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ALGEBRAS WITH HYPERCRITICAL TITS FORM

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Let Λ be a finite-dimensional basic and connected algebra over an algebraically closed field. Assume that the ordinary quiver Q_Λ of Λ has no oriented cycles. Let q_Λ be the Tits form of Λ. If Λ is tame, then q_Λ is weakly nonnegative [20]. Moreover the converse has been shown for some families of algebras [16, 20, 21]. We study the algebras Λ for which q_Λ is hypercritical, that is, every restriction of q_Λ is weakly nonnegative but q_Λ itself is not. We show that for a Schurian, $\tilde{A}$-free algebra Λ satisfying the (S)-condition, its Tits form q_Λ is not weakly nonnegative if and only if Λ has a full convex subalgebra Λ_0 such that q_{Λ_0} is hypercritical. A Schurian, $\tilde{A}$-free algebra Λ satisfying the (S)-condition has a hypercritical Tits form q_Λ if and only if Λ is concealed of a minimal wild hereditary algebra.

Let k be an algebraically closed field. Let Λ be a finite-dimensional, basic and connected k-algebra. Let $Q = Q_\Lambda$ be the ordinary quiver of Λ and let $\Lambda = kQ/N$ for an admissible ideal N of kQ (see [11]). In this work we assume that Q has no oriented cycles.

In [4] the *Tits form* q_Λ of Λ was introduced as the quadratic form q_Λ: $\mathbf{Z}^{Q_0} \to \mathbf{Z}$ given by

$$q_\Lambda(z) = \sum_{x \in Q_0} z(x)^2 - \sum_{x,y \in Q_0} z(x)z(y)\dim_k \mathrm{Ext}^1_\Lambda(S_x, S_y) + \sum_{x,y \in Q_0} z(x)z(y)\dim_k \mathrm{Ext}^2_\Lambda(S_x, S_y)$$

where Q_0 is the set of vertices of Q and S_x is the simple Λ-module associated with $x \in Q_0$. In [20] it was shown that a tame algebra Λ has a weakly nonnegative Tits form q_Λ (that is, $q_\Lambda(z) \geq 0$ whenever z has nonnegative coordinates). The converse of this result has been shown for some families of algebras [16, 20, 21].

This paper is in final form and no version of it will be submitted for publication elsewhere.

Given a quadratic form $q\colon \mathbf{Z}^n \to \mathbf{Z}$ of the shape

$$q(z) = \sum_{i=1}^{n} z(i)^2 + \sum_{i<j} a_{ij} z(i) z(j)$$

and a nonempty subset $I = \{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$ we denote by q^I the quadratic form qd_I, where $d_I\colon \mathbf{Z}^m \to \mathbf{Z}^n$ maps the kth natural basis vector e_k onto $e_{i_k} \in \mathbf{Z}^n$; q^I is called a *restriction* of q (see [15]). The quadratic form q is called *hypercritical* (von Höhne's terminology) if every restriction of q is weakly nonnegative but q itself is not. Clearly, a quadratic form q is not weakly nonnegative if and only if there is a restriction q^I which is hypercritical.

In Section 1 we make some remarks concerning weakly nonnegative quadratic forms. We show that it is easy to decide whether or not a quadratic form is weakly nonnegative. In particular, we give some characterizations of hypercritical forms (partly a reformulation of [25]).

We say that Λ is *good* if Λ is Schurian, satisfies the (S)-condition and is $\tilde{\mathbf{A}}$-free (see [3, 6]). We *conjecture* that if a good algebra Λ has a weakly nonnegative Tits form q_Λ, then Λ is tame.

In Section 2 we show that a good algebra Λ has a Tits form q_Λ which is not weakly nonnegative if and only if Λ has a convex (good) subalgebra Λ_0 such that q_{Λ_0} is hypercritical. Therefore, if our conjecture is true, there is a criterion for tameness using the good algebras with hypercritical Tits form similar to that given by Bongartz for representation-finite algebras [6].

In Section 3 we show that a good algebra Λ has hypercritical Tits form if and only if Λ is concealed of a minimal wild hereditary algebra A (that is, $A \xrightarrow{\sim} k\vec{\Delta}$ where $\vec{\Delta}$ is a quiver whose underlying graph Δ is a *hyperbolic diagram*). This result is the hypercritical counterpart of the critical result given in [14].

Given any matrix A we write $A \geqslant 0$ (resp. $A > 0$) if all elements of A are nonnegative (resp. positive).

Sometimes we will treat Λ as a k-category [7], therefore we may speak of full subalgebras, convex (= path closed) subalgebras,...

The category of finitely generated left Λ-modules is denoted by $\operatorname{mod}\Lambda$. An object $M \in \operatorname{mod}\Lambda$ is simply called a Λ-module. The *dimension vector* of $M \in \operatorname{mod}\Lambda$ is $\underline{\dim} M = (\dim_k xM)_{x \in Q_0}$.

The Auslander–Reiten quiver of Λ is denoted by Γ_Λ, the Auslander–Reiten translation by τ_Λ.

For notions and notations not explicitly given the reader is referred to [11] and [23].

1. Some remarks about weakly nonnegative quadratic forms

1.1. We start recalling some facts about the Tits form of an algebra. Let $\Lambda = kQ/N$ be as above. Let Q_0 be the set of vertices of Q and Q_1 the set of arrows. The trivial path of the vertex $x \in Q_0$ is also denoted by x. Let $L \subset \bigcup_{x,y \in Q_0} yNx$ be a minimal set of generators of the ideal N. For each couple $x, y \in Q_0$ we set $l(x, y) =$ cardinality of $L \cap yNx$. In [4] it is shown that the Tits form q_A: $\mathbf{Z}^{Q_0} \to \mathbf{Z}$ is given by

$$q_A(z) = \sum_{x \in Q_0} z(x)^2 - \sum_{(x \to y) \in Q_1} z(x)z(y) + \sum_{x,y \in Q_0} l(x, y)z(x)z(y).$$

Let A_A be the (symmetric) matrix associated to q_A, that is, $q_A(z) = \frac{1}{2}(zA_Az^t)$ (z^t denotes the transpose of the-row-matrix z). We get a bilinear form $(w, z)_A = wA_Az^t$.

1.2. By $\mathrm{ind}_d A$ we denote the (variety) of all indecomposable A-modules of k-dimension d. Following [9], we say that $\mathrm{ind}_d A$ is *parametrizable* if there is a finite family $M_1, \ldots, M_s$ of A-$k[x]$-bimodules such that M_i is a finitely generated free right $k[x]$-module and every module $N \in \mathrm{ind}_d A$ is of the form $N \simeq M_i \otimes_{k[x]} (k[x]/(x-\lambda))$ for some $i \in \{1, \ldots, s\}$ and $\lambda \in k$.

The algebra A is called *tame* if $\mathrm{ind}_d A$ is parametrizable for every $d \in \mathbf{N}$. In [20] the following result is shown:

PROPOSITION. *If A is a tame algebra then q_A is weakly nonnegative.* ∎

Although the converse is not true [5], it holds for some "good" classes of algebras:

(a) Tilted algebras [16]: let A be a tilted algebra; then A is tame iff q_A is weakly nonnegative.

