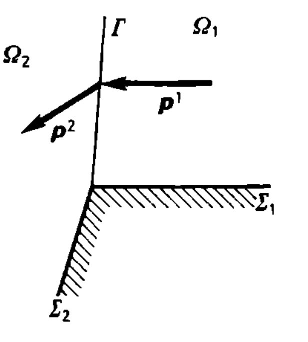

Fig. 8

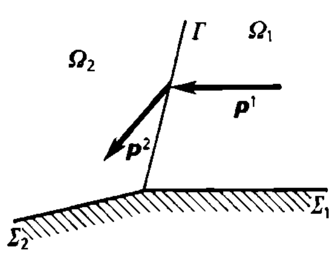

Fig. 9

5. First application to the problem of edge singularities in elasticity of composite materials

Equation (1.1) governs classical problems in electrostatics, electrodynamics and steady (i.e. independent of time) heat diffusion. The applications to these domains of the results of the preceding sections are obvious. We now give an application to a class of shear problems is elasticity. This application deserves some attention because the edge singularities in composite materials seem to be responsible for the failure of pieces made of composite materials (or for arising of plasticity regions or damaged regions). The question of arising of singularities is very controversial (see for instance [1], [9], [10]); in particular the existence of logarithmic singularities for some shear components in the edges of composite plates made of isotropic materials was announced in [2]. Our results show that for some related problems, anisotropy induces algebraic singularities (i.e. the stress tensor tends to infinity as $r^{\alpha-1}$ with $0 < \operatorname{Re} \alpha < 1$).

Let us consider a cylindrical domain of $\boldsymbol{R}^3$ (coordinates x_1, x_2, x_3), $\Omega \times \boldsymbol{R}$, where Ω is some domain of the (x_1, x_2) plane and $\boldsymbol{R}$ is the x_3 axis. We consider the elasticity system for the displacement vector $\boldsymbol{U}$:

$$\frac{\partial}{\partial x_j}(b_{ijlm} e_{lm}(U)) = 0, \qquad e_{lm}(U) = \frac{1}{2}\left(\frac{\partial U_l}{\partial x_m} + \frac{\partial U_m}{\partial x_l}\right), \tag{5.1}$$

satisfying

$$\begin{gathered} b_{ijlm} = b_{lmij} = b_{jilm}, \\ b_{ijlm} e_{lm} e_{ij} \geqslant \gamma \|e\|^2 \quad \forall e \text{ symmetric.} \end{gathered} \tag{5.2}$$

Then, if the coefficients b_{ijlm} are such that

$$b_{ij3n} = 0 \quad \text{for all } i, j, n \text{ equal to 1 or 2} \tag{5.3}$$

we may search for solutions of (5.1) where the displacement vector $\boldsymbol{U}$ has only the component U_3 nonzero, and $U_3 = U_3(x_1, x_2)$. The elasticity system (5.1) then becomes

$$\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial U_3}{\partial x_i}\right) = 0 \quad \text{in } \Omega \tag{5.4}$$

with the coefficients $a_{ij} = b_{3i3j}$. Of course (5.4) governs the shear stresses

$$\sigma_{13} \equiv a_{1i}\frac{\partial U_3}{\partial x_i}, \qquad \sigma_{23} \equiv a_{2i}\frac{\partial U_3}{\partial x_i} \tag{5.5}$$

with summation for $i = 1, 2$, and (5.4) reads

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} = 0, \tag{5.6}$$

and consequently the Neumann condition

$$(5.7) \qquad \sum_{i,j=1}^{2} n_j a_{ij} \frac{\partial U_3}{\partial x_i} = 0 \quad \Leftrightarrow \quad \sum_{i=1}^{2} n_i \sigma_{i3} = 0$$

on the boundary amounts to the condition that the lateral surface of the cylinder is free.

