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REGULARLY BISERIAL ALGEBRAS

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Let K be an algebraically closed field. Finite-dimensional regularly biserial K-algebras are introduced. In particular, we outline the proof of the result that every regularly biserial K-algebra is tame and its Auslander-Reiten invariant $\beta(A)$ is not greater than two.

Let A be a finite-dimensional algebra over an algebraically closed field K. The *Auslander–Reiten invariant* $\beta(A)$ is defined to be the largest possible number of indecomposable direct summands in the middle term of the Auslander–Reiten sequences which are neither projective nor injective. In [3] M. Auslander and I. Reiten proved that, if A is representation-finite and $\beta(A) \leqslant 2$, then A is biserial, that is, the radical of any indecomposable (left or right) projective A-module is a sum of at most two uniserial submodules whose intersection is zero or simple. In [17] A. Skowroński and J. Waschbüsch showed that, if A is representation-finite and biserial, then $\beta(A) \leqslant 2$. There are representation-infinite tame algebras A with $\beta(A) \leqslant 2$ which are not biserial (see [9]). On the other hand, as shown in [6, 18], every special biserial algebra A is tame and has $\beta(A) \leqslant 2$. Here we study a more general class of biserial algebras called regularly biserial algebras. A finite-dimensional K-algebra A over an algebraically closed field K is said to be *regularly biserial* if A is isomorphic to the bound quiver algebra KQ/I, where (Q, I) satisfies the following conditions:

(1) Every vertex in Q is a source (sink) of at most two arrows.

(2) If for an arrow α there are two different arrows β and γ such that $\beta\alpha \notin I$ and $\gamma\alpha \notin I$, then either $\alpha\beta - cu\gamma\alpha \in I$ for a path u and some $c \in K^* = K \setminus \{0\}$, or $\alpha\gamma - dv\beta\alpha \in I$ for a path v and some $d \in K^*$, and moreover in both cases $\beta\tau, \gamma\tau \in I$ for any arrow $\tau \neq \alpha$.

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(3) If for an arrow α there are two different arrows β and γ such that $\alpha\beta \notin I$ and $\alpha\gamma \notin I$, then either $\alpha\beta - c\alpha\gamma u \in I$ for a path u and some $c \in K^*$, or $\alpha\gamma - d\alpha\beta v \in I$ for a path v and some $d \in K^*$, and moreover in both cases $\tau\beta, \tau\gamma \in I$ for any arrow $\tau \neq \alpha$.

We prove that every regularly biserial algebra is biserial (Corollary 1). Let A be a regularly biserial algebra which without loss of generality (Lemma 1) may be assumed reduced, that is, every projective-injective indecomposable module is uniserial. We prove in Section 2 that A admits a standard bound quiver presentation $KQ/I \xrightarrow{\sim} A$. This allows us to prove in Section 3 (Theorem 1) that every reduced regularly biserial algebra A admits a simply connected Galois covering $\tilde{A} = K\tilde{Q}/\tilde{I} \to A = KQ/I$ with a free nonabelian fundamental group $\Pi(Q, I)$. This is the main result proved in this paper. Moreover, we outline in Section 4 the proof, contained in [13], of Theorem 2 stating that every regularly biserial algebra A is tame and has $\beta(A) \leqslant 2$. For this proof we give some generalizations (Theorems 3 and 4) of the main results of [6] concerning the Galois covering techniques for representation-infinite tame algebras.

The results presented here form a part of the doctoral thesis [13]. The author would like to thank Andrzej Skowroński for fruitful discussions concerning biserial algebras and Galois coverings.

1. Preliminaries

Let K be a fixed algebraically closed field. Recall from [4] that a *locally bounded category* is a K-category R satisfying the following conditions:

(a) Different objects are not isomorphic.
(b) The algebras $R(x, x)$ are local for any object $x \in R$.
(c) For every object $x \in R$,

$$\sum_{y \in R} \dim_K R(x, y) < \infty \quad \text{and} \quad \sum_{y \in R} \dim_K R(y, x) < \infty .$$

A *presentation* p of a locally bounded K-category R is a K-linear surjective functor p: $KQ \to R$ which is the identity on the objects and maps arrows α_i to $p(\alpha_i)$, where $p(\alpha_1), \ldots, p(\alpha_d) \in JR(x, y)$ are morphisms whose cosets modulo $J^2R(x, y)$ form a basis of $(JR/J^2R)(x, y)$. It is well known that p induces an isomorphism $KQ/I \approx R$, where Q is uniquely determined by R and $I = \ker(p)$ depends on the choice of p. The pair (Q, I) is called a *bound quiver* of R. We call a bound quiver (Q, I) *regularly biserial* if it satisfies conditions (1)–(3) of the definition of regularly biserial algebras.