(b) One-point extensions of tame concealed algebras [20]: let A_0 be a tame concealed algebra not of type $\tilde{\mathbf{A}}_n$, and let R be an indecomposable A_0-module; then $A_0[R]$ is tame iff $q_{A_0[R]}$ is weakly nonnegative.

(c) Iterated tubular algebras [21].

Therefore, the problem of knowing whether or not q_A is weakly nonnegative imposes itself. In the remaining part of this section we give some (maybe well-known) criteria for a quadratic form to be weakly nonnegative.

1.3. A *unit form* [15] is a quadratic form q: $\mathbf{Z}^n \to \mathbf{Z}$ of the shape

$$q(z) = \sum_{i=1}^{n} z(i)^2 + \sum_{i<j} a_{ij} z(i)z(j).$$

The unit form q is called *critical* if every restriction of q is weakly positive but q itself is not. By a theorem of Ovsienko [19] (see also [23] and [15]) we know that a critical form q is nonnegative and the radical $\mathrm{rad}\, q = \{z \in \mathbf{Z}^n$: $q(z) = 0\}$ is of the form $\mathbf{Z}z_0$ where z_0 is a vector with positive coordinates. We call z_0 the *positive generator* of $\mathrm{rad}\, q$.

For any nonempty subset $I = \{i_1, \ldots, i_m\}$ we get the restriction $q^I = qd_I$: $\mathbf{Z}^m \to \mathbf{Z}$. For a vector $z \in \mathbf{Z}^m$, we also write $z^0 = d_I(z)$ (the completion by zeroes). By A_q we denote the (symmetric) matrix associated with q, that is, $q(z) = \frac{1}{2}(zA_qz^t)$. We write $(z, w) = zA_qw^t$.

If $I = \{1, \ldots, n\} - \{i\}$ we write $q^{(i)}$ instead of q^I.

PROPOSITION. *Let q be a unit form. The following are equivalent:*

(a) *q is weakly nonnegative.*

(b) *For every restriction q^I of q, if q^I is critical with v the positive generator of* $\operatorname{rad} q^I$, *then $v^0 A_q \geqslant 0$.*

Proof. (a)$\Rightarrow$(b). Assume q^I is critical and $v^0 A_q$ has its jth component negative. Then $0 \leqslant 2v^0 + e_j \in \mathbf{Z}^n$ and $q(2v^0 + e_j) = 4q(v^0) + 2(vA_q e_j^t) + 1 < 0$.

(b)$\Rightarrow$(a). Assume (b) and let q be a counterexample with minimal n. Let $0 \leqslant z$ be such that $q(z) < 0$. As q is not weakly positive, there is a critical restriction q^I. Let v be the positive generator of $\operatorname{rad} q^I$. We can find a number $0 \geqslant a \in \mathbf{Q}$ such that $0 \leqslant z + av^0$ and $(z + av^0)(j) = 0$ for some $j \in \{1, \ldots, n\}$. Then (b) is satisfied for $q^{(j)}$ and $q^{(j)}(z + av^0) < av^0 A_q z^t \leqslant 0$, a contradiction. ∎

COROLLARY. *The unit form q is weakly nonnegative if and only if $0 \leqslant q(z)$ for every $z \in [0,12]^n$.*

Proof. This follows from the proof above and the fact that $z_0(i) \leqslant 6$ for the coordinates of the positive generator z_0 of $\operatorname{rad} q'$ of a critical form q' (see [15, 19]). ∎

COROLLARY. *Let q be a unit form which is not weakly nonnegative. The following are equivalent:*

(a) *q is hypercritical.*

(b) *For every restriction q^I of q, if q^I is critical with v the positive generator of* $\operatorname{rad} q^I$, *then there exists some $i \in \{1, \ldots, n\}$ such that $I \cup \{i\} = \{1, \ldots, n\}$ and* $(v^0, e_i) < 0$.

Proof. (a)$\Rightarrow$(b). By the Proposition there is a critical restriction q^J with ω the positive generator of $\operatorname{rad} q^J$ satisfying $(\omega^0, e_j) < 0$ for some $j \in \{1, \ldots, n\}$. Again by the Proposition, $J = \{1, \ldots, n\} - \{j\}$. If $v(j) = 0$, then $J = I$ and $v = \omega$. If $v(j) \neq 0$, we get

$$0 > v(j)(\omega^0, e_j) = (\omega^0, v^0) = \sum_{i=1}^{n} \omega(i)(v^0, e_i).$$

Hence, there is some $i \in \{1, \ldots, n\}$ with $(v^0, e_i) < 0$. It follows that $I = \{1, \ldots, n\} - \{i\}$.

(b)$\Rightarrow$(a). If a proper restriction q^I is not weakly nonnegative, there is a restriction q^J with $J \subsetneq I$ such that q^J is critical, a contradiction. ∎

1.4. The following result provides yet another combinatorial description of weakly nonnegative quadratic forms.

PROPOSITION [25]. *Let q: $\mathbf{Z}^n \to \mathbf{Z}$ be any quadratic form. The following are equivalent:*

(a) *q is weakly nonnegative.*

(b) *For any principal minor B of A_q, either* $\det B \geqslant 0$ *or the adjoint matrix* $\mathrm{ad}(B)$ *of B has at least one negative element.* ∎

We are indebted to W. W. Crawley-Boevey for pointing out the paper [25] to us. The main step in Zel'dich's proof is the following:

LEMMA [25]. *Let q: $\mathbf{Z}^n \to \mathbf{Z}$ be a hypercritical quadratic form. Then $q^{(i)}$ is nonnegative for every $i \in \{1, \ldots, n\}$.* ∎

1.5. We get some other characterizations of hypercritical forms. The equivalence (a)$\Leftrightarrow$(c) below is mainly a reformulation of 1.4 (in particular, it holds for any quadratic form).

PROPOSITION. *Let q: $\mathbf{Z}^n \to \mathbf{Z}$ be a unit form. The following are equivalent*:

(a) *q is hypercritical.*

(b) *There is a vector $z_0 \in \mathbf{Z}^n$ with $q(z_0) < 0$ and for every vector $z \in \mathbf{Z}^n$ with $q(z) \leqslant 0$ we have $z \geqslant 0$ or $0 \geqslant z$.*

(c) $\mathrm{ad}(A_q) \geqslant 0$, $\det A_q < 0$ *and for every proper principal minor B of A_q,* $\det B \geqslant 0$.

Proof. (a)$\Rightarrow$(b). As q is hypercritical, by 1.3, there is a critical restriction q^I of q with v the positive generator of $\mathrm{rad}\, q^I$ and a $j \in \{1,\ldots,n\}$ such that $(v^0, e_j) < 0$ and $I \cup \{j\} = \{1, \ldots, n\}$.