As an application of the results of Sect. 4, which are of course rigorous in the case of small anisotropy (Sect.3), we consider a composite in the framework of the present section, formed by two parts Ω_1, Ω_2. If there is anisotropy, it is easily seen by using the criterion of Sect. 4 that if the free boundaries AB, $A'B'$ are normal to the interface Γ, there is always a singularity, either at O or at O' (in Fig. 10 the singularity appears at O). But this may be avoided by taking free boundaries not normal to Γ (Fig.11).

Of course, this is a very particular problem of elasticity. We have no results for the general system of elasticity, but the preceding examples show that when anisotropy is involved, there is little chance to have no singularities all over the boundary.

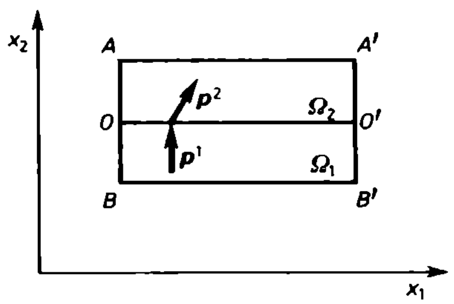

Fig. 10

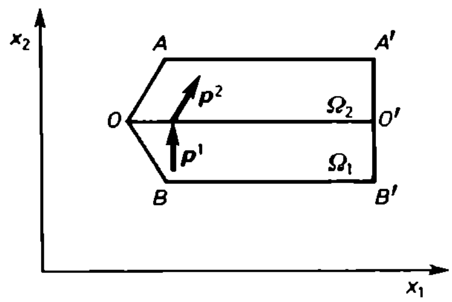

Fig. 11

6. Further study of the elasticity system in cylindrical domains

In Sect. 5 we proved that, for a particular elasticity system and for the displacement of the form $(0, 0, U_3)$, there are singularities. Now we perform a perturbation of the coefficients and we prove that, if the perturbation is sufficiently small, a singularity still occurs. Thus we prove the existence of an open domain in the space of coefficients (in fact of pairs of coefficients, as we have a system of coefficients in Ω_1 and another in Ω_2) such that singularities do appear. This shows that for arbitrary coefficients, the probability of arising of a singularity is not zero (for some definition of the probability associated with measure).

We consider, as before, an elastic body filling the half-space $x_2 > 0$ of $\boldsymbol{R}^3$ (coordinates x_1, x_2, x_3) and we shall search for solutions $U = (U_1, U_2, U_3)$ with $U = U(x_1, x_2)$ independent of x_3. The half-space $\Omega = \{x_2 > 0\}$ is formed by two quarters of the space, Ω_1, Ω_2, with different coefficients (see Fig. 12). The elasticity system (5.1) with $\partial/\partial x_3 = 0$ becomes (the e_{ij} are defined in (5.1))

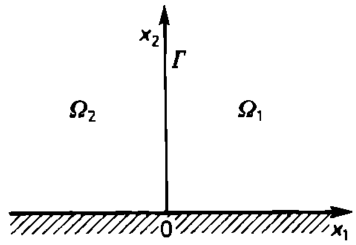

Fig. 12

$$
\begin{aligned}
&\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2} = 0,\\
(6.1)\qquad &\frac{\partial\sigma_{21}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2} = 0, \qquad \sigma_{ij} = b_{ijlm}\, e_{lm}(U),\\
&\frac{\partial\sigma_{31}}{\partial x_1}+\frac{\partial\sigma_{32}}{\partial x_2} = 0,
\end{aligned}
$$

or equivalently

$$\frac{\partial\sigma_{i\beta}}{\partial x_3} = 0$$

with Latin (resp. Greek) indices $= 1, 2, 3$ (resp. 1, 2), with the boundary and interface conditions

$$\sigma_{i\beta}\,\eta_\beta = 0 \quad \text{on } \partial\Omega, \tag{6.2}$$

$$[U] = 0, \quad [\sigma_{i\beta}\,\eta_\beta] = 0 \quad \text{on } \Gamma, \tag{6.3}$$

and the coefficients b_{ijlm} constant on each of the regions Ω_1, Ω_2.

