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DIVISOR CLASS GROUPS OF SOME 3-DIMENSIONAL SINGULARITIES

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The divisor class groups of the simple surface singularities A_n, D_n and E_n are well known. We consider the corresponding 3-dimensional singularities, given by the equations

$$
\begin{array}{lll}
A_n: & X_1^{n+1}+X_2^2+X_3^2+X_4^2 = 0, & n \geqslant 1, \\
D_n: & X_1^{n-1}+X_1X_2^2+X_3^2+X_4^2 = 0, & n \geqslant 4, \\
E_6: & X_1^3+X_2^4+X_3^2+X_4^2 = 0, & \\
E_7: & X_1^3+X_1X_2^3+X_3^2+X_4^2 = 0, & \\
E_8: & X_1^3+X_2^5+X_3^2+X_4^2 = 0. &
\end{array}
$$

In [2], Bingener and Storch computed their divisor class groups in characteristic 0. Here we give an independent way to calculate them in arbitrary characteristic $p \neq 2$ (cf. "Added in proof").

Let k be an algebraically closed field of characteristic $p \neq 2$, f one of the above equations, $A = k[[X_1, \ldots, X_4]]/(f)$ the corresponding complete local ring at the singular point. For $\operatorname{Spec}(A)$ we use the same notation A_n, D_n or E_n, respectively.

1. Picard groups of divisors on 3-folds

Let X be a smooth 3-fold, $E_1, \ldots, E_m$ proper divisors on X, all E_i smooth, irreducible and having only normal crossings such that $E_{i_1} \cap E_{i_2}$ is $\emptyset$ or a closed point (if $i_k \neq i_l$ for $k \neq l$).

This paper is in final form and no version of it will be submitted for publication elsewhere.

1.1 Definition. $(E_1, \ldots, E_m)$ is a *good configuration of divisors* if there exists a permutation $\pi \in S_n$ such that all curves $C = C^{(i)}$,

$$C^{(i)} = E_{\pi(i)} \cap \sum_{j=i+1}^{m} E_{\pi(j)} \qquad (i = 1, \ldots, m-1),$$

satisfy the following condition:

(*) The irreducible components $C_1, \ldots, C_l$ of C can be enumerated in such a way that C_j intersects precisely one of the curves $C_{j+1}, \ldots, C_l$ $(j = 1, \ldots, l-1)$.

Using this property, we obtain the Picard group of $E = E_1 + \ldots + E_m$ in the following way (cf. [1]):

1.2. Proposition. *Let* $\mathrm{Pic}^*(E) \subseteq \mathrm{Pic}(E_1) \times \ldots \times \mathrm{Pic}(E_m)$ *be the set of all* $(L_1, \ldots, L_m)$, L_i *an invertible sheaf on* E_i, *such that* $L_i|E_i \cap E_j \simeq L_j|E_i \cap E_j$ *for all* i, j. *If* $(E_1, \ldots, E_m)$ *satisfies* 1.1, *then*

$$\mathrm{Pic}(E) \to \mathrm{Pic}^*(E), \qquad L \mapsto (L|E_1, \ldots, L|E_m),$$

is bijective.

Proof. Put $\bar{E} = E_2 + \ldots + E_m$. Using the exact sequence

$$1 \to \mathcal{O}^*_{E_1+\bar{E}} \to \mathcal{O}^*_{E_1} \times \mathcal{O}^*_{\bar{E}} \to \mathcal{O}^*_{E_1 \cap \bar{E}} \to 1,$$

we obtain an exact sequence

$$H^0(\mathcal{O}^*_{E_1}) \times H(\mathcal{O}^*_{\bar{E}}) \to H^0(\mathcal{O}^*_{E_1 \cap \bar{E}}) \xrightarrow{\partial} \mathrm{Pic}(E) \to \mathrm{Pic}(E_1) \times \mathrm{Pic}(\bar{E}) \to \mathrm{Pic}(E_1 \cap \bar{E})$$

with ∂ vanishing if $E_1 \cap \bar{E}$ is connected (e.g. if the curve $E_1 \cap \bar{E}$ satisfies (*) from 1.1). Now the result follows by induction.

2. Application to the canonical resolutions of A_n, D_n, E_n

Let $X \to \mathrm{Spec}(A)$ be the canonical resolution of A_n, D_n or E_n, obtained by successively blowing up closed points (cf. [5]). Denote by $E = E_1 + \ldots + E_m$ its exceptional divisor.

This is shown explicitly in ([1], 2.4.1–2.4.7).