Let R be a locally bounded K-category. An *R-module* is a contravariant K-linear functor M from R to the category of K-vector spaces. The *dimension vector* $\underline{\dim}(M)$ of an R-module M is defined to be $(\dim_K M(x))_{x \in R}$ and the *dimension* of M as $\dim(M) = \sum_{x \in R} \dim_K M(x)$.

An R-module M is called *locally finite-dimensional* if $\dim_K M(x) < \infty$ for every object $x \in R$ and *finite-dimensional* if $\dim(M) < \infty$. We denote by $\mathrm{MOD}\, R$ the category of all R-modules. $\mathrm{Mod}\, R$ (resp. $\mathrm{mod}\, R$) denotes the full subcategory of $\mathrm{MOD}\, R$ consisting of all locally finite-dimensional (resp. finite-dimensional) R-modules. $\mathrm{Ind}\, R$ (resp. $\mathrm{ind}\, R$) denotes the full subcategory of $\mathrm{Mod}\, R$ (resp. $\mathrm{mod}\, R$) consisting of all indecomposable R-modules. $(\mathrm{Ind}\, R)/\approx$ (resp. $(\mathrm{ind}\, R)/\approx$) denotes the set of isoclasses of the objects from $\mathrm{Ind}\, R$ (resp. $\mathrm{ind}\, R$).

For a full subcategory C of $\mathrm{Mod}\, R$ we denote by $[C]$ the two-sided ideal in $\mathrm{Mod}\, R$ consisting of all morphisms in $\mathrm{Mod}\, R$ which factor through morphisms from C.

Let M be an R-module. The *support* of M is the full subcategory $\mathrm{supp}(M)$ of R formed by all objects $x \in R$ with $M(x) \neq 0$. M is called *sincere* if $\mathrm{supp}(M) = R$.

Following Yu. Drozd [7] (cf. [5]) a finite (the number of objects is finite) locally bounded K-category R is *tame* if, for any dimension d, there exists a finite family of functors F_i: $\mathrm{mod}\, A_i \to \mathrm{mod}\, R$, $i = 1, \ldots, n_d$, where $A_i = K$ or A_i is a rational algebra $K[T]_f$ of dimension 1, which satisfies the following conditions:

(a) For any $1 \leqslant i \leqslant n_d$, $F_i = - \otimes_{A_i} Q_i$, where Q_i is an A_i-R-bimodule which is a finitely generated free left A_i-module.

(b) Every indecomposable R-module M of dimension d is of the form $M \approx F_i S$ for some i and some simple A_i-module S.

A locally bounded K-category R is tame if so is every finite full subcategory.

2. Regularly biserial quivers

For a biserial algebra A consider a decomposition $A_A = P \oplus Q$, where P is a direct sum of indecomposable projective-injective nonuniserial A-modules and Q has no such direct summands. Then by [3], $\mathrm{soc}_A(P)$ is a two-sided ideal in A and any right (left) indecomposable projective-injective $A/\mathrm{soc}_A(P)$-module is uniserial. We call the algebra $A_{\mathrm{red}} = A/\mathrm{soc}_A(P)$ the *reduced form* of A. Recall that for an algebra B the *Auslander–Reiten invariant* $\alpha(B)$ is the largest possible number of all indecomposable direct summands in the middle term of the Auslander–Reiten sequences. From [3] we have the following lemma.

LEMMA 1. (a) $\beta(A) \leqslant 2$ *iff* $\alpha(A_{\mathrm{red}}) \leqslant 2$.

(b) *The nonuniserial projective-injective A-modules are the only indecomposable A-modules which are not A_{red}-modules. In particular, A is tame iff so is A_{red}.*

Now we shall choose a possibly nice presentation of a regularly biserial reduced algebra.