Let $z \in \mathbf{Z}^n$ be such that $q(z) \leqslant 0$. If $z(j) = 0$, then $z \in \mathbf{Z}v^0$. Assume that $z(j) > 0$. Let $i \neq j$ and let $a \in \mathbf{Q}$ be such that $(z + av^0)(i) = 0$. By Lemma 1.4, $0 \leqslant q(z + av^0) \leqslant a(z, v^0) = az(j)(v^0, e_j)$. Thus $a \leqslant 0$ and $z(i) \geqslant 0$.

(b)$\Rightarrow$(a). If q is not hypercritical, there is a vector $0 \leqslant z$ but not $0 < z$ such that $q(z) < 0$. Let j be any index with $z(j) = 0$. Choose $n \in \mathbf{N}$ with $nq(z) \leqslant (z, e_j) - 1$. Then

$$q(nz - e_j) = n^2 q(z) - n(z, e_j) + 1 \leqslant 0.$$

(a)$\Rightarrow$(c). Since $q^{(i)}$ is nonnegative for every $i \in \{1,\ldots,n\}$, it follows that $\det B \geqslant 0$ for every proper principal minor B of A_q (see [9]). If also $\det A_q \geqslant 0$, then q would be nonnegative. Therefore $\det A_q < 0$.

Let $\mathrm{ad}(A_q) = (v_1^t, \ldots, v_n^t)$ with $v_i(j) < 0$. Let $0 < z \in \mathbf{Z}^n$ be such that $q(z) < 0$. There exists a number $0 < a \in \mathbf{Q}$ such that $(z + av_i)(j) = 0$. Then by 1.4,

$$\begin{aligned} 0 \leqslant q(z + av_i) &< a(z, v_i) + a^2 q(v_i) \\ &= az(i) \det A_q + a^2 \det A_{q^{(i)}} \det A_q < 0, \end{aligned}$$

a contradiction. Thus $\mathrm{ad}(A_q) \geqslant 0$.

(c)$\Rightarrow$(a). Since $\det B \geqslant 0$ for every principal minor of $A_{q^{(i)}}$, it follows that $q^{(i)}$ is nonnegative, $i = 1, \ldots, n$. Since $\mathrm{ad}\,(A_q) \geqslant 0$, by Frobenius' theorem [12], there exist a number $0 \leqslant r$ and a vector $0 \leqslant z \neq 0$ with $\mathrm{ad}(A_q) z^t = rz^t$. Then $rq(z) = rzA_q z^t = zA_q \mathrm{ad}(A_q) z^t = \det A_q zz^t < 0$. Thus $q(z) < 0$. ∎

2. Good algebras with weakly nonnegative Tits form

2.1. Let $\Lambda = kQ/N$ be as in the introduction. For every $x \in Q_0$, we denote by P_x the indecomposable projective Λ-module associated with x, that is, $P_x = \Lambda x$. The module P_x is said to have a *separated radical* if the supports of any two nonisomorphic indecomposable summands of $\operatorname{rad} P_x$ are contained in different connected components of the subquiver Q_x of Q obtained by deleting all those vertices j such that there exists a path from j to x. If all the indecomposable projective Λ-modules have separated radical, then Λ is said to satisfy the *(S)-condition* [3].

The algebra Λ is said to be *Schurian* if $\dim_k y\Lambda x \leqslant 1$ for each couple $x, y \in Q_0$. It is said to be $\tilde{\mathbf{A}}$*-free* if there exists no full subalgebra Λ' of Λ such that $\Lambda' \xrightarrow{\sim} kQ'$ where the underlying graph of Q' is $\tilde{\mathbf{A}}_m$ $(m \geqslant 1)$.

In this work we call an algebra Λ *good* if it is Schurian, $\tilde{\mathbf{A}}$-free and satisfies the (S)-condition. The following facts about good algebras are relevant:

(a) A representation-finite algebra Λ is simply connected if and only if it is good [3].

(b) A simply connected, Schurian, $\tilde{\mathbf{A}}$-free algebra is good. Therefore the study of a large class of algebras may be reduced to the study of good algebras using covering techniques (see [1, 18]).

(c) If Λ is good, the opposite algebra Λ^{op} is good.

(d) If Λ' is a full convex subalgebra of a good algebra Λ, Λ' is good [6, 8].

(e) If Λ is a good representation-infinite algebra, then Λ contains a full convex subalgebra which is tame concealed of type $\tilde{\mathbf{D}}_n$ or $\tilde{\mathbf{E}}_p$ ($p = 6$, 7 or 8) [5, 6, 14].

2.2. The aim of this section is to prove the following:

THEOREM. *Let Λ be a good algebra. Then the Tits form q_Λ is not weakly nonnegative if and only if there is a full convex subalgebra Λ_0 of Λ such that q_{Λ_0} is hypercritical.*

The "if" part of the statement is clear: the quiver Q_{Λ_0} of Λ_0 is a full subquiver of Q $(= Q_\Lambda)$. Let I be the set of vertices of Q_{Λ_0}; then $q_{\Lambda_0} = q_\Lambda^I$ and q_Λ is not weakly nonnegative.

The "only if" part will be shown after some technical lemmata.

2.3. LEMMA. *Let $\Lambda = kQ/N$ be a good algebra. Then Q does not admit full subquivers of the form shown in Fig.* 1.

Proof. Assume that Λ is a counterexample with a minimal number of vertices. Consider a full subquiver of Q as in Fig. 1 such that x is maximal in the order in Q given by the oriented paths. By minimality, the full convex hull of α in Λ is Λ itself (notation: $\Lambda = [\alpha]$). Since P_x has separated radical, we get

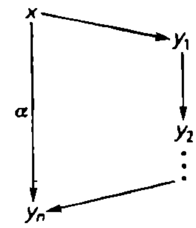

Fig. 1

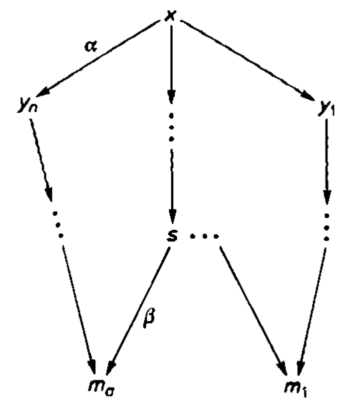

Fig. 2

Fig. 2 with $m_i \Lambda x \neq 0$. Since $s \in [\alpha]$, there is a path from s to y_n and we get a contradiction with the maximality of x. ∎

2.4. The following lemma is similar to [15, 3.3].

LEMMA. *Let $\Lambda = kQ/N$ be a good algebra. Assume in Q we have a walk*

$$x_0 \xrightarrow{\alpha_1} x_1 \overset{\alpha_2}{—} \dots — x_{s-1} \overset{\alpha_s}{—} x_s, \quad s \geqslant 2$$

(where the edges α_i are arrows in some direction) satisfying:

(a) *There is a path (= oriented walk) from x_0 to x_s.*
(b) *There are no arrows connecting x_i and x_j if $|i-j| \geqslant 2$.*
(c) *$l(x_i, x_j) = 0$, for every $\{i, j\} \neq \{0, s\}$.*

Then the walk has the shape

$$x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \dots \to x_{s-1} \xrightarrow{\alpha_s} x_s.$$

Proof. Assume the result fails and let Λ be a counterexample. Therefore, there is a full subquiver Q' of Q with $0 \leqslant s_1 < s_2 < \dots < s_{t-1} < s_t = s$, $1 \leqslant t$ odd, of the shape shown in Fig. 3 and satisfying:

1) There is a path $x_0 = y_0 \xrightarrow{\beta_1} y_1 \xrightarrow{\beta_2} \dots \xrightarrow{\beta_r} y_r = x_s$, $\quad r \geqslant 1$.
2) $l(x_i, x_j) = 0$, for any two vertices x_i, x_j connected by a path in Q'.

