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ON A CLASS OF TAME SYMMETRIC ALGEBRAS HAVING ONLY PERIODIC MODULES

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Let K be an algebraically closed field, and let Λ be a finite-dimensional K-algebra which is connected, symmetric and tame and whose Cartan matrix is nonsingular. Assume that the stable Auslander–Reiten quiver of Λ consists only of tubes. In this paper it is proved that, if in addition the algebra Λ/J^3 is special biserial, then Λ has at most three simple modules.

In the process of classifying tame blocks in [2, 3, 4] we obtained various families of tame symmetric algebras having a nonsingular Cartan matrix. One of the important steps in that project was to derive bounds for the number of simple modules from the graph structure of the stable Auslander–Reiten quiver. The question arises whether such bounds might exist for more general classes of algebras.

Here we are interested in finite-dimensional K-algebras Λ, where K is an algebraically closed field, having the following properties:

(1) Λ is connected, symmetric, tame.
(2) The Cartan matrix of Λ is nonsingular.
(3) The stable Auslander–Reiten quiver of Λ consists only of tubes.

It has been suggested by A. Skowroński that such an algebra may always have only very few simple modules. In fact, in the special case when all tubes of Λ have rank $\leqslant 2$, Λ has at most three simple modules. This has been proved in [5].

Here we shall investigate the number of simple modules for another class of algebras, those where Λ/J^3 is special (see [10]), or a "string algebra".

This paper is in final form and no version of it will be submitted for publication elsewhere.

We shall assume throughout the paper that Λ is basic; let $\Lambda = KQ/I$ where Q is a quiver, and I is an admissible ideal. Denote by J the radical of Λ. Recall that Λ/J^3 is *special* if the following conditions are satisfied:

(0.1)(a) The number of arrows starting, respectively ending, at any fixed vertex of Q is bounded by 2.

(b) For any arrow α of Q there are at most one arrow β and at most one arrow γ such that $\alpha\beta$ and $\gamma\alpha$ do not lie in J^3.

Our aim here is to prove the following:

THEOREM. *Assume that Λ is an algebra with quiver Q satisfying conditions* (1) *to* (3). *Suppose also that*

(i) *For any indecomposable projective Λ-module P,* $(\operatorname{rad} P)/(\operatorname{soc} P)$ *is indecomposable.*

(ii) *Λ/J^3 is special.*

Then Λ has at most three simple modules.

The condition in (i) means that the projective modules are attached to the ends of the tubes. Condition (ii) is important for having such a small bound. We include an example due to A. Skowroński for an algebra satisfying (1) to (3) and (i), with four simple modules (see Chapter 4).

The methods of proof are largely combinatorical, exploiting the hypothesis (2). In fact, a nonsingular Cartan matrix seems to be a very strong property. The condition (3) is difficult to use directly if the rank of tubes is not necessarily bounded. However, we were able to make use of modules which cannot lie in tubes [see (3.8), (3.9)]. It might be of interest that modules of this kind occur for algebras of "semidihedral type", lying at ends of $\mathbf{Z}D_\infty$-components.

We note that one can also classify the algebras Λ satisfying the hypotheses of the Theorem. One discovers that they are algebras of quaternion type, as defined in [4]. In particular, the simple Λ-modules must lie in tubes of rank $\leqslant 2$. On the other hand, there are algebras of quaternion type where Λ/J^3 is not special (for example, family II in [4 II]).

If Λ satisfies the hypothesis of the Theorem then so does Λ^{op}; we shall use this occasionally. Concerning notation, Q_0 is the set of vertices of Q, and Q_1 is the set of arrows. If e is a vertex of Q we define $H(e\Lambda) := (\operatorname{rad} e\Lambda)/(\operatorname{soc} e\Lambda)$. We write $s^{-1}(e)$ [and $t^{-1}(e)$] for the set of arrows starting [terminating] at e.

The stable AR-quiver of Λ is denoted by $\Gamma_s(\Lambda)$. If M is a module then $|M|$ is its composition length. We write $l(\pi)$ for the length of a cyclic permutation.

We assume from now on that Λ is a basic algebra satisfying (1) such that Λ/J^3 is special.

Concerning basic facts and further terminology, we refer to [1, 6, 8].

1. K-bases

In this chapter, we study the Loewy series and the quiver of Λ.

(1.0)(a) Each path in the quiver gives rise to some element in the algebra which we also call "path". The paths from e to f span $e\Lambda f$. In fact, there is a K-basis of Λ consisting of paths which is compatible with the radical series of Λ; this is true for any finite-dimensional K-algebra. Recall that

(b) If Λ is symmetric then $\mathrm{soc}(e\Lambda)$ is simple and lies in $e\Lambda e$.

(1.1) LEMMA. *Let e be a vertex of Q. Then*

(a) *For all* $k \geqslant 1$, $\dim(eJ^k/eJ^{k+1}) \leqslant 2$.

(b) *If* $\dim(eJ^k/eJ^{k+1}) = 1$ *for some k then* $\dim(eJ^{k+1}/eJ^{k+2}) \leqslant 1$.

Proof. This follows immediately, using (0.1), if one constructs a K-basis as in (1.0).

(1.2) LEMMA. *For every vertex e of Q, the number of arrows starting at e is the same as the number of arrows ending at e, and this is* 1 *or* 2.

Proof. Assume that there is only arrow starting at e. Then, by (1.1), the module $e\Lambda$ is uniserial. Since $e\Lambda$ is also the injective hull for the simple module S_e, there is only one arrow ending at e. Dually, if there is only one arrow ending at e then there is also only one arrow starting at e. Now condition (a) of (0.1) ensures that there is only one alternative.

(1.3) *A remark on the socle series.* Assume that there are two arrows starting at e. Then, by (1.1), there is an integer $t > 1$ such that for $1 \leqslant i < t$, the module eJ^i/eJ^{i+1} has length 2 but $|eJ^t/eJ^{t+1}| \leqslant 1$. Since Λ is symmetric, $\mathrm{soc}(e\Lambda) = eJ^{t+r}$ for some $r \geqslant 0$; and by (1.2) we know that $\mathrm{soc}_2(e\Lambda)$ has length 3. If $r = 0$ then $\mathrm{soc}_2(e\Lambda) = eJ^{t-1}$, and if $r > 0$ then there is some $w \in eJ^{t-1} - eJ^t$ such that $\mathrm{soc}_2(e\Lambda) = eJ^{t+r-1} + w\Lambda$.