Before going on, we note that (6.1)–(6.3) is associated with the classical sesquilinear form of three-dimensional elasticity, i.e. the fact that we search for solutions depending only on x_1, x_2 modifies Ω, but not the structure of the form:

$$
(6.4)\qquad
\begin{aligned}
&b(U, V) = \int_\Omega b_{ijlm}\, e_{lm}(U)\, e_{ij}(\bar{V})\, dx,\\
&\text{Neumann and jump conditions} \quad \Leftrightarrow \quad U, V \in H^1(\Omega).
\end{aligned}
$$

Indeed, from (6.1) by multiplying by $\bar{V}(x_1, x_2)$ and integrating over Ω we have, on account of (6.2), (6.3),

$$
(6.5)\qquad
\begin{aligned}
\int_\Omega -\frac{\partial\sigma_{ij}}{\partial x_j}\bar{V}_i\, dx &= \int_\Omega -\frac{\partial\sigma_{i\beta}}{\partial x_\beta}\bar{V}_i\, dx = \int_\Omega \sigma_{i\beta}\frac{\partial\bar{V}_i}{\partial x_\beta}dx\\
&= \int_\Omega \sigma_{ij}\frac{\partial\bar{V}_i}{\partial x_j}dx = \int_\Omega \sigma_{ij}\, e_{ij}(\bar{V})\, dx.
\end{aligned}
$$

Conversely, the computations (6.5) may be performed in the opposite sense, and we see that (6.4) is the corresponding sesquilinear form as we claimed.

7. Unperturbed and perturbed system

We define an unperturbed system in the framework of Sect. 5, i.e. we define the (unperturbed) elasticity coefficients $\tilde{b}_{ijlm}(x)$ such that $\tilde{b}_{ij3m}$ is zero whenever i, j, m are in $\{1, 2\}$. But this does not determine the unperturbed system for our purposes. We also take

$$\tilde{b}_{3i3j} = a_{ij} \tag{7.1}$$

where the a_{ij} are the coefficients of an elliptic equation with a singular solution, for instance

$$a_{11} = a_{22} = 1, \qquad a_{12}(x) = \begin{cases} \varepsilon \text{ (small, } > 0) & \text{if } x_1 > 0, \\ 0 & \text{if } x_1 < 0, \end{cases} \tag{7.2}$$

and for the coefficients involving x_1, x_2, we take the coefficients corresponding to a "two-dimensional" isotropic elasticity in x_1, x_2; this gives

$$\begin{aligned}
&\tilde{b}_{3131} = 1, \quad \tilde{b}_{3232} = 1, \\
&\tilde{b}_{3132} = \begin{cases} \varepsilon \text{ (small, } > 0) & \text{if } x_1 > 0, \\ 0 & \text{if } x_1 < 0, \end{cases} \\
&\tilde{b}_{3333} > 0, \quad \tilde{b}_{3322} > 0, \quad \tilde{b}_{3311} > 0, \\
&\tilde{b}_{3313} = 0, \quad \tilde{b}_{3323} = 0, \\
&\tilde{b}_{1131} = \tilde{b}_{1132} = \tilde{b}_{1231} = \tilde{b}_{1232} = \tilde{b}_{2231} = \tilde{b}_{2232} = 0, \\
&\tilde{b}_{1111} = 2\mu + \lambda = \tilde{b}_{2222}, \\
&\tilde{b}_{1122} = \lambda, \quad \tilde{b}_{1212} = 2\mu, \\
&\tilde{b}_{1112} = b_{2212} = 0.
\end{aligned}$$

Remark 7.1. It is easily seen that, by joining the coefficients which are obtained by the symmetries (5.2), we have all the coefficients. In addition, with appropriate values of the coefficients marked > 0, the unperturbed coefficients satisfy the positivity condition (5.2); indeed, they may be obtained as a small perturbation (in ε) of an isotropic elasticity (it suffices to take $\tilde{b}_{3333} = 2\mu + \lambda$, $\tilde{b}_{3311} = \tilde{b}_{3322} = \lambda$ and $2\mu = 1$ to be consistent with the assigned values $\tilde{b}_{3131} = \tilde{b}_{3232} = 1$).

