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HETEROCLINIC CONNECTIONS OF STATIONARY SOLUTIONS OF SCALAR REACTION-DIFFUSION EQUATIONS

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§ 1. Introduction

We consider the flow of a one-dimensional reaction-diffusion equation

$$u_t = u_{xx} + f(u) \tag{1.1}$$

on the interval $x \in (0, 1)$ with Dirichlet conditions

$$u(t, 0) = u(t, 1) = 0. \tag{1.2}$$

Let v, w denote stationary, i.e. t-independent solutions. We say that v *connects to* w if there exists an orbit $u(t, x)$ of (1.1), (1.2) such that

$$\lim_{t \to -\infty} u(t, \cdot) = v, \qquad \lim_{t \to +\infty} u(t, \cdot) = w. \tag{1.3}$$

In this report we consider the following question:

(*) Given v, which stationary solutions w does it connect to?

To fix the technical setting for our investigation we assume

$$f \in C^2, \quad f(0) = 0, \quad f'(0) > 0, \quad \overline{\lim_{|s| \to \infty}} f(s)/s < \pi^2. \tag{1.4}$$

As a solution space we consider (cf. [8])

$$u(t, \cdot) \in X := H^2 \cap H_0^1.$$

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Introducing a parameter α, Chafee and Infante [4] studied the bifurcation behavior of stationary solutions of

$$u_t = u_{xx} + \alpha f(u) \tag{1.1$_\alpha$}$$

with boundary condition (1.2) under the additional assumption

$$sf''(s) > 0 \quad \text{for } s \neq 0. \tag{1.5}$$

Partial answers to (*) were obtained by Conley and Smoller [5, 16] using Conley's index, and by Henry [8] using invariant manifold theory. Later, Henry [9] solved problem (*) by an ingenious transversality argument. However, the convexity assumption (1.5) was crucial in all these results. We present a new approach which dispenses with (1.5) and contains the previous results. Dropping (1.5) greatly increases the complexity of the problem because it introduces many additional stationary solutions – the nontrivial stationary branches of $(1.1)_\alpha$ are not globally parametrized over α any more.

There are several ingredients to our analysis. The *gradient structure* of (1.1) guarantees that all orbits tend to equilibrium via the functional

$$V(u) := \int_0^1 \left(\tfrac{1}{2}u_x^2 - F(u)\right) dx, \qquad F'(s) := f(s),$$

$$\frac{d}{dt} V\big(u(t, \cdot)\big) = -\int_0^1 u_t^2 \, dx$$

(cf. [8]). Another (discrete) functional is the zero number z. For continuous φ: $[0, 1] \to \boldsymbol{R}$, the *zero number* $z(\varphi)$ is the maximal integer $n \leqslant \infty$ such that there exist $0 < x_0 < x_1 < \ldots < x_n < 1$ with

$$\varphi(x_i)\,\varphi(x_{i+1}) < 0 \qquad (0 \leqslant i < n);$$

$z(0) := 0$. By maximum principle arguments [2, 11, 14], $t \to z\big(\tilde{u}(t, \cdot)\big)$ is decreasing along solutions $\tilde{u}(t, \cdot)$ of

$$\tilde{u}_t = \tilde{u}_{xx} + g(x, \tilde{u}), \tag{1.6}$$

$$\tilde{u}(t, 0) = \tilde{u}(t, 1) = 0 \tag{1.2}$$

provided $g(x, 0) = 0$ for all x. As an example consider a hyperbolic stationary solution v of (1.1), (1.2). By "hyperbolic" we mean that zero is not an eigenvalue of the linearization L at v,

$$Lu := u_{xx} + f'\big(v(x)\big)u, \tag{1.7}$$

$$u(0) = u(1) = 0. \tag{1.8}$$

Let $W^{\mathrm{u}}(v)$ resp. $W^{\mathrm{s}}(v)$ denote the unstable resp. stable manifold of v [8] and let $i(v) := \dim W^{\mathrm{u}}(v)$ denote the *instability index* (Morse index) of v. If u is a solution of (1.1), (1.2) then $\tilde{u} := u - v$ is a solution of (1.6), (1.2) with