A pair of nonzero parallel paths $(\alpha, \beta_t \ldots \beta_1)$, $t \geqslant 1$, in a bound quiver

(Q, I) is said to be a pair of *weakly commutative* paths in (Q, I) if either $\alpha\gamma \notin I$, $\alpha\gamma - a\beta_t \ldots \beta_1\gamma \in I$ for some arrow γ and some $a \in K^*$, or $\delta\alpha \notin I$, $\delta\alpha - b\delta\beta_t \ldots \beta_1 \in I$ for some arrow δ and some $b \in K^*$.

LEMMA 2. *Let (Q, I) be a regularly biserial quiver such that KQ/I is a finite-dimensional algebra. Then*:

(1) *(Q, I) contains no two pairs of weakly commutative paths of the form $(\alpha_m, \beta_1 \ldots \beta_t)$ and $(\beta_t, \alpha_1 \ldots \alpha_m)$.*

(2) *(Q, I) contains no two pairs of weakly commutative paths of the form $(\alpha_m, \beta_t \ldots \beta_1)$ and $(\beta_t, \alpha_m \ldots \alpha_1)$.*

Proof. For the proof of (1) suppose there are two pairs of weakly commutative paths $(\alpha_m, \beta_1 \ldots \beta_t)$ and $(\beta_t, \alpha_1 \ldots \alpha_m)$. Suppose that $\beta_t\tau \notin I$, $\beta_t\tau - a\alpha_1 \ldots \alpha_m\tau \in I$; the proof for the case $\varkappa\beta_t \notin I$, $\varkappa\beta_t - d\varkappa\alpha_1 \ldots \alpha_m \in I$ is completely analogous. The regular biseriality of (Q, I) implies that, if $\delta \neq \tau$, then $\alpha_m\delta$, $\beta_t\delta \in I$. Consequently, either $\alpha_m\tau - b\beta_1 \ldots \beta_t\tau \in I$ for some $b \in K^*$, or there is an arrow ε such that $\varepsilon\alpha_m - c\varepsilon\beta_1 \ldots \beta_t \in I$ for some $c \in K^*$. In the first case we have

$$\begin{aligned} 0 \neq \beta_t\tau &= a\alpha_1 \ldots \alpha_m\tau = ab\alpha_1 \ldots \alpha_{m-1}\beta_1 \ldots \beta_t\tau \\ &= \ldots = (ab)^n(\alpha_1 \ldots \alpha_{m-1}\beta_1 \ldots \beta_{t-1})^n\beta_t\tau \end{aligned}$$

in KQ/I for any $n \in \mathbf{N}$, a contradiction to the fact that KQ/I is finite-dimensional. In the second case we get $\varepsilon = \alpha_{m-1}$, by the regular biseriality of (Q, I), and we have

$$\begin{aligned} 0 \neq \beta_t\tau &= a\alpha_1 \ldots \alpha_{m-1}\alpha_m\tau = ac\alpha_1 \ldots \alpha_{m-1}\beta_1 \ldots \beta_t\tau \\ &= \ldots = (ac)^n(\alpha_1 \ldots \alpha_{m-1}\beta_1 \ldots \beta_{t-1})^n\beta_t\tau \end{aligned}$$

in KQ/I for any $n \in \mathbf{N}$, again a contradiction.

The proof of (2) is similar and we omit it.

LEMMA 3. *Let (Q, I) be a regularly biserial quiver such that KQ/I is a finite-dimensional K-algebra. Then for any pair of nonzero parallel paths $\alpha_n \ldots \alpha_1\gamma$, $\beta_m \ldots \beta_1\gamma$ there is $a \in K^*$ such that $\alpha_n \ldots \alpha_1\gamma - a\beta_m \ldots \beta_1\gamma \in I$ and for any pair of nonzero parallel paths $\tau\alpha_n \ldots \alpha_1$, $\tau\beta_m \ldots \beta_1$ there is $b \in K^*$ such that $\tau\alpha_n \ldots \alpha_1 - b\tau\beta_m \ldots \beta_1 \in I$.*

Proof. Suppose that there are two different nonzero parallel paths $\alpha_n \ldots \alpha_1\gamma$, $\beta_m \ldots \beta_1\gamma$ such that $\alpha_n \ldots \alpha_1\gamma - a\beta_m \ldots \beta_1\gamma \notin I$ for any $a \in K^*$. We can assume that $\alpha_1 \neq \beta_1$. Thus, by the regular biseriality of (Q, I), either there is a path u such that $\alpha_1\gamma - a_1u\beta_1\gamma \in I$ for some $a_1 \in K^*$, or there is a path v such that $\beta_1\gamma - b_1v\alpha_1\gamma \in I$ for some $b_1 \in K^*$. In the first case we consider the paths $\alpha_n \ldots \alpha_2u\beta_1\gamma$ and $\beta_m \ldots \beta_1\gamma$, in the second $\beta_m \ldots \beta_2v\alpha_1\gamma$ and $\alpha_n \ldots \alpha_1\gamma$. Observe that each of these pairs is a pair of noncommutative nonzero paths.