Now let $c_{ijlm}(x)$ be any coefficients (constant on each of the regions Ω_1, Ω_2) satisfying the symmetry properties in (5.2) (but not necessarily the positivity condition). Let η be a small (real or complex) parameter. We consider the perturbed coefficients

$$b_{ijlm}(x) = \tilde{b}_{ijlm}(x) + \eta c_{ijlm}(x) \tag{7.3}$$

which for small η satisfy the positivity and symmetry conditions (5.2).

Now, we establish an uncoupling property of the unperturbed system. We consider solutions of two forms:

$$(1) \quad U(x_1, x_2) = (U_1, U_2, 0), \qquad (2) \quad U(x_1, x_2) = (0, 0, U_3), \tag{7.4}$$

with, of course, the unperturbed coefficients $\tilde{b}_{ijlm}$. It is easily seen that in case (1) we have $\sigma_{31} = \sigma_{32} = 0$ (and $\sigma_{33} \neq 0$, independent of x_3, but this is of no importance) and in case (2), $\sigma_{11} = \sigma_{12} = \sigma_{22} = 0$. Consequently, in system (6.1)–(6.3) there is an uncoupling between the first two components and the third one. Of course, the third component (and the third equation) is associated with the problem of Sect. 5, which has a singular solution (i.e. with $0 \leqslant \operatorname{Re} \alpha \leqslant 1$), and components 1, 2 are associated with a plane-deformation elasticity problem with isotropic constant coefficients in the half-plane, which has no singular solution, since the standard regularity theory holds for it. Then the singular solution for U_3 with $U_1 = U_2 = 0$ constitutes a singular solution for the whole system, and zero is a simple eigenvalue for it. We thus have

PROPOSITION 7.1 *The unperturbed system* (6.1)–(6.3) *with coefficients* $\tilde{b}_{ijlm}$ *is such that, when searching for solutions of the form*

$$U(x_1, x_2) = r^{\alpha} \boldsymbol{u}(\theta) \tag{7.5}$$

with $0 \leqslant \operatorname{Re} \alpha \leqslant 1$, *it has a simple solution, i.e. there exists an* α *such that the corresponding eigenspace for* $\boldsymbol{u}(\theta)$ *is one-dimensional. In addition, the considered value of* α *is real.*

8. Perturbation in η. Existence of singular solutions

We study the properties of the implicit eigenvalue perturbation in order to prove that for small η, the perturbed problem with coefficients given in (7.3) does have singular solutions. In general, for coefficients depending only on θ, we start from (6.4) and we take

$$\begin{aligned} U &= r^{\alpha} \boldsymbol{u}(\theta), \qquad \boldsymbol{u}(\theta) \in (H^1(0, \pi))^3, \\ V &= \varphi(r) \boldsymbol{v}(\theta), \qquad \varphi \in \mathscr{D}(0, \infty),\ \boldsymbol{v} \in (H^1(0, \pi))^3, \end{aligned} \tag{8.1}$$

and performing the change (2.9) as in Sect. 2, we arrive at a form $a(\eta, \alpha; \boldsymbol{u}, \boldsymbol{v})$ which is sesquilinear and bounded on $(H^1)^3$ and holomorphic in η, α (i.e., for fixed $\boldsymbol{u}$, $\boldsymbol{v}$ it is a holomorphic function). Let us admit for the moment that the form a satisfies a coerciveness condition of the type

$$a(\eta, \alpha; \boldsymbol{u}, \boldsymbol{u}) + \lambda |\boldsymbol{u}|^2_{L^2} \geqslant c \|\boldsymbol{u}\|^2_{H^1} \tag{8.2}$$