$g(x, \tilde{u}) := f(\tilde{u}+v(x)) - f(v(x))$, and $z(\tilde{u}(t, \cdot))$ is decreasing. Using this fact, it was proved in [2] that

$$(1.9) \qquad \begin{aligned} z(u_0 - v) &< i(v) \quad \text{for any } u_0 \in W^u(v), \\ z(u_0 - v) &\geqslant i(v) \quad \text{for any } u_0 \in W^s(v) \setminus \{v\}. \end{aligned}$$

As another relation between i and z we mention

$$(1.10) \qquad i(v) \in \{z(v), z(v)+1\}$$

for $v \not\equiv 0$.

For hyperbolic stationary v we define

$$(1.11) \qquad \Omega(v) := \{w \mid v \text{ connects to } w \not\equiv v\}$$

and for $0 \leqslant k < i(v)$

$\bar{v}_k$ is the stationary solution $\tilde{v}$ with $z(\tilde{v}) = k$ such that $\tilde{v}'(0) > |v'(0)|$ is minimal,

$\underline{v}_k$ is the stationary solution $\tilde{v}$ with $z(\tilde{v}) = k$ such that $\tilde{v}'(0) < -|v'(0)|$ is maximal.

With this notation we can state our main result.

THEOREM [3]. *Let f satisfy assumption* (1.4) *and let v be a hyperbolic stationary solution of* (1,1), (1.2). *Then v connects to other stationary solutions as follows.*

(i) *If $v \equiv 0$, or if $v \not\equiv 0$ and $i(v) = z(v)$, then*

$$\Omega(v) = \{\underline{v}_k, \bar{v}_k \mid 0 \leqslant k < z(v)\}.$$

(ii) *If $v'(0) > 0$ and $i(v) = z(v)+1$, then*

$$\Omega(v) = \Omega_1 \cup \Omega_2 \cup \Omega_3$$

where

$\Omega_1 = \{\bar{v}_k \mid 0 \leqslant k \leqslant z(v)\}$,

$\Omega_2 = \{\underline{v}_k \mid 0 \leqslant k < z(v)\}$, *and either*

$\Omega_3 = \{\underline{v}_k \mid k = z(v)\}$ *or*

Ω_3 *consists of one or several stationary solutions w with $-v'(0) \leqslant w'(0) < v'(0)$.*

(iii) *If $v'(0) < 0$ and $i(v) = z(v)+1$, then a corresponding statement holds with $f(s)$ replaced by $-f(-s)$.*

In the remaining sections we illustrate some aspects of the proof of the theorem. In § 2 we use topological degree theory to show that for any stationary v, any $\sigma \in \{-1, +1\}$ and any $0 \leqslant k < i(v)$ there exists a stationary

w such that v connects to w and

$$z(w-v)=k,\quad \sigma\big(w'(0)-v'(0)\big)>0. \tag{1.12}$$

In § 3 we identify those w to coincide with the $\bar{v}_k$, $\underline{v}_k$ introduced above. However, in one case ((ii), $i(v)=z(v)+1$, $k=z(v)$, $\sigma=-1$) our analysis is not complete and this accounts for the awkward alternative for Ω_3. Finally, § 4 indicates some further generalizations and open problems.

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§ 2. Existence of connections

Let the assumptions of the theorem be satisfied. For a hyperbolic stationary v we take a small n-sphere Σ^n in its unstable manifold $W^u(v)$, $i(v)=n+1$. Below, we construct a continuous map

$$y\colon \Sigma^n\to S^n$$

of Σ^n into the standard n-sphere in $\boldsymbol{R}^{n+1}$ such that, given $u_0\in\Sigma^n$ and $y(u_0)$, we can reconstruct $z\big(u(t,\cdot)-v\big)$ all along the positive semi-orbit through u_0. By a homotopy argument we show that y is not homotopic to a constant, and hence surjective. Picking u_0 such that $y(u_0)=\sigma e_k$ and $w:=\lim_{t\to+\infty}u(t,\cdot)$ we establish the existence of a connection from v to w satisfying (1.12). For simplicity of presentation we restrict our attention to the special case $v\equiv 0$ which does require the crucial arguments.