2. The permutation associated to Λ

Assume that Λ satisfies condition (i) of the Theorem.

(2.1) DEFINITION. Define a map π: $Q_1 \to Q_1$ as follows:

$$\pi(\gamma) = \begin{cases} \delta & \text{if } \gamma\delta \notin J^3, \\ \gamma & \text{if } \gamma\delta \in J^3 \text{ for all arrows } \delta. \end{cases}$$

This is a map, by condition (b) of (0.1).

(2.2) PROPOSITION. *The map π is a permutation.*

Proof. We shall prove that π is surjective. Assume for contradiction that $e \xrightarrow{\delta} f$ is an arrow in Q such that $\gamma\delta$ lies in J^3 for all arrows γ. Then $J\delta \subseteq J^3 f$.

(1) *There is another arrow ending at* f: Otherwise, we would have $J^3 f = J^2\delta$ and then $J\delta \subseteq J^2\delta$ and $J\delta = 0$. This would imply that δ lies in $\operatorname{soc}\Lambda$; moreover, Λf would have length 2. Since Λ is connected and symmetric, Λ would be the local 2-dimensional algebra which is of finite type.

Let η be the other arrow ending at f. Then $J^2 f = J\eta$. Since Λ/J^3 is special it follows that

(2) $|J^2 f/J^3 f| \leqslant 1$.

Note also that, by (1.2), $|\operatorname{soc}_2(\Lambda f)/\operatorname{soc}(\Lambda f)| = 2$.

We now have $|Jf/J^2 f| = 2$ but $|J^2 f/J^3 f| \leqslant 1$. By (1.3) applied to the algebra Λ^{op}, there is an element $\omega \in Jf\backslash J^2 f$ which lies in $\operatorname{soc}_2(\Lambda f)$. Hence $H(\Lambda f)$ is a direct sum, a contradiction.

(2.3) We express π as a product of disjoint cycles. Each of these cycles corresponds to a closed path in Q. Take such a cycle, $\sigma = (\alpha_0, \ldots, \alpha_n)$ say; then $t(\alpha_j) = s(\alpha_{j+1})$ for all j. We say that *the cycle goes through e*, or *e occurs in* σ, if $e = t(\alpha_j)$ for some arrow α_j in σ.

In order to have control over the Cartan matrix we need "multiplicities" of the cycles of π.

(2.4) LEMMA. *Let* $\sigma = (\alpha_0, \ldots, \alpha_r)$ *be a cycle of* π *where* $\alpha_i\colon e_i \to e_{i+1}$. *Define elements* $w_i \in \Lambda$ *by* $w_i := (\alpha_i \alpha_{i+1} \ldots \alpha_r \alpha_0 \ldots \alpha_{i-1})$. *Let m be the largest integer such that* $w_0^m \neq 0$. *Then* $\langle w_i^m \rangle = \operatorname{soc}(e_i\Lambda)$, *for all i*.

Thus m depends only on σ; we say that m is the *multiplicity* of σ.

Proof. Assume first that $i = 0$. Let W be the set of all paths around the cycle σ starting with α_0. Then the last nonzero $w \in W$ spans $\operatorname{soc}(e_0\Lambda)$ (see e.g. (1.1)), and $w = we_0$ [(1.0)(b)].

Suppose that $w \neq w_0^m$. Since $w_0^m \neq 0$ but $w_0^{m+1} = 0$ it follows that $w = w_0^m x$ where $x = \alpha_0 \ldots \alpha_s$ and $s < r$. Let $y = \alpha_{s+1} \ldots \alpha_r$; then $w = (xy)^m x$. We may assume that $s+1 \geqslant r-s$. [Otherwise, we interchange x and y. Note that $(yx)^m y$ also spans $\operatorname{soc}(e_0\Lambda)$.] Then $\operatorname{soc}(e_0\Lambda) = e_0 J^k$ where $k = m(r+1)+(s+1)$. Let λ be a symmetrizing bilinear form of Λ; then $0 \neq \lambda[w] = \lambda[x(xy)^m]$. On the other hand, $x(xy)^m \in e_0 J^{k+1} = 0$, a contradiction.

This shows that $\langle w_0^m \rangle = \operatorname{soc}(e_0\Lambda)$. Now we also have $w_i^m \neq 0$ and $w_i^{m+1} = 0$ [factorize w_0 and w_i, and use λ]. Then the arguments for the case $i = 0$ apply, and the statement follows.

(2.5) *A basis for* $e\Lambda$. (a) If there is only one arrow, α_0 say, starting at e then there is a unique cycle through e, call this $\sigma = (\alpha_0, \ldots, \alpha_r)$; and necessarily $\alpha_i e = 0$ for $i \neq r$. Thus if m is the multiplicity of σ then $e\Lambda$ has a K-basis

$$\{e, \alpha_0, \alpha_0\alpha_1, \ldots, (\alpha_0\alpha_1 \ldots \alpha_r)\alpha_0, \ldots, (\alpha_0\alpha_1 \ldots \alpha_r)^m\}.$$

(b) Now assume that there are two arrows, α_0 and β_0 say, starting at e. Let $\sigma = (\alpha_0, \ldots, \alpha_r)$ and $\sigma' = (\beta_0, \ldots, \beta_s)$ be the cycles through e. [We do not exclude here that $\sigma' = \sigma$.] Then if m and m' are the multiplicities of σ and σ' respectively then $e\Lambda$ has a K-basis

$$\{e, \alpha_0, \alpha_0\alpha_1, \ldots, (\alpha_0\alpha_1\ldots\alpha_r)^m, \beta_0, \beta_0\beta_1, \ldots, (\beta_0\ldots\beta_s)^{m'-1}\beta_0\ldots\beta_{s-1}\}.$$

3. On the paths of Q determined by π

We assume in the following that Λ is an algebra satisfying conditions (1) and (ii) of the Theorem. We also assume that the map π defined in (2.1) is a permutation.

If e is a vertex of Q then we know that $|s^{-1}(e)| = |t^{-1}(e)| = 1$ or 2, by (1.2).