for some λ, c and η (resp. α) sufficiently small (resp. close to the value α_1 corresponding to the unperturbed coefficients $\tilde{b}$). Then a is a form of type (b) of Kato [5]. As the imbedding $H^1 \subset L^2$ is compact, the standard perturbation theory holds for eigenvalues, and as 0 is a simple eigenvalue of the problem with $\eta = 0$, $\alpha = \alpha_1$, the eigenvalue $\lambda(\eta, \alpha)$ exists and is holomorphic for $|\eta|$, $|\alpha - \alpha_1|$ sufficiently small, and the problem amounts to the study of the equation

$$\lambda(\eta, \alpha) = 0. \tag{8.3}$$

Of course, the derivative $\partial\lambda/\partial\alpha$ for $\eta = 0$, $\alpha = \alpha_1$ is different from zero since it coincides with the corresponding value for the coefficients $\tilde{b}$, where it is different from zero for small ε. Thus (8.3) defines $\alpha(\eta)$ which is holomorphic and takes the value α_1 for $\eta = 0$. As $0 < \alpha_1 < 1$, $\alpha(\eta)$ also satisfies this inequality, and we obtain

PROPOSITION 8.1. *For $|\eta|$ sufficiently small, the perturbed problem has singular solutions.*

In order to complete the proof, we have to prove (8.2). Because of the holomorphy it suffices to prove that the unperturbed form (i.e. for $\eta = 0$, $\alpha = \alpha_1$) satisfies (8.2). Moreover, as the solutions $(U_1, U_2, 0)$ and $(0, 0, U_3)$ are uncoupled, the associated form $a(0, \alpha, u, v)$ is the sum of the forms corresponding to $(u_1, u_2, 0)$, $(v_1, v_2, 0)$ and $(0, 0, u_3)$, $(0, 0, v_3)$. The latter is coercive on H^1 by Sect. 2. Thus it suffices to prove that the form corresponding to $(u_1, u_2, 0)$, $(v_1, v_2, 0)$ is coercive on $(H^1(0, \pi))^2$. Moreover, as we have at our disposal the term λ of (8.2), we may only consider the terms containing first order derivatives with respect to θ (i.e. for this computation, we only consider in (2.9) the terms containing $\partial/\partial\theta$). Moreover, as the problem in the plane x_1, x_2 is a standard plane elasticity problem, the sesquilinear form (6.4) takes the form (for some $c_1, c_2 > 0$)

$$\int_\Omega (c_1 e_{11}(U) e_{11}(V)\, dx + c_1 e_{22}(U) e_{22}(V) + c_2 e_{12}(U) e_{12}(V))\, dx,$$

and this gives, with (2.6), (2.7) and the change (2.9) (where, as explained before, we only consider the terms containing $\partial/\partial\theta$), for $U = V$,

$$\int_0^\pi c_1 (\sin\theta)^2 (v_1')^2 + c_2 (\cos\theta)^2 (v_2')^2 + \tfrac{1}{4} c_2 (\cos\theta \bar{v}_1' - \sin\theta \bar{v}_2')^2\, d\theta$$

$$\geqslant \operatorname{Inf}(c_1, \tfrac{1}{4} c_2) \int_0^\pi (|v_1'|^2 + |v_2'|^2 - |\sin 2\theta|\, |v_1'|\, |v_2'|)\, d\theta$$

$$\geqslant c_3 \int_0^\pi (|v_1'|^2 + |v_2'|^2)\, d\theta$$

for some $c_3 > 0$, and (8.2) follows.

Remark 8.1. Proposition 8.1 deals with a single parameter η when the coefficients $c_{ijlm}(x)$ are given (see (7.3)). It is clear that we may take these coefficients to be constant on Ω_1 and Ω_2 and the corresponding values of the coefficients as parameters. This proves the property announced at the beginning of Sect. 6 about the existence of an open domain in the space of pairs of coefficients giving a singularity.

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Added in proof (June 1987). A more elaborate version of this work along with new results is: D. Leguillon and E. Sanchez-Palencia, *Computation of Singular Solutions in Elliptic Problems and Elasticity*, Masson, Paris 1987.