We construct the y-map. For $u_0\in X$, $u_0\not\equiv 0$, $z(u_0)=n$, with orbit $u(t,\cdot)$, define $t_k\in[0,\infty]$ to be the first time such that the zero number $z\big(u(t,\cdot)\big)$ drops to the k-level or below:

$$t_k:=\inf\{t\geqslant 0\mid z\big(u(t,\cdot)\big)\leqslant k\},\quad \tau_k:=\tanh t_k\in[0,1]. \tag{2.1a}$$

Note that $0=\tau_n\leqslant\tau_{n-1}\leqslant\ldots\leqslant\tau_0$. Further we define

$$\varphi_0(\tau_0):=\tau_0,\quad \psi_0(t_0):=(1-\varphi_0^2)^{1/2}, \tag{2.1b}$$

and for $k\geqslant 1$, $\tau_{k-1}>0$,

$$\varphi_k:=\tau_k/\tau_{k-1},\quad \psi_k:=(1-\varphi_k^2)^{1/2}, \tag{2.1c}$$

$$\sigma_k:=\begin{cases}\operatorname{sign}u_x(t,0),\ t\in(t_k,t_{k-1}) & \text{if } \varphi_k<1,\\ 0 & \text{if } \varphi_k=1.\end{cases} \tag{2.1d}$$

The sign σ_k is well defined because $u_x(t,0)\neq 0$ for $\varphi_k<1$, $t_k<t<t_{k-1}$, by the maximum principle. The components of the map $y=y(u_0)=(y_0,\ldots,y_n)$ are defined as

$$y_0:=\sigma_0\psi_0,\quad y_k:=\sigma_k\psi_k\cdot\varphi_0\cdot\ldots\cdot\varphi_{k-1}. \tag{2.2}$$

In case $z(u_0) = k < n$, i.e. $0 = t_n = \ldots = t_k < t_{k-1} \leqslant \ldots \leqslant t_0$ we extend the definition of y by putting $y_{k+1} = \ldots = y_n = 0$; $y_0, \ldots, y_k$ are well defined above.

To illustrate the meaning of y suppose $y = \sigma e_k$, where e_k denotes the kth unit vector, $\sigma \in \{-1, +1\}$. This implies $\psi_0 = \ldots = \psi_{k-1} = 0$, $\varphi_0 = \ldots = \varphi_{k-1} = 1$, $\psi_k = 1$, $\varphi_k = 0$, $\sigma_k = \sigma$. Hence $t_0 = \ldots = t_{k-1} = \infty$, $t_k = 0$ and $z(u(t, \cdot)) \equiv k$, $\sigma u_x(t, 0) > 0$ for all $t > 0$. Let $w := \lim_{t \to +\infty} u(t, \cdot)$. Then w is stationary and satisfies (1.12) by the above observations. In general y determines all t_k, σ_k, and therefore $z(u(t, \cdot))$, uniquely.

Let $\mathscr{F}$ denote the set of C^2-functions f satisfying assumption (1.4) endowed with the strong Whitney topology [10].

LEMMA 2.1. *y: $\{u_0 \mid z(u_0) \leqslant n,\ u_0 \not\equiv 0\} \times \mathscr{F} \to S^n$ is a continuous mapping.*

The proof of Lemma 2.1 uses the fact that given $u_0 \not\equiv 0$ with $z(u_0) < \infty$ the set

$$G := \{t \geqslant 0 \mid u(t, \cdot) \text{ has only simple zeros}\}$$

is open and dense in $[0, \infty)$. This fact again follows from maximum principles (cf. [2, 11, 13]).

Now consider the trivial solution $v \equiv 0$ of $(1.1)_\alpha$. Let $\alpha_n := (n+1)^2 \pi^2 / f'(0)$, $n \geqslant 0$, denote the bifurcation points of the trivial solution. Then $i(v \equiv 0) = n+1$ for $\alpha_n < \alpha < \alpha_{n+1}$, and by Σ^n we denote a (small) n-sphere in $W^u(v \equiv 0)$.