Among all the subquivers of Q with these properties we choose one with

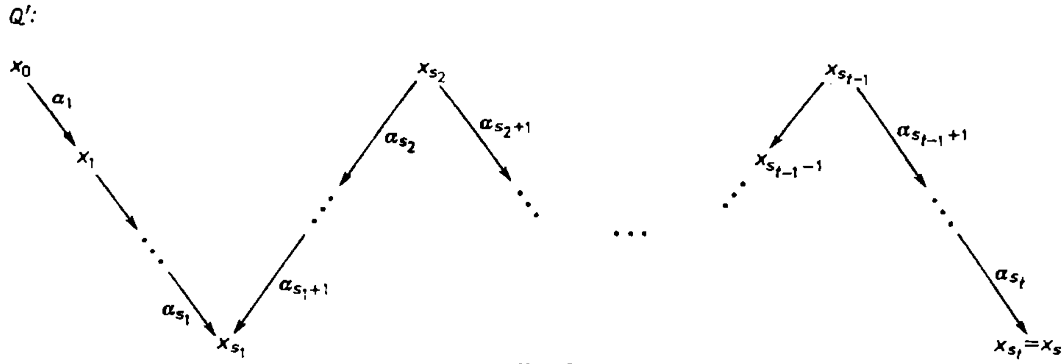

Fig. 3

minimal number of vertices. Then $0 < s_1, s_{t-1} < s$ and the full convex hull Λ' of Q' in Λ has as quiver the union of Q' and of a quiver Q'' with unique source x_0, unique sink x_s and such that $Q' \cap Q'' = \{x_0, x_s\}$. As Λ' is again good (2.1), we may assume $\Lambda = \Lambda'$.

Since $x_{s_{t-1}-1}$ and x_s are connected in $Qx_{s_{t-1}}$, by the (S)-condition we get the situation of Fig. 4 with $m_i \Lambda z_0 \neq 0 \neq m_i N z_0$ for $i = 1, \ldots, a$. Therefore,

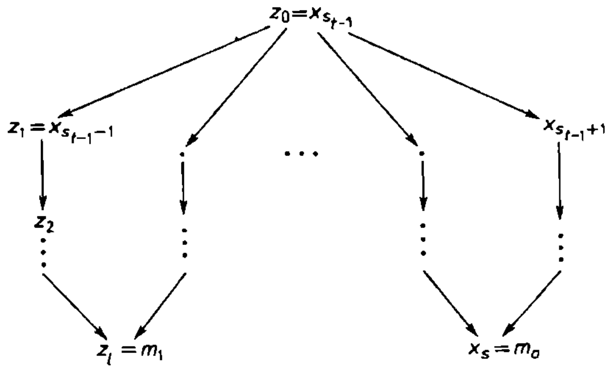

Fig. 4

there is some $j \in \{1, \ldots. l\}$ such that $l(x_{s_{t-1}}, z_j) \neq 0$ and the path $x_{s_{t-1}} = z_0 \rightarrow z_1 \rightarrow \ldots \rightarrow z_j$ cannot be contained in Q'. Assume $z_0 = x_{s_{t-1}}$, $z_1, \ldots, z_p$ are in Q' and z_{p+1} is not.

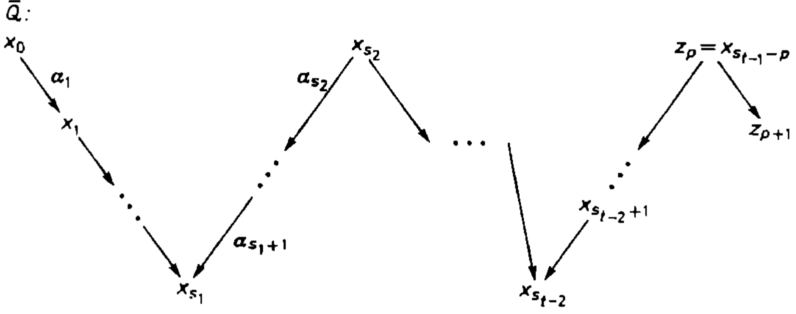

Fig. 5

We consider the quiver $\bar{Q}$ of Fig. 5 which satisfies:

1) There is a path $x_0 \to \ldots \to z_{p+1}$, since $\Lambda = \Lambda'$.

2) $l(x_i, x_j) = 0$ for any two vertices x_i, x_j connected by a path in $\bar{Q}$ (that $l(z_p, z_{p+1}) = 0$ follows from 2.3).

By construction $\bar{Q}$ has less vertices than Q', a contradiction. ∎

2.5. Let Λ_0 be a tame concealed algebra and let v be the positive generator of $\operatorname{rad} q_{\Lambda_0}$. Let R be an indecomposable Λ_0-module. Consider the one-point extension $\Lambda = \Lambda_0[R]$. For the new vertex x in Q_Λ we get $\operatorname{rad} P_x = R$.

LEMMA. *With the above notation, R is a preprojective Λ_0-module if and only if $(v^0, e_x)_\Lambda < 0$.*

Proof. Let V be any simple regular Λ_0-module with $\underline{\dim} V = v$. If R is preprojective, then $\operatorname{gl\,dim} \Lambda \leqslant 2$. Since $e_x = \underline{\dim}\, P_x - \underline{\dim}\, R$, we have

$$(v^0, e_x)_\Lambda = \sum_{i=0}^{1} (-1)^i \dim \operatorname{Ext}_\Lambda^i(S_x, V) = -\dim \operatorname{Hom}_\Lambda(R, V) < 0.$$

Conversely, assume that $(v^0, e_x)_\Lambda < 0$. Since $\operatorname{gl\,dim} \Lambda_0[R] \leqslant 3$ we get

$$\begin{aligned} 0 > (v^0, e_x)_\Lambda &= \sum_y v^0(y) \Big(\sum_{i=0}^{2} (-1)^i \dim \operatorname{Ext}_\Lambda^i(S_x, S_y) \Big) \\ &\geqslant \sum_{i=1}^{3} (-1)^i \dim \operatorname{Ext}_1^i(S_x, V) \\ &= -\sum_{i=0}^{1} (-1)^i \dim \operatorname{Ext}_\Lambda^i(R, V) \\ &\geqslant -\dim \operatorname{Hom}_\Lambda(R, V) \end{aligned}$$

and therefore R is preprojective. ∎

2.6. We get a first insight into the structure of good algebras Λ with hypercritical Tits form.

LEMMA. *Let Λ be a good algebra with a hypercritical Tits form q_Λ. Then there is a full convex subalgebra Λ_0 of Λ which is tame concealed and an indecomposable Λ_0-module R such that Λ has one of the following forms: $\Lambda = \Lambda_0[R]$ (one-point extension of Λ_0 by R) and R is Λ_0-preprojective, or $\Lambda = [R]\Lambda_0$ (one-point coextension of Λ_0 by R) and R is Λ_0-preinjective.*