(3.1) LEMMA. *Assume that e and f are distinct vertices of Q such that $|s^{-1}(e)| = 2$ and $|s^{-1}(f)| = 2$. Suppose that all arrows starting or ending at e or f lie in the same cycle of π. Then for all primitive idempotents g of Λ, $\dim(e\Lambda g) = \dim(f\Lambda g)$, hence the Cartan matrix of Λ is singular.*

Proof. With the notation of (2.5), we have $\sigma = \sigma'$ and $\beta_0 = \alpha_s$ for some s with $0 < s \leqslant r$; then $\beta_i = \alpha_{s+i}$. Using the K-basis given in (2.5)(b) we obtain $\dim(e\Lambda g) = 2m \,\#\{i\colon \alpha_i g = \alpha_i\}$, even for $g = e$. Now, this does not depend on e but only on the fact that all arrows at e lie in σ, and that $|s^{-1}(e)| = 2$. These conditions are also satisfied by f, and we deduce that $\dim(f\Lambda g)$ is the same.

(3.2) LEMMA. *Assume that $e, f \in Q_0$ where $e \neq f$, with $|s^{-1}(e)| = |s^{-1}(f)| = 2$. Suppose that π has two distinct cycles which both go through e and f. Then $\dim(e\Lambda g) = \dim(f\Lambda g)$, for all primitive idempotents g of Λ, hence the Cartan matrix of Λ is singular.*

The proof of (3.2) is similar to that of (3.1) and is omitted.

(3.3) LEMMA. *Let e be a vertex of Q such that $|s^{-1}(e)| = 2$. Assume that*

(a) *The arrows at e lie in two cycles π_0, π_1 of π.*

(b) *If $t(\alpha) = e$ and $\pi(\alpha) \neq \beta$ then $\alpha\beta = 0$.*

Then $H(e\Lambda)$ is a direct sum.

Proof. Let $\pi_0 = (\alpha_0, \ldots, \alpha_n)$ and $\pi_1 = (\beta_0, \ldots, \beta_r)$ where α_0 and β_0 are the arrows starting at e. Then $\mathrm{rad}(e\Lambda) = \alpha_0\Lambda + \beta_0\Lambda$, and we have to show that $\alpha_0\Lambda \cap \beta_0\Lambda \subset \mathrm{soc}(e\Lambda)$.

Now by the hypothesis, $\alpha_n\beta_0 = 0$ and also $\beta_r\alpha_0 = 0$. Let $x \in \alpha_0\Lambda \cap \beta_0\Lambda$; then $x = \alpha_0 y = \beta_0 z$. Consequently $Jx = \langle \alpha_n x, \beta_r x\rangle = \langle \alpha_n\beta_0 z, \beta_r\alpha_0 y\rangle = 0$, therefore $x \in \mathrm{soc}\Lambda \cap e\Lambda \subset \mathrm{soc}(e\Lambda)$, as required.

(3.3.1) The hypotheses of (3.3) are satisfied for the quiver Q of Fig. 1 and $\pi = (\alpha_0\alpha_1)(\beta_0\alpha_1)\ldots$

(3.4) LEMMA. *Let Q be a quiver containing the one in Fig. 2. Assume that*

(a) *The arrows at e form cycles $\pi_0 = (\alpha)$, $\pi_1 = (\beta_r, \beta_0, \ldots)$.*

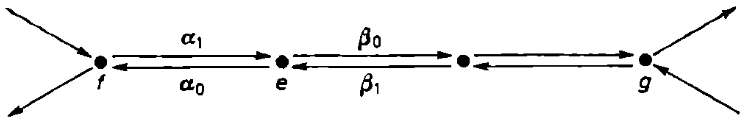

Fig. 1

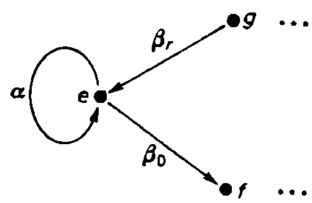

Fig. 2

(b) π_1 *does not go twice through* g *or* f *(thus* $g \neq f$*).*

Then $H(e\Lambda)$ *is a direct sum.*

Proof. It suffices to consider the case when π_1 does not go twice through f. We wish to apply (3.3).

(1) *We may assume that* $\alpha\beta_0 = 0$: Using the K-basis in (2.5) one gets an expression $\alpha\beta_0 = \sum c_i(\beta_0 \ldots \beta_r)^i \beta_0$ where $c_i \in K$. This holds since β_0 is the only arrow in π_1 ending at f. Set $\alpha' = \alpha - \sum c_i(\beta_0 \ldots \beta_r)^i$, and replace α by α'.

(2) $\beta_r \alpha = 0$: We use the K-basis in (2.5) for $g\Lambda$. Thus there are a_i $[b_i]$ in K such that

$$\beta_r \alpha = \sum a_i(\beta_r \beta_0 \ldots \beta_{r-1})^i \beta_r \quad [+\sum b_j(\beta_s \ldots \beta_{s-1})^j \beta_s \ldots \beta_r].$$

Here the second sum appears only in the case when π_1 goes twice through g; then β_s is the other arrow starting at g.

Since $\beta_r \alpha \beta_0 = 0$ it follows that $a_i = 0$ $[= b_j]$, for all i, j. (Note that a nonzero element $\beta_s \ldots \beta_r \beta_0$ does not lie in $\operatorname{soc} \Lambda$.) Now the statement follows from (3.3).

(3.5) LEMMA. *Assume that* $e \in Q_0$ *is a fixed vertex and that* $f \in Q_0$. *Then* $e\Lambda f \neq 0$ *if and only if* π *has a cycle* σ *such that* e *and* f *both occur in* σ.

This follows immediately from (2.5).

(3.6) LEMMA. *Assume that* Q *contains a quiver as in Fig. 3 such that*

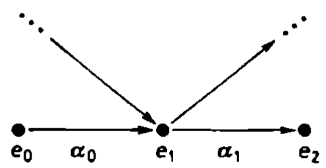

Fig. 3

$|s^{-1}(e_0)| = |t^{-1}(e_2)| = 1$ *and that* $\pi(\alpha_0) \neq \alpha_1$. *Then we may assume that* $\alpha_0\alpha_1 = 0$.

Proof. We have $\alpha_0\alpha_1 \in J^3$ since $\pi(\alpha_0) \neq \alpha_1$. The hypothesis on the number of arrows starting at e_0 and ending at e_2 implies that $e_0 J^3 e_2 = \alpha_0 J \alpha_1$, consequently $\alpha_0\alpha_1 = \alpha_0 x \alpha_1$ for some $x \in J$. Set $\alpha_0' = \alpha_0(1-x)$ and replace α_0 by α_0'.