LEMMA 2.2. *y: $\Sigma^n \to S^n$ is essential, i.e. y is not homotopic to a constant.*

Proof. We prove the lemma by induction on n. For $n = 0$, i.e. $\alpha_0 < \alpha < \alpha_1$, the unstable manifold is one-dimensional and tangent to the first (positive) eigenfunction Φ_0 of the linearization L at $v \equiv 0$. Hence $\sigma_0 = \pm \operatorname{sign} \Phi_0'(0) = \pm 1$ depending on whether we start in direction $+\Phi_0$ or $-\Phi_0$. By (1.9), $t_0 = 0$, and $y = y_0 = \sigma_0$ is bijective, hence essential.

Suppose now the lemma is already proved for $n-1$. By Lemma 2.1 all y-maps for $\alpha \in (\alpha_n, \alpha_{n+1})$ are homotopic. Hence it is sufficient to prove that y is essential for $\alpha = \alpha_n + s$, $0 < s \ll 1$, by the homotopy invariance of degree (cf. [6]). Let $\lambda_0 > \ldots > \lambda_n > \ldots$ denote the (Sturm–Liouville) eigenvalues of the linearization (1.7), (1.8) at $v \equiv 0$ with eigenfunctions $\Phi_0, \ldots, \Phi_n, \ldots$, $\Phi_j'(0) > 0$. As $s = \alpha - \alpha_n$ increases through zero, λ_n also increases through zero and the dimension of the unstable manifold increases by 1. Let Σ_s^{n-1} resp. Σ_s^n denote a small sphere in $W^u(v \equiv 0)$ at $\alpha = \alpha_n + s$ for $-1 \ll s < 0$ resp. $0 \leqslant s \ll 1$. Moreover, Σ_s^{n-1} can be continued to $0 \leqslant s \ll 1$ to become a (topological) equator in Σ_s^n. Indeed, for $0 \leqslant s \ll 1$ there is an invariant submanifold $W_{n-1}^u(v \equiv 0)$ of $W^u(v \equiv 0)$ with dimension n, tangent to $\langle \Phi_0, \ldots, \Phi_{n-1} \rangle$, which consists of those u_0 with

$$\lim_{t \to -\infty} \frac{u(t, \cdot)}{|u(t, \cdot)|} \in \langle \Phi_0, \ldots, \Phi_{n-1} \rangle.$$

Take Σ_s^{n-1} to be a small sphere in W_{n-1}^{u}. By [2], $z \leqslant n-1$ on W_{n-1}^{u}. Again by [2], $u_0 \in W^u \setminus W_{n-1}^{u}$ implies

$$\lim_{t \to -\infty} \frac{u(t, \cdot)}{|u(t, \cdot)|} = \pm \Phi_n.$$

To utilize the above observations, let

$$y_s \colon \Sigma_s^n \to S^n$$

denote the restricted y-map for $0 < s \ll 1$ ($z \leqslant n$ on Σ_s^n by (1.9)),

$$\Sigma_s^{n,\pm} := \{u_0 \in \Sigma_s^n \mid \lim_{t \to -\infty} u(t, \cdot)/|u(t, \cdot)| = \pm \Phi_n\} \cup \Sigma_s^{n-1}$$

the (closed) hemispheres,

$$S_{\pm}^n := \{(y_0, \ldots, y_n) \in S^n \mid \pm y_n \geqslant 0\}$$

the standard hemispheres, and

$$S^{n-1} := \{(y_0, \ldots, y_n) \in S^n \mid y_n = 0\}$$

the standard equator. Then

$$\begin{aligned} y_s \colon \Sigma_s^n &\to S^n, \\ \Sigma_s^{n,+} &\to S_+^n, \\ \Sigma_s^{n,-} &\to S_-^n, \\ y_s^{n-1} := y_s|_{\Sigma_s^{n-1}} \colon \Sigma_s^{n-1} &\to S^{n-1}. \end{aligned}$$