Proof. By 2.1(e) there is a full convex subalgebra Λ_0 of Λ such that Λ_0 is tame concealed. Then q_{Λ_0} is a critical restriction of q_Λ. If v is the positive generator of $\operatorname{rad} q_{\Lambda_0}$, by 1.3, there is a vertex x satisfying $(v^0, e_x)_\Lambda < 0$ and $Q_0 = (Q_{\Lambda_0})_0 \cup \{x\}$. Therefore, x is extremal in Q.

Assume that x is a source in Q. By the (S)-condition, $R = \operatorname{rad} P_x$ is Λ_0-indecomposable. Thus 2.5 says that R is Λ_0-preprojective. The other case is dual. ∎

2.7. *Proof of* 2.2. We just have to show the "only if" part of the result. So, assume that q_A is not weakly nonnegative. There is a restriction q_A^I of q_A which is hypercritical. By 1.3, there is an index $y \in I$ such that for $J = I - \{y\}$, the restriction q_A^J is critical and if v is the positive generator of $\operatorname{rad} q_A^J$ then $(v^0, e_y)_A < 0$.

Using 2.4 we may repeat the argument in [15, 3.4] to get $q_A^J = q_{A'}$, for some tame concealed algebra A'. Observe that the quiver $Q_{A'}$ of A' is a full subquiver of Q and if $A' = kQ_{A'}/N'$, then $\dim yN'x = \dim yNx$ for $x, y \in (Q_{A'})_0$. We distinguish three situations:

(a) A' is not full convex in A. In this case there exists a proper full convex subalgebra A_1 of A such that q_{A_1} is not weakly nonnegative; the result follows by induction. The proof of this claim is done by an easy inspection of the list of tame concealed algebras using the following remarks:

(1) If there is a path $x = x_0 \xrightarrow{\alpha_1} x_1 \to \dots \xrightarrow{\alpha_s} x_s = y$ with $x, y \in (Q_{A_1})_0$ and no x_i ($i \in \{1, \dots, s-1\}$) in $Q_{A'}$, then there is a walk in $Q_{A'}$ of the shape

$$x = y_0 \leftarrow y_1 \leftarrow \dots \leftarrow y_p \to y_{p+1} \to \dots \to y_q \leftarrow y_{q+1} \leftarrow \dots \leftarrow y_l = y$$

and with $l(y_p, y_q) \neq 0$. Moreover, there is only one path joining y_p and y_q in $Q_{A'}$. This is checked in each case using 2.4 and the (S)-condition.

(2) In the cases allowed by (1), the result is shown using 1.3. Here we give a typical example: let A be given by the quiver with relations shown in Fig. 6.

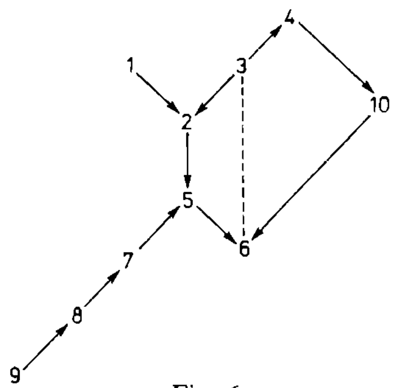

Fig. 6

Let $J = \{1, 2, \dots, 8, 9\}$. Then q_A^J is a critical form given as $q_A^J = q_{A'}$ for the tame concealed algebra $A' = A/A(10)A$ which is not full convex in A. But $A_1 = A/A(9)A$ is convex in A and q_{A_1} is not weakly nonnegative (since for $I = \{1, 2, \dots, 7, 10\}$ we get $q_{A_1}^I$ critical and if v is the positive generator of $\operatorname{rad} q_{A_1}^I$, then $(v^0, e_8) = -1$).

(b) Let Q^I be the full subquiver of Q with vertices in I. Assume that Q^I is convex in Q. Then consider the full (convex) subalgebra A_0 of A with quiver Q^I. Clearly, $q_A^I = q_{A_0}$.

(c) A' is full convex in A and Q^I is not convex in Q. Since $(v^0, e_y)_A < 0$, we may assume the existence of an arrow $y = y_0 \xrightarrow{\alpha_0} x_0$ with $x_0 \in J$.

Without loss of generality we may assume that Λ' is cofinal in Λ, that is, if $t \in J$ and there is a path from t to s in Q, then $s \in J$. Applying the (S)-condition we find a sequence of arrows $y_i \xrightarrow{\alpha_i} x_i$, and vertices $s_{i1}, \ldots, s_{ir_i}$, $m_{i1}, \ldots, m_{ir_i - 1}$ $(1 \leqslant r_i \leqslant 4)$ for $i = 0, \ldots, m$ (see Fig. 7) satisfying: s_{ij}, $m_{ij} \in J$, $y_i \in Q_0 - J$ and

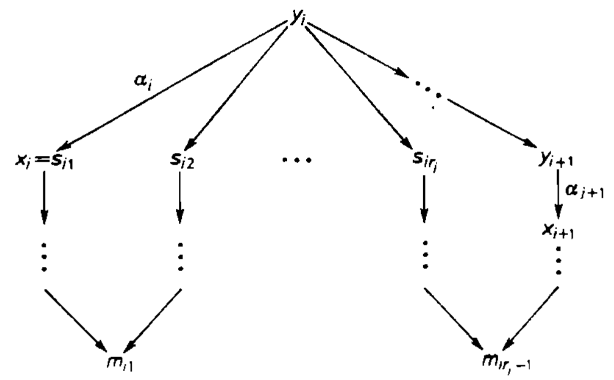

Fig. 7

$m_{ij} \Lambda y_i \neq 0$. If we choose a maximal sequence with these properties, then $J \cup \{y_m\}$ is convex in Q. Therefore, if $(v^0, e_{y_m})_\Lambda < 0$, we are done (case (b)).

Assume that $(v^0, e_{y_m})_\Lambda \geqslant 0$. For each $i = 1, \ldots, m$, let R_i be the unique indecomposable Λ'-module which is a direct summand of $\mathrm{Hom}_\Lambda(\Lambda', \mathrm{rad}\, P_{y_i})$ satisfying $R_i(x_i) \neq 0$. Then $\mathrm{Hom}_{\Lambda'}(R_{i+1}, R_i) \neq 0$ for $i = 0, \ldots, m-1$.