(3.7) *Remark.* Let $e = \sum e_i$ where the sum is taken over the distinct idempotents occurring in the cycle π_0 of π, and define Λ_e to be the algebra $e\Lambda e$. Assume that e_i is a vertex such that all arrows (in Q) at e_i lie in π_0. Then $H(e_i\Lambda_e)$ is indecomposable if and only if $H(e_i\Lambda)$ is indecomposable.

(3.8) LEMMA. *Assume the quiver of* Λ *has the form shown in Fig.* 4 *and that*

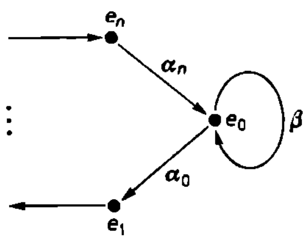

Fig. 4

Λ *is defined by the relations*

(i) $\alpha_n\alpha_0 = 0$,

(ii) $(\alpha_0 \ldots \alpha_n\beta)^k J = 0$,

(iii) $\beta^2 = (\alpha_0 \ldots \alpha_n\beta)^{k-1}\alpha_0 \ldots \alpha_n + d(\alpha_0 \ldots \alpha_n\beta)^k$,

(n, $k \geqslant 1$ *and* $d \in K$). *Then the stable quiver* $\Gamma_s(\Lambda)$ *has a component which is not a tube.*

Proof. Let $H = H(e_0\Lambda)$; then there is an AR-sequence

$$0 \to \mathrm{rad}(e_0\Lambda) \to e_0\Lambda \oplus H \to e_0\Lambda/\mathrm{soc}(e_0\Lambda) \to 0.$$

Moreover, H is indecomposable; this can be seen from the K-basis in (2.5); and therefore H lies at the end of a component. Define $U := \beta\Lambda/\beta\Lambda \cap \alpha_0\Lambda$ and $V := \alpha_0\Lambda/\mathrm{soc}(e_0\Lambda)$. Thus there is a short exact sequence

$$0 \to V \to H \to U \to 0.$$

We shall prove that this must be an AR-sequence; thus the component of $\Gamma_s(\Lambda)$ is not a tube.

(1) $\Omega U \cong \beta\Lambda$: We define a projective cover $\bar{\pi}$: $e_0\Lambda \to U$ by setting $\bar{\pi}(x) = \beta x + (\alpha_0\Lambda \cap \beta\Lambda)$. Then $\beta\Lambda \subseteq \mathrm{Ker}\,\bar{\pi}$. Using (2.5) [and the relation (iii)] we can calculate dimensions and see that equality must hold.

(2) $\Omega(\beta\Lambda) \cong (\beta\alpha_0)\Lambda$: We define a projective cover π: $e_0\Lambda \to \beta\Lambda$ by setting

$\pi(x) = \beta x$. By (i)–(iii) we have $\beta\alpha_0 \Lambda \subset \mathrm{Ker}\,\pi$, and equality follows from comparing dimensions.

(3) $(\beta\alpha_0)\Lambda \cong V$: Left multiplication by β induces an epimorphism $\alpha_0\Lambda \to (\beta\alpha_0)\Lambda$ whose kernel is $\mathrm{soc}(e_0\Lambda)$.

Hence $\Omega^2(U) \cong V$, and there is an AR-sequence

$$(*) \qquad 0 \to V \to X \xrightarrow{\varkappa} U \to 0.$$

We have to show that $H \cong X$.

Clearly X and H have the same composition factors, and there is a nonsplit epimorphism $p\colon H \to U$. Since $(*)$ is an AR-sequence, we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\varkappa"] & U \arrow[r] & 0 \\ & H \arrow[ul, dashed, "\psi"] \arrow[u, "p"'] & \end{tikzcd}$$

We claim that ψ must be an isomorphism. It suffices to show that ψ is one-to-one or just that $\mathrm{Ker}\,\psi \cap \mathrm{soc}\,H = 0$. If $\mathrm{Ker}\,\psi \cap \mathrm{soc}\,H$ were $\neq 0$ then ψ would map $\mathrm{soc}_2(\alpha_0\Lambda)/\mathrm{soc}(\alpha_0\Lambda)$ to zero. But then, from the structure of H [using (iii)] it would follow that $\mathrm{Im}\,\psi \cong U \oplus V'$ where V' is some quotient of V; this would imply that $\varkappa$ splits.

(3.9) LEMMA. *Assume that the quiver of Λ is as shown in Fig. 5 and that Λ is*

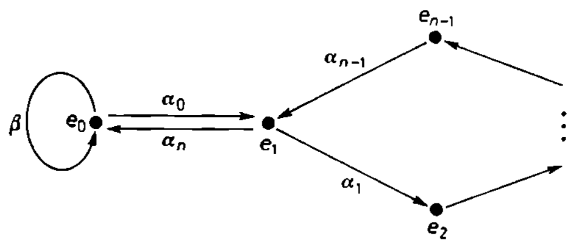

Fig. 5

defined by the relations

(i) $\alpha_{n-1}\alpha_1 = 0$,
(ii) $\beta^s = (\alpha_0 \ldots \alpha_n)^m$ *and* $\beta^s J = 0$,
(iii) $\beta\alpha_0 = (\alpha_0 \ldots \alpha_n)^{m-1}\alpha_0 \ldots \alpha_{n-1}$ *and* $\alpha_0\alpha_n = \beta^{s-1} + d\beta^s$,
(iv) $\alpha_n\beta = (\alpha_1 \ldots \alpha_n\alpha_0)^{m-1}\alpha_1 \ldots \alpha_n$

($m \geqslant 1$, $n > 3$, $s \geqslant 4$ and $d \in K$). Then the stable quiver $\Gamma_s(\Lambda)$ has a component which is not a tube.

The proof is the same as that of (3.8), taking $H = H(e_1\Lambda)$, with $U = \alpha_n\Lambda/\alpha_n\Lambda \cap \alpha_1\Lambda$ and $V = \alpha_1\Lambda/\mathrm{soc}(e_1\Lambda)$. [Here $\Omega U \cong \beta\Lambda$ and $\Omega(\beta\Lambda) \cong (\alpha_0\alpha_1\Lambda) \cong V$.]