Now y_s^{n-1} continues through $s = 0$ and is essential by induction hypothesis and the homotopy invariance of the degree $\deg y_s^{n-1} \neq 0$. Let us consider the Mayer–Vietoris sequence for y_s, [6]:

$$\begin{tikzcd} 0 \arrow[r] & H_n(\Sigma_s^n) \arrow[r] \arrow[d, "\deg y_s"] & H_{n-1}(\Sigma_s^{n-1}) \arrow[r] \arrow[d, "\deg y_s^{n-1}"] & H_{n-1}(\Sigma_s^{n,+}) \oplus H_{n-1}(\Sigma_s^{n,-}) \arrow[d] \\ 0 \arrow[r] & H_n(S^n) \arrow[r] & H_{n-1}(S^{n-1}) \arrow[r] & H_{n-1}(S_+^n) \oplus H_{n-1}(S_-^n) \end{tikzcd}$$

The homology of hemispheres is trivial, the other homologies are just $\boldsymbol{Z}$, hence

$$\deg y_s = \deg y_s^{n-1} \neq 0$$

and y_s is essential [6]. This completes the induction step and the proof of the lemma. ∎

As was indicated above, y essential implies that v connects to a w such that (1.12) holds. In general, additional preparations are needed for $v \not\equiv 0$. First we approximate f by generic $f_n \in \mathscr{F}$ such that each f_n displays only standard saddle-node and (at $v \equiv 0$) quadratic bifurcations [1, 12, 18]. It is sufficient to establish connections for these generic f_n. Now we consider the y-maps associated to $u(t, \cdot) - v$. By exchange of stability, we may continue spheres Σ_s^n and their associated maps y_s, even across bifurcation points, along continuous paths of stationary solutions provided $i(v)$ does not change. At saddle-node bifurcations we resort to a Mayer–Vietoris argument as outlined above for $v \equiv 0$. Thus we obtain Lemma 2.2 all along the connected component of the trivial solution in the (α, v) bifurcation diagram. But all other components can be connected to $(\alpha, 0)$ artificially, if we introduce an additional homotopy f_β from $f_0 = f$ to some f_1 satisfying the Chafee–Infante assumption (1.5) in addition to (1.4). The details will be given elsewhere.

§ 3. Excluding connections

Let assumption (1.4) on f hold and suppose v is a hyperbolic stationary solution of (1.1), (1.2). Let $\sigma \in \{-1, 1\}$, $0 \leqslant k < i(v)$. In § 2 we have shown that v connects to a stationary w such that

$$z(w-v) = k, \quad \sigma\big(w'(0) - v'(0)\big) > 0. \tag{1.12}$$

Now we present criteria which rule out certain connections of equilibria. These criteria are applied to identify w in accordance with our theorem.

Excluding connections boils down to two basic lemmata.

LEMMA 3.1. *Let $v_1 \not\equiv v_2$ be two stationary solutions of* (1.1), (1.2). *Then*

$$|v_1'(0)| \geqslant |v_2'(0)| \;\Rightarrow\; z(v_1 - v_2) = z(v_1). \tag{3.1}$$

LEMMA 3.2. *Let v, $\tilde{w}$, w be three distinct stationary solutions such that $\tilde{w}'(0)$ is between $v'(0)$ and $w'(0)$. Assume*

$$z(v-\tilde{w}) \leqslant z(w-\tilde{w}). \tag{3.2}$$

Then v does not connect to w.

The proof of the first lemma involves a phase plane analysis of the Hamiltonian system which describes stationary solutions of (1.1). In fact the v_2-trajectory cannot lie outside the v_1-trajectory, and $v_1(x)$ intersects $v_2(x)$ precisely once between any two consecutive zeros of $v_1'(x)$.

The proof of the second lemma is indirect. If $u(t, \cdot)$ connects v to w, then $z\big(u(t, \cdot) - \tilde{w}\big)$ is decreasing and hence constant by (3.2). This implies $\operatorname{sign}\big(v'(0) - \tilde{w}'(0)\big) = \operatorname{sign}\big(w'(0) - \tilde{w}(0)\big)$ (cf. the definition of σ_k in (2.1d)) which is a contradiction.