By construction, $\Lambda_m = \Lambda'[R_m]$ is a full convex subalgebra of Λ. Thus $(v^0, e_{y_m})_{\Lambda_m} \geqslant 0$ and by 2.5, R_m is not Λ'-preprojective. We conclude that R_0 is not Λ'-preprojective.

Let $e = \sum_{x \in Q_0 - (J \cup \{y_0\})} x$ and $\bar{\Lambda} = \Lambda / \Lambda e \Lambda$. Then Λ' is a full convex subalgebra of $\bar{\Lambda}$ and $\bar{\Lambda} = \Lambda'[\bar{R}_0]$ where $\bar{R}_0 = \mathrm{rad}\, \bar{P}_{y_0}$ and $\bar{P}_{y_0}$ is the indecomposable projective $\bar{\Lambda}$-module associated with y_0 (observe that $\bar{R}_0$ is possibly decomposable). Clearly,

$$0 > (v^0, e_{y_0})_\Lambda \geqslant (v^0, e_{y_0})_{\bar{\Lambda}}.$$

Therefore, by the argument given in 2.5, there is an indecomposable direct summand R' of $\bar{R}_0$ which is Λ'-preprojective. Since there is an arrow $y_0 \xrightarrow{\alpha'} x'$ with $R'(x') \neq 0$, we may start with $x_0 = x'$. But then we get $\mathrm{Hom}_{\Lambda'}(R_0, R') \neq 0$ and R' cannot be a preprojective Λ'-module. This contradiction finishes the proof. ■

2.8. The following consequence of 2.2 shows that in fact there is a close relation between weak nonnegativity of q_Λ and tameness of Λ.

PROPOSITION. *Let Λ be a good algebra. The following are equivalent:*

(a) *q_Λ is weakly nonnegative.*

(b) *For every full convex subalgebra Λ_0 of Λ such that Λ_0 is a one-point extension or a one-point coextension of a tame concealed algebra, Λ_0 is tame.*

Proof. (a) $\Rightarrow$ (b). Since Λ_0 is full convex in Λ, q_{Λ_0} is weakly nonnegative. By [20], Λ_0 is tame.

(b) $\Rightarrow$ (a). If q_Λ is not weakly nonnegative, by 2.2 there is a full convex subalgebra Λ_0 of Λ such that q_{Λ_0} is hypercritical. By 2.6, Λ_0 is a one-point extension or a one-point coextension of a tame concealed algebra and by [20], Λ_0 should be wild. ∎

2.9. For good algebras we get an improvement of [20, 1.3].

PROPOSITION. *Let Λ be a good algebra. Assume that* $\operatorname{ind}_d \Lambda$ *is parametrizable for every $d \leqslant 61$. Then q_Λ is weakly nonnegative.*

Proof. Assume that q_Λ is not weakly nonnegative. By 2.2 there is a full convex subalgebra Λ_0 of Λ such that q_{Λ_0} is hypercritical. It is enough to show that $\operatorname{ind}_d \Lambda_0$ is not parametrizable for some $d \leqslant 61$.

By 2.6 we may assume $\Lambda_0 = \Lambda'[R]$ where Λ' is tame concealed and R is an indecomposable preprojective Λ'-module. Let v be the positive generator of $\operatorname{rad} q_{\Lambda'}$ and let $y \in (Q_{\Lambda_0})_0$ be such that $R = \operatorname{rad} P_y$.

Let $(V_\lambda)_{\lambda \in k}$ be a family of indecomposable simple regular Λ'-modules such that $\underline{\dim}\, V_\lambda = v$ for every $\lambda \in k$ and $V_\lambda \not\cong V_\mu$ for $\lambda \neq \mu$. Then we may choose morphisms $0 \neq \varphi_\lambda \in \operatorname{Hom}_{\Lambda'}(R, V_\lambda)$, for every $\lambda \in k$.

The $\Lambda_0 = \Lambda'[R]$-modules can be written as triples (V, W, φ) where V is a Λ_0-module, W is a k-vector space and $\varphi\colon W \to \operatorname{Hom}_{\Lambda'}(R, V)$ is k-linear. For each couple $\lambda, \mu \in k$ we define the Λ_0-module

$$V_{\lambda\mu} = \big(V_\lambda \oplus V_\mu,\ k,\ (\varphi_\lambda, \varphi_\mu)\colon k \to \operatorname{Hom}_{\Lambda'}(R, V_\lambda) \times \operatorname{Hom}_{\Lambda'}(R, V_\mu)\big).$$

It is easy to show that $V_{\lambda\mu}$ is indecomposable and $V_{\lambda\mu} \not\cong V_{\lambda'\mu'}$ for $\{\lambda, \mu\} \neq \{\lambda', \mu'\}$. Therefore $\operatorname{ind}_d \Lambda_0$ is not parametrizable for

$$d = 2 \sum_{x \in (Q_{\Lambda'})_0} v(x) + 1.$$

If Λ' is tame concealed of type $\tilde{\mathbf{E}}_p$ ($p = 6, 7, 8$), then a simple calculation shows $d \leqslant 61$. If Λ' is tame concealed of type $\tilde{\mathbf{D}}_n$, using the fact that $q_{\Lambda'[R]}$ is hypercritical we get $n \leqslant 8$ and $d \leqslant 29$. ∎

3. Algebras with hypercritical Tits form

3.1. In this section we prove the following result.

THEOREM. *Let $\Lambda = kQ/N$ be a good algebra. The following are equivalent*:

(a) *q_Λ is hypercritical.*

(b) *Λ is wild, but for every vertex $y \in Q_0$, the quotient $\Lambda/\Lambda y \Lambda$ is tame concealed or representation-finite; if $\Lambda/\Lambda y \Lambda$ is tame concealed, then y is extremal in Q.*

(c) *Λ is concealed of a minimal wild hereditary algebra A, that is, $A = k\vec{\Delta}$ where the underlying graph Δ of $\vec{\Delta}$ is a hyperbolic diagram. Moreover, Δ is a tree.*

This theorem corresponds to that shown by Happel and Vossieck in the critical case. The assumption that Λ is good (or something similar) is necessary in our theorem as the following example of Kerner [17] shows:

Let $\Lambda = \Lambda_0[R]$ where Λ_0 is the path algebra given by the quiver of Fig. 8 and R is the indecomposable regular Λ_0-module of Fig. 9. Then Λ satisfies (b) but it is not concealed and q_Λ is weakly nonnegative.

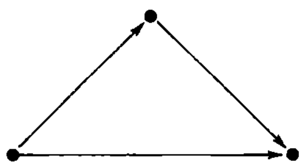

Fig. 8

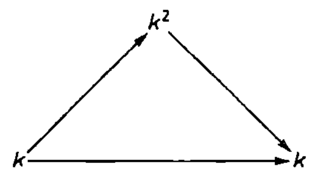

Fig. 9

3.2. An algebra $\Lambda = kQ/N$ is said to be *minimal wild* if Λ is wild but every quotient $\Lambda/\Lambda y\Lambda$ is tame ($y \in Q_0$). If $\Lambda = k\vec{\Delta}$ is a minimal wild hereditary algebra and Δ is a tree, then Δ is one of the diagrams of Fig. 10 (which are called *hyperbolic tree diagrams*; here, $\tilde{\tilde{\mathbf{D}}}_n$ has $n+2$ vertices and $n \leqslant 8$).