In preparation for the proof of the Theorem, we shall study algebras Λ with (1) and (2) having the following properties:

(3.10)(a) The quiver Q of Λ contains the one illustrated in Fig. 6.

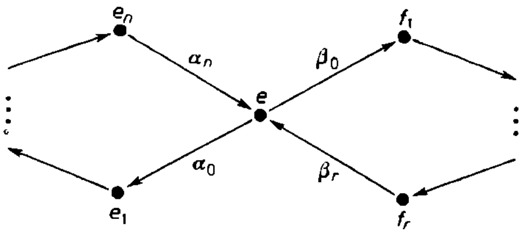

Fig. 6

(b) The associated permutation π has cycles $\pi_0 = (\alpha_n, \alpha_0, \ldots)$ and $\pi_1 = (\beta_0, \ldots, \beta_r)$ which intersect only at e.

(c) Any two vertices $\in \{e, e_n, e_1, f_1, f_r\}$ are distinct except that $e_n = e_1$ when $l(\pi_0) = 2$, or that $f_r = f_1$ when $l(\pi_1) = 2$.

(3.11) LEMMA. *Assume Λ satisfies* (3.10). *Suppose also that $\beta_r \alpha_0 \neq 0$. Then there is an arrow $e_1 \xrightarrow{\gamma} f_r$.*

Proof. Suppose $\beta_r \alpha_0 \neq 0$. By (3.5), there is an arrow γ starting at e_1 which lies in a cycle, π_2 say, through e_1 and f_r. Let $g = t(\gamma)$; we have to show that $g = f_r$.

Note that π_2 is not a loop. By (3.2), π_2 does not go through any other vertex of π_0 or π_1. Suppose $g \neq f_r$; then g does not occur in π_0 or π_1 and therefore $\alpha_0 \gamma = 0$, by (3.5). Using the basis for $f_r \Lambda$ as in (2.5) we find that if $\pi_2 = (\gamma_0 \gamma_1 \cdots \gamma_s \gamma_{s+1} \cdots \gamma_w)$ where $s(\gamma_0) = f_r$ and $\gamma_s = \gamma$ for $s < w$, then $\beta_r \alpha_0 = \sum c_i (\gamma_0 \cdots \gamma_w)^i \gamma_0 \cdots \gamma_{s-1}$ for unique $c_i \in K$. Now $0 = \beta_r \alpha_0 \gamma$, hence all c_i are 0 [using (2.5)] and therefore $\beta_r \alpha_0 = 0$, a contradiction.

(3.12) LEMMA. *Let Λ be an algebra satisfying* (3.10). *Assume that there is an arrow $e_1 \xrightarrow{\gamma} f_r$. If the quiver has more than* 3 *vertices then $\alpha_0 \gamma$ lies in* $\operatorname{soc}_2 \Lambda$.

Proof. We shall prove that $J^2(\alpha_0 \gamma) = 0$.

Let $\pi_2 = (\gamma, \ldots)$ be the cycle of π containing γ. Then by (3.2), π_2 intersects π_0, π_1 only in e_1, f_r respectively. In particular, no other cycles go through e, e_1 or f_r. Note also that, by (2.5),

$$(*) \qquad \alpha_0 \gamma \in e \Lambda f_r \subset \beta_0 \Lambda.$$

Case 1. $e_1 \neq e_n$ and $f_1 \neq f_r$: There is no cycle through e_n and f_r, so $\alpha_n \alpha_0 \gamma = 0$ [see (3.5)], and

$$J^2(\alpha_0 \gamma) = J \beta_r \alpha_0 \gamma = \langle \beta_{r-1} \beta_r \alpha_0 \gamma, \gamma \beta_r \alpha_0 \gamma \rangle.$$

There is no cycle through f_{r-1} and e_1, hence $\beta_{r-1} \beta_r \alpha_0 = 0$. Moreover,

$\gamma\beta_r\beta_0 = 0$ since there is no cycle through e_1 and f_1. Hence, using (∗), we see that $\gamma\beta_r\alpha_0\gamma \in \gamma\beta_r\beta_0 \Lambda = 0$.

Case 2. $e_1 = e_n$ but $f_1 \neq f_r$: Then, by (3.10) (c), π_0 is a 2-cycle. By (∗), $\alpha_1\alpha_0\gamma \in \alpha_1\beta_0\Lambda$. There is no cycle through e_1 and f_1, therefore $\alpha_1\beta_0 = 0$. Hence

$$J^2\alpha_0\gamma = J\beta_r\alpha_0\gamma = \langle \beta_{r-1}\beta_r\alpha_0\gamma,\ \gamma\beta_r\alpha_0\gamma \rangle.$$

The first generator is 0, as above, and $\gamma\beta_r\alpha_0\gamma$ lies in $\gamma\beta_r\beta_0\Lambda \subset e_1\Lambda f_1\Lambda = 0$.

Case 3. $e_1 = e_n$ and $f_1 = f_r$: Then π_0 and π_1 are both 2-cycles. Now, if Λ has more than 3 simple modules then, by (1.2), π_2 is not a 2-cycle, and there is no arrow $f_1 \to e_1$. Thus, by (3.11) [applied to α_1, β_0] we have $\alpha_1\beta_0 = 0$. There is an expression

$$\alpha_0\gamma = \sum c_i(\beta_0\beta_1)^i\beta_0 \qquad (c_i \in K).$$

Hence $\alpha_1\alpha_0\gamma \in \alpha_1\beta_0\Lambda = 0$ and therefore

$$J^2\alpha_0\gamma = J(\beta_1\alpha_0\gamma) = \langle (\beta_0\beta_1\alpha_0\gamma),\ (\gamma\beta_1\alpha_0\gamma) \rangle.$$

(i) We have $\beta_0\beta_1\alpha_0 \in eJ^3e_1$. Hence there are $r_i \in K$ such that $\beta_0\beta_1\alpha_0 = [\sum r_j(\alpha_0\alpha_1)^j]\alpha_0$. Then

$$\beta_0\beta_1\alpha_0\gamma = [\sum r_j(\alpha_0\alpha_1)^j][\sum c_i(\beta_0\beta_1)^i\beta_0] \in \Lambda\alpha_1\beta_0\Lambda = 0.$$

(ii) We have $\gamma\beta_1 \in e_1\Lambda e \subset \Lambda\alpha_1$, by (2.5); consequently $\gamma\beta_1\alpha_0\gamma \in \Lambda\alpha_1\beta_0\Lambda = 0$.