We illustrate the contribution of these lemmata to our theorem for case (ii), $v'(0) > 0$, $i(v) = z(v) + 1$.

First suppose $0 \leqslant k < z(v)$ and $\sigma = -1$. Then $z(v-w) \neq z(v)$, hence $|v'(0)| < |w'(0)|$ by contraposition of Lemma 3.1. By $v'(0) > 0$, $\sigma = -1$, this means

$$w'(0) < -v'(0), \qquad z(w) = z(w-v) = k. \tag{3.3}$$

On the other hand, v connects to w. Therefore w is the stationary solution satisfying (3.3) with maximal $w'(0)$, i.e. $w = \underline{v}_k$, by Lemma 3.2. The case $\sigma = +1$ is analogous.

Now suppose $k = z(v)$ and $\sigma = +1$. Then $0 < v'(0) < w'(0)$, v connects to w, and $w = \bar{v}_k$ by Lemmata 3.1, 3.2 as before. However, the case $\sigma = -1$ is quite different this time. If $w'(0) < -v'(0)$, then $w = \underline{v}_k$ as above; moreover, v does not connect to any stationary $\tilde{w}$ with $\tilde{w}'(0)$ between $v'(0)$ and $w'(0)$, or else $z(\tilde{w}) \leqslant z(v)$ and using Lemma 3.1

$$z(v-\tilde{w}) \leqslant \max(z(v), z(\tilde{w})) = z(v) = z(w) = z(w-\tilde{w})$$

which contradicts Lemma 3.2. If $-v'(0) \leqslant w'(0) < v'(0)$, we are unable to identify w by the means above. Hopefully you will not accept our apologies and try it yourself.

§ 4. Generalizations

We try to drop assumptions or sharpen our conclusions. Consider assumption (1.4) first. Certainly $f(0) = 0$ may be perturbed by an f-homotopy f_β to $f(0) \neq 0$. We briefly indicated such a procedure at the end of § 2, keeping $f(0) = 0$ there. A more thorough account is given in [3]. Dropping $\overline{\lim} f(s)/s < \pi^2$ may force some stationary branches of $(1.1)_\alpha$ to become unbounded for finite α. If $\bar{v}_k$ or $\underline{v}_k$ happen to have escaped that way, we will observe a trajectory $u(t, \cdot)$ with $z(u(t, \cdot)-v) \equiv k$ in $W^u(v)$ which becomes unbounded – possibly in finite time.

Our approach does not tell anything geometric about the "number" of trajectories connecting v to w. This problem was solved by Henry [9], also without convexity assumption (1.5). Assume some additional regularity of f, and suppose v and w are hyperbolic, v connects to w. Then the set of connecting orbits

$$C(v, w) := W^u(v) \cap W^s(w)$$

is a manifold of dimension

$$\dim C(v, w) = i(v) - i(w);$$

in fact $W^u(v)$ and $W^s(v)$ intersect transversely. The case of v, w isolated but not necessarily hyperbolic is also analyzed in [9]. Using this information it should be possible to work out the connections of any isolated, not necessarily hyperbolic v.

We may change the boundary conditions (1.2) to Neumann or mixed type conditions. Then the zero number $z(u(t, \cdot)-v)$ remains a decreasing functional and the topological considerations of § 2 apply *cum grano salis*. In § 3, Lemma 3.1 becomes false, e.g. for Neumann conditions, and we have to use some other ordering of stationary solutions, e.g. by their boundary values $v(0)$. This should yield results similar to our theorem.

Again, we emphasize that we were unable to determine the connections in Ω_3 more precisely. We have some additional but incomplete information in various special cases. Each alternative of Ω_3 does occur for distinct choices of f. An explicit class of nonlinearities f where our theorem describes all connections is revealed in [15], cf. also [17].

Finally, nothing global is known for higher dimensions of the space variable x or of u (systems). We lack an analogue of the zero number $z(\varphi)$. Within the class of rotationally symmetric solutions in a ball, the problem seems tractable. However, introducing polar coordinates this is essentially the one-dimensional case again.

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