Assume that we have constructed a pair of noncommutative paths (w, y) such that $(w, \alpha_n \dots \alpha_1 \gamma)$ and $(y, \beta_m \dots \beta_1 \gamma)$ are pairs of commutative paths and moreover $w = w'\varrho\lambda u'$ and $y = y'\mu\lambda u'$, where ϱ, λ, μ are arrows in (Q, I) and u' is a composition of at least t arrows. Then either there is a path r such that $\mu\lambda - cr\varrho\lambda \in I$ for some $c \in K^*$, or there is a path s such that $\varrho\lambda - ds\mu\lambda \in I$ for some $d \in K^*$, because (Q, I) is regularly biserial. In the first case we put $w_1 = w'\varrho\lambda u'$ and $y_1 = y'r\varrho\lambda u'$, in the second $w_1 = w's\mu\lambda u'$ and $y_1 = y'\mu\lambda u'$. In this way we have constructed a pair of noncommutative nonzero paths (w_1, y_1) such that $(w_1, \alpha_n \dots \alpha_1 \gamma)$ and $(y_1, \beta_m \dots \beta_1 \gamma)$ are pairs of commutative paths, $w_1 = w'_1 \varrho_1 \lambda_1 u'_1$ and $y_1 = y'_1 \mu_1 \lambda_1 u'_1$ and $\lambda_1 u'_1$ is a composition of at least $t+1$ arrows. Consequently, we can construct inductively paths satisfying the above properties and such that $\lambda_1 u'_1$ is a nonzero path in (Q, I) of an arbitrarily large length, which gives a contradiction to the finite-dimensionality of KQ/I.

The proof of the second assertion is similar.

LEMMA 4. *For any regularly biserial algebra A there is a regularly biserial quiver (Q, I) such that $A \approx KQ/I$ and the following conditions are satisfied:*

(1) *If (Q, I) contains a pair of weakly commutative paths $(\alpha, \beta_t \dots \beta_1)$ such that $\tau\alpha \notin I$ and $\tau\alpha - b\tau\beta_t \dots \beta_1 \in I$ for some $b \in K^*$ and for an arrow τ, then there exist two arrows $\gamma_1 \neq \gamma_2$ such that either $\beta_1\gamma_1$, $\alpha\gamma_2 \in I$ or $\alpha\gamma_2$, $\alpha\gamma_1 \in I$ or $\beta_1\gamma_2$, $\beta_1\gamma_1 \in I$.*

(2) *Every pair of weakly commutative paths $(\alpha, \beta_t \dots \beta_1)$ in (Q, I) is a pair of weakly commutative paths in (Q, I') for any presentation $KQ/I' \xrightarrow{\sim} A$.*

Proof. Let $A \approx KQ_A/I_A$, where (Q_A, I_A) is a regularly biserial quiver, and let p: $KQ_A \to A$ be a presentation with $\ker(p) = I_A$. Moreover, assume that there is a pair of weakly commutative paths $(\alpha, \beta_t \dots \beta_1)$ such that $\tau\alpha - b\tau\beta_t \dots \beta_1 \in I_A$, $\tau\alpha \notin I_A$ for some τ and some $b \in K^*$. Thus we have $\varepsilon\alpha$, $\varepsilon\beta_t \in I_A$ for any arrow $\varepsilon \neq \tau$ by the definition of the regularly biserial quiver. If there exists an arrow $\gamma_1 \in Q_A$ such that $\alpha\gamma_1$, $\beta_1\gamma_1 \notin I_A$, then either there exists a path u such that $\alpha\gamma_1 - cu\beta_1\gamma_1 \in I_A$ for some $c \in K^*$, or there exists a path v such that $\beta_1\gamma_1 - dv\alpha\gamma_1 \in I_A$ for some $d \in K^*$. In the second case we get a contradiction to Lemma 2. So in the first case u must be of the form $u = \beta_t u'$. Indeed, if $u = \alpha u'$, then $u = \alpha\gamma_1 u''$, where $u' = \gamma_1 u''$ and we get

$$0 \neq \alpha\gamma_1 = c\alpha\gamma_1 u''\beta_1\gamma_1 = c^2\alpha\gamma_1 u''\beta_1\gamma_1 u''\beta_1\gamma_1 = \dots = c^n\alpha\gamma_1(u''\beta_1\gamma_1)^n$$

in KQ_A/I_A for any $n \in \mathbb{N}$, contrary to the finite-dimensionality of A.