A good algebra Λ satisfying (a) (to (c)) will be called a *hypercritical algebra*. This differs from our previous terminology: in [20], a good algebra Λ satisfying (b) was called a *wild hyperbolic algebra* (equivalence with (c) would also justify this name).

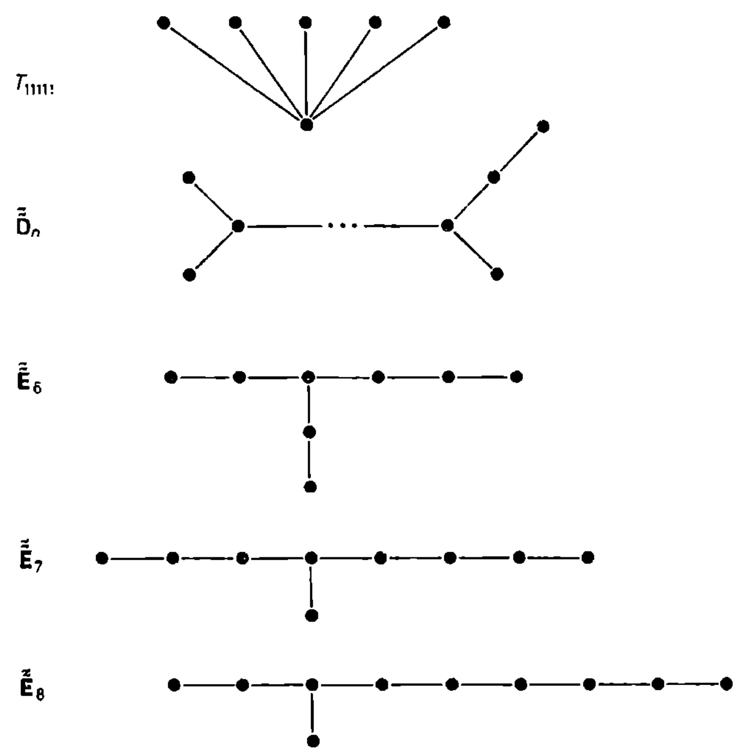

Fig. 10

The possible frames of hypercritical algebras (using the characterization (c)) have been classified independently by Lersch and Unger.

3.3. We need some elementary facts about concealed algebras for the proof of 3.1.

LEMMA. *Let Λ be a concealed algebra. Let e be any idempotent of Λ. Then $\Lambda/\Lambda e\Lambda$ is a product of representation-infinite concealed and representation-finite algebras.*

Proof. By [23, 4.3(6)] the Auslander–Reiten quiver Γ of $\Lambda/\Lambda e\Lambda$ has preprojective (resp. preinjective) components such that their union contains all indecomposable projective (resp. injective) $\Lambda/\Lambda e\Lambda$-modules. If Λ' is a representation-infinite factor of $\Lambda/\Lambda e\Lambda$, then $\Gamma_{\Lambda'}$ contains a preprojective component with all indecomposable projective Λ'-modules and a preinjective component with all indecomposable injective Λ'-modules. By [24, lecture 2], Λ' is concealed. ■

3.4. LEMMA. *Let Λ be a concealed algebra of a minimal wild hereditary algebra $A = k\vec{\Delta}$. If Λ is $\tilde{\mathbf{A}}$-free, then Δ is a tree.*

Proof. If Δ is not a tree, there is a vertex $a \in \Delta_0$ such that $\Delta - \{a\} = \tilde{\mathbf{A}}_m$ for some $m \geqslant 3$. By [13], there is a vertex $y \in Q_0$ such that $\Lambda/\Lambda y\Lambda$ is concealed of type $\tilde{\mathbf{A}}_m$. Then $\Lambda/\Lambda y\Lambda \xrightarrow{\sim} k\vec{\Delta}'$, where $\vec{\Delta}'$ is a quiver with underlying graph $\Delta' = \tilde{\mathbf{A}}_m$. It is easy to show that Λ is not $\tilde{\mathbf{A}}$-free. ■

3.5. The next lemma is a consequence of [22] (see also [17]). We include here a proof for the sake of completeness. Another proof could be done using APR-tilts [2].

LEMMA. *Let Λ be a concealed algebra of the hereditary algebra A. Then Λ is minimal wild if and only if A is minimal wild.*

Proof. Let ${}_AT$ be a preprojective tilting module such that $\Lambda = \operatorname{End}_A(T)$. Then we get a torsion pair $(\mathscr{G}, \mathscr{F})$ in $\operatorname{mod} A$ as follows:

$$\mathscr{G} = \{{}_AM\colon \operatorname{Ext}_A^1(T, M) = 0\}, \qquad \mathscr{F} = \{{}_AM\colon \operatorname{Hom}_A(T, M) = 0\},$$

and a torsion pair $(\mathscr{X}, \mathscr{Y})$ in $\operatorname{mod}\Lambda$ given by

$$\mathscr{X} = \{{}_\Lambda N\colon T \otimes_\Lambda N = 0\}, \qquad \mathscr{Y} = \{{}_\Lambda N\colon \operatorname{Tor}_1^\Lambda(T, N) = 0\}.$$

Moreover, there are equivalences

$$\Sigma = \operatorname{Hom}_A(T, -)\colon \mathscr{G} \xrightarrow{\sim} \mathscr{Y}, \qquad \operatorname{Ext}_A^1(\mathrm{T}, -)\colon \mathscr{F} \xrightarrow{\sim} \mathscr{X}.$$

Clearly, Λ is wild if and only if A is wild.

Assume that Λ is minimal wild and let $a \in (Q_A)_0$ be such that $\bar{A} = A/AaA$ is a (hereditary) wild algebra. We identify $\operatorname{mod}\bar{A}$ with the full subcategory of $\operatorname{mod} A$ with objects $\{x \in \operatorname{mod} A\colon \operatorname{Hom}_A(Aa, x) = 0\}$. There exists $n_1 \geqslant 0$ such that $\tau_A^{-n_1}(Aa) \in \mathscr{G}$ and $n_2 \geq 0$ such that $\Lambda y = \tau_A^{n_2} \Sigma \tau_A^{-n_1}(Aa)$ is indecomposable projective. By [22], there is a full exact embedding φ: $\mathscr{U} \to \mathrm{mod}\,\bar{\Lambda}$, where $\mathscr{U}$ is a wild bimodule category and the image of φ is contained in the regular part of $\mathrm{mod}\,\bar{\Lambda}$. Then

$$\tau_A^{n_2} \Sigma \tau_A^{-n_1} \varphi\colon\ \mathscr{U} \to \mathrm{mod}\,\Lambda/\Lambda y \Lambda$$

if a full exact embedding from $\mathscr{U}$ to $\Lambda/\Lambda y \Lambda$. Therefore $\Lambda/\Lambda y \Lambda$ is wild, a contradiction.