4. Proof of the Theorem

Let Λ be an algebra satisfying the conditions (1) to (3) and also (i) and (ii) of the Theorem.

(1) *We may assume that* $|s^{-1}(e)| = 2$ *for some vertex* e *of* Q: Suppose not, then by (1.2), $|s^{-1}(e)| = |t^{-1}(e)| = 1$ for all $e \in Q_0$. Since Λ is symmetric (and connected), any two vertices of Q must be joined by some path. Hence Q is of the form $\tilde{A}_n$, and all arrows have the same orientation. Consequently Λ is of finite type, a contradiction.

(2) π *must have at least two cycles*: Assume for contradiction that π is just one cycle. Then π goes twice through any vertex e with $|s^{-1}(e)| = 2$. By (3.1) [and (1)], there is therefore a unique such vertex e.

Assume first that there is no loop attached to e. Then Q contains a subquiver of Fig. 7 and $\pi = (\alpha_n\alpha_0 \ldots, \beta_m\beta_0 \ldots)$. Now e_n, e_1, f_1 and f_m are all

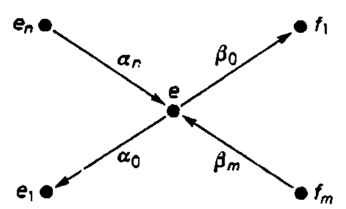

Fig. 7

distinct from e. By the uniqueness of e, we have therefore $|s^{-1}(e_n)| = |s^{-1}(f_1)| = 1$; moreover, $\pi(\alpha_n) \neq \beta_0$. Hence, by (3.6), we may assume that $\alpha_n\beta_0 = 0$. Similarly, without loss of generality, $\beta_m\alpha_0 = 0$. [This does not affect the other zero relation.] Therefore the algebra Λ itself is special, and $H(e\Lambda)$ is a direct sum, a contradiction.

Now assume that there is a loop attached to the vertex e. If e has two loops then e is the only vertex of Q, and there is nothing to do. So we may assume that the quiver is as shown in Fig. 8 and $\pi = (\alpha_n\beta\alpha_0\ldots)$, with $e_n \neq e_0$ and $e_1 \neq e_0$. Our aim is to show that Λ satisfies the relations in (3.8).

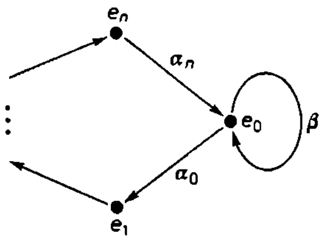

Fig. 8

By (3.6), we can take α_n, α_0 satisfying $\alpha_n\alpha_0 = 0$. Moreover, if k is the multiplicity of π then the relation (ii) in (3.8) holds. It remains to determine the relation for β^2.

(2.a) *Without loss of generality, β^2 lies in $e_0\Lambda\alpha_n$*: Since $\beta^2 \in J^3$, we may write, using (2.5), $\beta^2 = x\alpha_n + y\alpha_n\beta$ with $x, y \in J$. Set $\beta' := \beta - y\alpha_n$ and replace β by β'.

(2.b) *β^2 lies in* $\operatorname{soc}_2(\alpha_0\Lambda)$: We deduce from (2.a) that $\beta^2\alpha_0 = 0$. By (2.a) and (2.5) there are c_i, $d_i \in K$ such that

$$(*) \qquad \beta^2 = \sum c_i(\alpha_0\ldots\alpha_n\beta)^i\alpha_0\ldots\alpha_n + \sum d_i(\beta\alpha_0\ldots\alpha_n)^i.$$

Hence $\beta^3 = \sum c_i(\beta\alpha_0\ldots\alpha_n)^{i+1} = \sum c_i(\alpha_0\ldots\alpha_n\beta)^{i+1} + \sum d_i(\beta\alpha_0\ldots\alpha_n)^i\beta$. Using the K-basis in (2.5) it follows that all c_i and d_i are 0 except possibly c_{k-1} and d_k; this proves (2.b).

The element c_{k-1} in $(*)$ is $\neq 0$; otherwise, $H(e_0\Lambda)$ would be a direct sum. Then we may assume that $c_{k-1} = 1$, and we obtain the relation (iii) in (3.8).

Now, by (3.8), we have a contradiction to the hypothesis. This completes the proof of (2).

Thus π must have at least two cycles; and then there is a vertex of Q which occurs in two distinct cycles.

Quivers having two intersecting cycles which are not loops. Suppose Q is such a quiver. Then Q contains a subquiver as in (3.10). We shall now see that the conditions there hold.

The cycles π_0, π_1 intersect only once, by (3.2). Moreover,

(3) *$e_1 \neq f_1$ and $e_n \neq f_r$, also $e_1 \neq f_r$ and $e_n \neq f_1$*: Suppose $e_1 = f_1$. Then we see from (3.2) that $e_1\Lambda$ and $e\Lambda$ have the same composition factors.

(4) *If* $l(\pi_0) \geqslant 3$ *then* $s(\alpha_n)$ *and* $t(\alpha_0)$ *are distinct. If* $l(\pi_1) \geqslant 3$ *then* $s(\beta_r) \neq t(\beta_0)$: Suppose that $e_n = e_1$. By the hypothesis, all arrows at e_1 must lie in π_0. Hence there is no cycle through e_1 and $t(\beta_0)$ or $t(\beta_r)$. By (3.5), $\alpha_n \beta_0 = 0$ and $\beta_r \alpha_0 = 0$. Now we deduce from (3.3) that $H(e_0 \Lambda)$ is a direct sum, a contradiction.

Hence the hypotheses of (3.10) hold, and we have the results from (3.11) and (3.12) available.

We will now show that $H(e\Lambda)$ is close to being a direct sum. Let $A := \{\alpha_0, \alpha_0\alpha_1, \ldots, \omega_0^s\}$ and $B := \{\beta_0, \beta_0\beta_1, \ldots, \omega_1^m\}$ where ω_0 and ω_1 are the products over all arrows in π_0, π_1 starting with α_0, β_0 respectively, and s, m are the multiplicities of π_0 and π_1.

Let δ be any arrow.