Now $\mathrm{Pic}(E) \simeq \mathrm{Pic}^*(E) \subseteq \mathbf{Z}^s$ as a subgroup by 1.2 (all $\mathrm{Pic}(E_i)$ are free abelian groups), and the conditions $L_i|E_i \cap E_j \simeq L_j|E_i \cap E_j$ can be expressed by linear equations (cf. [1], 3.). We obtain

2.2. Corollary. $\mathrm{Pic}(E)$ *is a free abelian group,*

$$\mathrm{rk}(\mathrm{Pic}(E)) = \left[\frac{n+3}{2}\right] \quad \textit{for } A_n, \quad \mathrm{rk}(\mathrm{Pic}(E)) = \left[\frac{3n-1}{2} + 2(-1)^n\right] \quad \textit{for } D_n,$$

$\mathrm{rk}(\mathrm{Pic}(E)) = 5, 11, 12$ *for* E_6, E_7, E_8, *respectively.*

Now denote by $\mathrm{Cl}(A) := \mathrm{Cl}(\mathrm{Spec}(A))$ the group of Weil divisors modulo linear equivalence. We compute $\mathrm{Cl}(A)$ in terms of $\mathscr{E}$ (= free abelian group generated by $E_1, \ldots, E_m$), $\mathrm{Pic}(E)$ and the canonical homomorphism $\varphi \colon \mathscr{E} \to \mathrm{Pic}(E)$, sending E_i to $\mathcal{O}_X(X_i)|E$.

2.3. *Remark.* Denote by $X^\wedge$ the completion of X along E. The canonical morphism $X^\wedge \to X$ induces an isomorphism

$$\psi \colon \mathrm{Pic}(X) \to \mathrm{Pic}(X^\wedge), \qquad L \mapsto L \otimes_{\mathcal{O}_X} \mathcal{O}_{X^\wedge}.$$

(Apply Grothendieck's "Théorème d'existence" to the projective morphism $X \to \mathrm{Spec}(A)$.)

2.4. *Remark.* $\mathrm{Cl}(A) = \mathrm{Cl}(U) = \mathrm{Pic}(U)$, where $U = X - E = \mathrm{Spec}(A) - \{m_A\}$. ($U$ is regular.)

2.5. *Remark.* The sequence

$$0 \to \mathscr{E} \xrightarrow{\alpha} \mathrm{Pic}(X) \xrightarrow{\beta} \mathrm{Pic}(U) \to 0$$

(given by the canonical homomorphism $E_i \mapsto \mathcal{O}_X(E_i)$ and the restriction $L \mapsto L|U$) is exact.

Proof. β is surjective since the open subset $U \subseteq X$ contains all points of codimension 1. α is injective since for any invertible sheaf $L = \mathcal{O}_X(\sum \mu_i E_i)$ with $L \simeq \mathcal{O}_X$ we find an element g in the quotient field of A having the divisor $\sum \mu_i E_i$. A is normal, i.e. if $\mathrm{Supp}(g)$ is contained in the exceptional locus, $\mathrm{Supp}(g)$ must be empty, i.e. all $\mu_i = 0$. Now the exactness of the sequence is obvious.

2.6. THEOREM. *There is an exact sequence*

$$0 \to \mathscr{E} \xrightarrow{\varphi} \mathrm{Pic}(E) \to \mathrm{Cl}(A) \to 0,$$

where φ denotes the above map.

Proof. By [4], the canonical morphisms

$$\mathrm{Pic}(\mu_1 E_1 + \ldots + \mu_m E_m) \to \mathrm{Pic}(E)$$

are isomorphisms if $\mu = (\mu_1, \ldots, \mu_m) \in \mathbf{N}^m$ is such that $\mu_1 E_1 + \ldots + \mu_m E_m$ has an ample conormal bundle and μ is "symmetric". Therefore, the completion gives an isomorphism $\tau \colon \mathrm{Pic}(X^\wedge) \xrightarrow{\sim} \mathrm{Pic}(E)$. Put $\varphi = \tau \circ \psi \circ \alpha$. Thus we obtain 2.6 from 2.4 and 2.5.

2.7. COROLLARY.

$\mathrm{Cl}(A) \cong 0$ *if A is of type A_n (n even), E_6 or E_8.*

$\mathrm{Cl}(A) \cong \mathbf{Z}$ *if A is of type A_n (n odd), D_n (n odd) or E_7.*

$\mathrm{Cl}(A) \cong \mathbf{Z}^2$ *if A is of type D_n (n even).*

Proof. The elements $(\mathcal{O}_X(E_i)|E_1, \ldots, \mathcal{O}_X(E_i)|E_m)$ in $\mathrm{Pic}^*(E)$ $(i = 1, \ldots, m)$ generate a subgroup H such that

$$\mathrm{Cl}(A) \simeq \mathrm{coker}(\varphi) \simeq \mathrm{Pic}^*(E)/H.$$

The computation is done explicitly.

References

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[4] M. Roczen, *The Picard group of the canonical resolution of a 3-dimensional simple singularity*, in: Singularities, Banach Center Publ. 20, PWN 1988, 379–383.

[5] —, *Some properties of the canonical resolutions of the 3-dimensional singularities A_n, D_n, E_n over a field of characteristic $\neq 2$*, in: Lecture Notes in Math. 1056, Springer, 1984, 297–365.

Added in proof (May 1990). A complete list of the simple singularities is given in Grenel–Kröning (Preprint MPI Bonn, 88-19). The canonical resolutions can be shown to be the same as in [5]. Therefore, our result remains true for all 3-dimensional simple singularities in arbitrary characteristic.