Assume now that $\bar{\Lambda} = \Lambda/\Lambda y \Lambda$ is wild. By 3.3, we may assume that $\bar{\Lambda}$ is concealed. Therefore there exists a hereditary wild algebra B such that the regular parts of $\mathrm{mod}\,\bar{\Lambda}$ and $\mathrm{mod}\,B$ are equivalent. Thus, there is a full exact embedding φ: $\mathscr{U} \to \mathrm{mod}\,\bar{\Lambda}$ where $\mathscr{U}$ is a wild bimodule category and the image of φ is contained in the regular part of $\mathrm{mod}\,\bar{\Lambda}$. The proof follows as in the converse case. ∎

3.6. *Proof of* 3.1. (a) ⇒ (b). Assume that q_Λ is hypercritical. By 2.6, there is a full convex subalgebra Λ_0 of Λ and a vertex $y \in Q_0$ such that $q_{\Lambda_0} = q_\Lambda^{(y)}$ is critical and if v is the positive generator of $\mathrm{rad}\,q_{\Lambda_0}$, then $(v^0, e_y)_\Lambda < 0$. Moreover, assume $\Lambda = \Lambda_0[R]$ with $R = \mathrm{rad}\,P_y$. We divide the proof in several steps.

(1) Λ is concealed. Since Λ satisfies the (S)-condition, Γ_Λ has a preprojective component $\mathscr{P}$. First we observe that $\mathscr{P}$ has no injective modules. Indeed, if $x \in (Q_{\Lambda_0})_0$ then $I_x^0 \subsetneq I_x$, where I_x (resp. I_x^0) is the indecomposable injective Λ-module (resp. Λ_0-module) associated to the vertex x. Since I_x^0 is not Λ_0-preprojective, $I_x \notin \mathscr{P}$. Clearly, $I_y \notin \mathscr{P}$.

Assume $0 \neq e = \sum_{P_x \notin \mathscr{P}} x$. Then $\bar{\Lambda} = \Lambda/\Lambda e \Lambda$ has $\mathscr{P}$ as a preprojective component of $\Gamma_{\bar{\Lambda}}$. Since $\bar{\Lambda}$ is a full convex subalgebra of Λ, $q_{\bar{\Lambda}}$ is a proper restriction of q_Λ and hence $q_{\bar{\Lambda}}$ is weakly nonnegative. By [21, 1.3] (see also [17]), $\bar{\Lambda}$ is a domestic tubular coextension of a tame concealed algebra Λ'. Let ω be the positive generator of $\mathrm{rad}\,q_{\Lambda'}$. Then, by 1.3, there exists a vertex $t \in Q_0$ such that $(\omega^0, e_t)_\Lambda < 0$ and the module $R' = \mathrm{rad}\,P_t$ is an indecomposable preprojective Λ'-module. But $\mathscr{P}$ is the preprojective component of Γ_Λ, therefore $R' \in \mathscr{P}$ yields a contradiction. We have shown that all indecomposable projective Λ-modules belong to $\mathscr{P}$.

Let $\mathscr{I}$ be a preinjective component of Γ_Λ. Since $\mathscr{I}$ has no projective modules, the proof above can be dualized to show that $\mathscr{I}$ contains all the indecomposable injective Λ-modules. By [24], Λ is a concealed algebra.

(2) Λ is minimal wild. Let $y \in Q_0$. By 3.3, $\Lambda/\Lambda y \Lambda$ is a product of concealed and representation-finite algebras. Let Λ' be a representation-infinite factor of $\Lambda/\Lambda y \Lambda$. Let X be an indecomposable Λ'-module.

If X is Λ'-preinjective, $q_{\Lambda'}(\underline{\dim}\,X) = 1$. If X is not Λ'-preinjective, then since Λ' is concealed, $\mathrm{pdim}_{\Lambda'} X \leq 1$. Therefore

$$q_{\Lambda'}(\underline{\dim}\,X) = \dim \mathrm{Hom}_{\Lambda'}(X, X) - \dim \mathrm{Ext}^1_{\Lambda'}(X, X).$$

Observe that X is not Λ-preinjective, thus $\operatorname{pdim}_\Lambda X \leqslant 1$ and

$$0 \leqslant q_\Lambda^{(y)}(\underline{\dim} X) = q_\Lambda(\underline{\dim} X)$$
$$= \dim \operatorname{Hom}_\Lambda(X, X) - \dim \operatorname{Ext}_\Lambda^1(X, X) = q_{\Lambda'}(\underline{\dim} X).$$

Since gldim $\Lambda' \leqslant 2$, [23, 2.3] implies that $q_{\Lambda'}$ is weakly nonnegative. Again [17, 21] say that Λ' is tame (concealed).

(3) Λ is a wild hyperbolic algebra. Let $x \in Q_0$ and assume that $\bar{\Lambda} = \Lambda/\Lambda x \Lambda$ is not representation-finite. We have to show that x is extremal and $\bar{\Lambda}$ is connected.

There exists an indecomposable $\bar{\Lambda}$-module X such that $q_{\bar{\Lambda}}$ $(\underline{\dim} X) = 0$. By (2), $q_\Lambda^{(x)}(\dim X) = 0$. Therefore, by 1.3, $q_\Lambda^{(x)}$ is critical. By [15] (using 2.4) we get $q_\Lambda^{(x)} = q_{\Lambda'}$ for some tame concealed algebra Λ'. By (1) in 2.7, Λ' is a full convex subalgebra of Λ. Hence, x is extremal in Q and $\bar{\Lambda} = \Lambda'$.

(b) $\Rightarrow$ (a). By 2.1(e) there exists a full convex subalgebra Λ_0 of Λ such that Λ_0 is tame concealed. By hypothesis, we may assume that $\Lambda = \Lambda_0[R]$ for some indecomposable Λ_0-module R. Since Λ is wild, q_Λ is not weakly nonnegative [20].

Let $z \in \mathbf{N}^{Q_0}$ be such that $z(y) = 0$. Then $\bar{\Lambda} = \Lambda/\Lambda y \Lambda$ is tame and by [20, 1.3], $q_{\bar{\Lambda}}$ is weakly nonnegative. By comparing the coefficients of $q_{\bar{\Lambda}}$ and $q_\Lambda^{(y)}$ we get

$$0 \leqslant q_{\bar{\Lambda}}(z) \leqslant q_\Lambda^{(y)}(z).$$

Therefore, q_Λ is hypercritical.

(a) $\Rightarrow$ (c). If q_Λ is hypercritical, then Λ is concealed and minimal wild. Then 3.5 implies that Λ is concealed of a minimal wild hereditary algebra A. If $A = k\vec{\Delta}$, 3.4 says that Δ is a tree.

(c) $\Rightarrow$ (a). Since Λ is wild concealed, q_Λ is not weakly nonnegative [17, 21]. By 3.5, each quotient $\Lambda/\Lambda y \Lambda$ is tame for $y \in Q_0$. Then the proof follows as in (b) $\Rightarrow$ (a). ∎

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