(5) *If* $x \in A$, *then* $x\delta \in A$ *or* $x\delta$ *is zero, unless* $x = \alpha_0$ *and* δ *is an arrow* $e_1 \to f_r$: We have $x = xe_i$ for a unique primitive idempotent e_i, and we may take δ starting at e_i, and δ not occurring in π_0. Let $g = t(\delta)$; then g does not occur in π_0, by (3.1). If g does not occur in π_1 either, then $e\Lambda g = 0$, by (3.5), and $x\delta = 0$. Hence we may assume that $g = f_j$ for some j but $f_j \neq e$.

Consider first the case when $x = \alpha_0$ and $f_j \neq f_r$. Then there is no arrow $f_j \to e$. Applying (3.11) [with α_0, δ instead of β_r, α_0] shows that $\alpha_0\delta = 0$.

Now assume that x has length $\geqslant 2$; write $x\delta = x_1\alpha_{i-1}\delta$ for some $x_1 \in A$. If $\alpha_{i-1}\delta = 0$ then we are done. Otherwise, there is an arrow, γ say, from f_j [$= t(\delta)$] to $s(\alpha_{i-1}) = e_{i-1}$, by (3.11).

Suppose $\gamma \neq \beta_j$; then π has a cycle $(\delta, \gamma, \ldots)$ [$\neq \pi_0$] which intersects π_0 in e_{i-1} and in e_i. By (3.2) we can only have $e_i = e_{i-1}$, that is, α_{i-1} is a loop. But then $|s^{-1}(e_i)| \geqslant 3$, a contradiction to (1.2).

It follows that $\gamma = \beta_j$ and hence $\beta_j = \beta_r$, $f_j = f_r$ and $\alpha_{i-1} = \alpha_0$. Now, x has length $\geqslant 2$, hence $x\delta = (\alpha_0 \ldots \alpha_n)^k \alpha_0 \delta$ and $\alpha_0\delta$ lies in $J^2\mathrm{soc}_2(\Lambda)$, by (3.12). Hence $x\delta = 0$, as required. Similarly

(5*) *If* $x \in B$ *then* $x\delta \in B$, *or* $x\delta = 0$, *unless* $x = \beta_0$ *and* δ *is an arrow* $f_1 \to e_n$.

Hence there must be an arrow $e_1 \to f_r$, or an arrow $f_1 \to e_n$.

(6) *The structure of* $H(e\Lambda)$ *in case there is an arrow* $e_1 \xrightarrow{\gamma} f_r$ *but no arrow* $f_1 \to e_n$: In (5), (5*) [and (3.11)] the modules $\alpha_0\alpha_1\Lambda$ and $\beta_0\Lambda$ are completely described, in particular, their intersection is $\mathrm{soc}(e\Lambda)$. Moreover, by (3.12), there is some $c \in K$ such that

$$(*) \qquad \alpha_0\gamma = c(\beta_0 \ldots \beta_r)^{m-1}\beta_0 \ldots \beta_{r-1}.$$

Since $H(e\Lambda)$ is not a direct sum, we must have $c \neq 0$, and then we may assume that $c = 1$.

(6.a) *The component of* $\Gamma_s(\Lambda)$ *containing* $H(e\Lambda)$ *is not a tube*: Let $H := H(e\Lambda)$. There is the usual AR-sequence

$$0 \to \mathrm{rad}(e\Lambda) \to e\Lambda \oplus H \to e\Lambda/\mathrm{soc}(e\Lambda) \to 0;$$

moreover, H is indecomposable and lies therefore at the end of some component. Define $U := \alpha_0 \Lambda / \alpha_0 \Lambda \cap \beta_0 \Lambda$ and $V := \beta_0 \Lambda / \mathrm{soc}(e\Lambda)$. Then one shows, using (*), by the method of (3.8) that $\Omega U \cong \gamma\Lambda$ and that $\Omega(\gamma\Lambda) \cong \beta_r \beta_0 \Lambda \cong V$. Thus there is an AR-sequence $0 \to U \to X \to V \to 0$, and as in (3.8), one proves that $X \cong H$.

By (6) and (6.a), there must be an arrow $e_1 \xrightarrow{\gamma} f_r$ and also an arrow $f_1 \xrightarrow{\eta} e_n$. Let π_2 be the cycle of π containing γ, and π_3 the cycle of π containing η.

(7) $\pi_2 = \pi_3$: Suppose not. For any vertex g with $g \in \{e_1, e_n, f_1, f_r\}$, the composition factors of $g\Lambda$ are determined by the cycles π_i ($0 \leqslant i \leqslant 3$) and their multiplicities. Using (2.5), it follows that $e_1 \Lambda \oplus f_1 \Lambda$ and $e_n \Lambda \oplus f_r \Lambda$ have the same composition factors. Thus the Cartan matrix is singular, a contradiction.

Now, since $\pi_2 = \pi_3$, we must have $e_1 = e_n$ and $f_1 = f_r$, by (3.1). Moreover, π_0 and π_1 are 2-cycles, by (4). But π_2 must also be a 2-cycle, by the definition of π_2 and π_3, since no other arrows can start or end at e_1 or f_r. Thus the quiver has three vertices only and is of the form shown in Fig. 9.

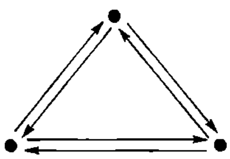

Fig. 9

Quivers in which one of two intersecting cycles is always a loop. Now we have to study the possibility that, whenever two cycles of π intersect, then one of them is a loop.

Suppose that Λ has more than one simple module; then the quiver must have just one cycle, π_0 say, of length > 1, and otherwise only loops. We know from (2) that there are at least two cycles, hence Q must contain a quiver as in Fig. 10 and $\pi_0 = (\alpha_n \alpha_0 \ldots)$. It follows from (3.4) that the cycle π_0 must go twice

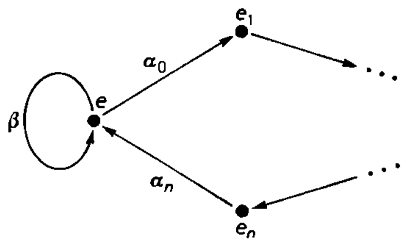

Fig. 10

through e_1 and e_n. On the other hand, by (3.1), the cycle π_0 can only have at most one self-crossing. It follows that $e_n = e_1$. This argument applies to any loop. Hence, if Q has another loop then this loop must have distance $\leqslant 1$ from e_1, and it follows that Q is one of the quivers of Fig. 11. Hence in this case, Λ has two or three simple modules.

Now consider a quiver as in Fig. 10 with only one loop. Assume (for

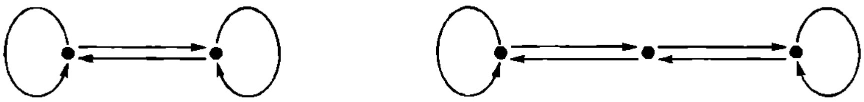

Fig. 11

contradiction) that Q has more than three vertices. Then Q is a quiver as in (3.9).

(8) *If $H(e_1 \Lambda)$ is not a direct sum then Λ is defined by the relations in* (3.9): We have $\pi(\alpha_{n-1}) \neq \alpha_1$ and moreover $|s^{-1}(e_{n-1})| = |s^{-1}(e_2)| = 1$. Hence, by (3.6), we may assume that $\alpha_{n-1}\alpha_1 = 0$. Then one shows that $\beta\alpha_0 \in \mathrm{soc}_2(\alpha_0 \Lambda)$, by using arguments as in (2.a) and (2.b). The rest is straightforward.

Now, by (3.9), $\Gamma_s(\Lambda)$ has a component which is not a tube. We deduce that this cannot occur, and the proof is complete.

We remark that the quivers with one loop in the last part left are just those of Fig. 12. Thus altogether there are just six possible quivers.

Fig. 12

EXAMPLE (A. Skowroński [9]). Let $\Lambda = KQ/I$ be the bound quiver algebra given by the quiver

$$Q\colon\quad 4 \underset{\sigma}{\overset{\nu}{\rightleftarrows}} 2 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 1 \underset{\mu}{\overset{\gamma}{\rightleftarrows}} 3$$

and the ideal generated by $\alpha\beta - \mu\gamma$, $\beta\alpha - \nu\sigma$, $\sigma\beta\mu$, $\gamma\alpha\nu$. Then Λ satisfies the conditions (1) to (3). Moreover, the stable AR-quiver consists only of 1-tubes and 3-tubes.

Proof. Consider the tubular algebra A of type (3, 3, 3) illustrated in Fig. 13 with $\alpha\beta = \mu\gamma$, $\beta\alpha = \nu\sigma$, $\sigma\beta\mu = 0$, obtained from the tame concealed algebra C of Fig. 14 with $\alpha\beta = \mu\gamma$, of type (2, 2, 3), by two one-point extensions using the simple regular C-modules of Fig. 15, lying in different tubes in Γ_C of rank 2. Then the repetitive algebra $\hat{A}$ of A is of the form shown in Fig. 16, with $\alpha\beta = \mu\gamma$, $\beta\alpha = \nu\sigma$, $\sigma\beta\mu = 0$, $\gamma\alpha\nu = 0$. It follows from the results of [7] that $\hat{A}$ is locally support-finite of polynomial growth, nondomestic, and that $\Gamma_{\hat{A}}$ consists only of 1-tubes and 3-tubes. Consider the shift $g\colon \hat{A} \to \hat{A}$ given by

$$1 \to 2 \to 6 \to 8 \to 1 \to 2 \to 6 \to \dots$$

$$3 \to 4 \to 7 \to 5 \to 3 \to 4 \to 7 \to \dots$$

Then g^4 is the Nakayama shift. Let $B = \hat{A}/(g^2)$; then B is nondomestic, standard self-injective of polynomial growth, and Γ_B consists only of 1-tubes

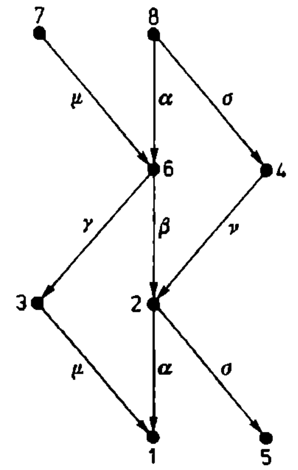

Fig. 13

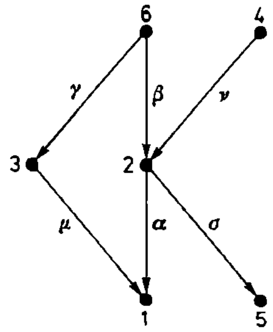

Fig. 14

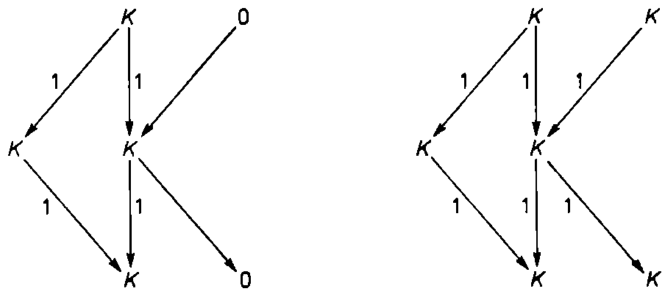

Fig. 15

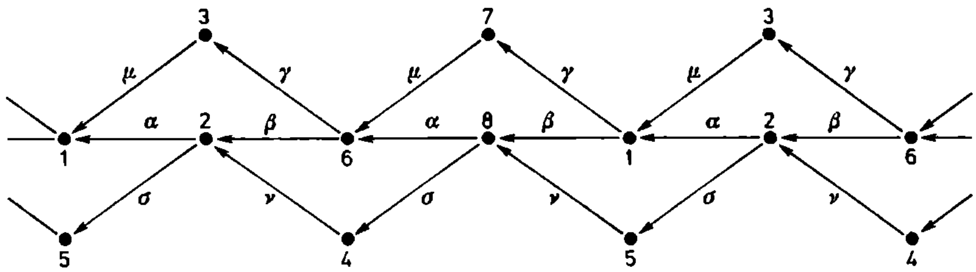

Fig. 16

and 3-tubes. The Cartan matrix of B is

$$\begin{bmatrix} 3 & 2 & 2 & 1 \\ 2 & 3 & 1 & 2 \\ 2 & 1 & 3 & 0 \\ 1 & 2 & 0 & 3 \end{bmatrix},$$

hence is nonsingular. Moreover, B is symmetric and $B \cong \Lambda$.

Note that $\Lambda/\mathrm{rad}^3\Lambda$ is not special biserial.

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