Suppose $\beta_t \dots \beta_1 \gamma \in I_A$. Then $\beta_t \dots \beta_1 \notin I_A$ and $\beta_t u'\beta_1 \notin I_A$, so, by Lemma 3, one obtains $\beta_t \dots \beta_1 - f\beta_t u'\beta_1 \in I_A$ for some $f \in K^*$. Then $\beta_t \dots \beta_1\gamma_1 - f\beta_t u'\beta_1\gamma_1 \in I_A$ and $\beta_t u'\beta_1\gamma_1 \notin I_A$, so $\beta_t \dots \beta_1\gamma_1 \notin I_A$, a contradiction. This shows that $\beta_t \dots \beta_1\gamma_1 \notin I_A$ and by Lemma 3 there is $e \in K^*$ such that $\alpha\gamma_1 - e\beta_t \dots \beta_1\gamma_1 \in I_A$. Moreover, for any $\gamma_2 \neq \gamma_1$, we have $\alpha\gamma_2$, $\beta_1\gamma_2 \in I_A$.

Summarizing: $\alpha\gamma_1 - e\beta_t \dots \beta_1\gamma_1 \in I_A$ for some $e \in K^*$, and for $\gamma_2 \neq \gamma_1$ we have either $\alpha\gamma_1$, $\beta_1\gamma_2 \in I_A$ or $\alpha\gamma_1$, $\alpha\gamma_2 \in I_A$ or $\beta_1\gamma_1$, $\beta_1\gamma_2 \in I_A$ or $\alpha\gamma_1$, $\beta_1\gamma_2 \in I_A$.

Consequently, for $\alpha\gamma_1, \beta_1\gamma_1 \notin I_A$ we choose a presentation p_0: $KQ_A \to A$ in the following way: $p_0(\varkappa) = p(\varkappa)$ for $\varkappa \neq \alpha$ and $p_0(\alpha) = p(\alpha - b\beta_t \dots \beta_1)$. It is obvious that $\tau\alpha \in \ker(p_0) = I_0$ and $\alpha\gamma_1 - (e-b)\beta_t \dots \beta_1\gamma_1 \in I_0$. The bound quiver (Q_A, I_0) is regularly biserial. Proceeding in this way, by induction on the number of pairs of weakly commutative paths $(\alpha, \beta_t \dots \beta_1)$ such that $\tau\alpha - b\tau\beta_t \dots \beta_1 \in I_0$ for some τ and some $b \in K^*$, one obtains a regularly biserial quiver (Q, I_1) which satisfies condition (1).

In order to choose a regularly biserial quiver which satisfies (2) and (1) consider the quiver (Q, I_1) chosen above. Let p': $kQ \to A$ be a presentation of A with $\ker(p') = I_1$. Then for any pair of weakly commutative paths $(\alpha, \beta_t \dots \beta_1)$ in (Q, I_1) we have either $\tau\alpha - b\tau\beta_t \dots \beta_1 \in I_1$ for some $b \in K^*$ and $\alpha\gamma \in I_1$, or $\beta_1\gamma \in I_1$ for any arrow γ, or $\alpha\gamma - a\beta_t \dots \beta_1\gamma \in I_1$ for some $a \in K^*$ and $\tau_1\alpha \in I_1$, or $\tau_1\beta_t \in I_1$ for any arrow τ_1. Thus in the case of $\beta_t \dots \beta_1\varepsilon \in I_1$ for any arrow ε (resp. $\varrho\beta_t \dots \beta_1 \in I_1$ for any arrow ϱ) we choose a presentation p'_0: $kQ \to A$ as follows: $p'_0(\varkappa) = p'(\varkappa)$ for $\varkappa \neq \alpha$ and $p'_0(\alpha) = p'(\alpha - b\beta_t \dots \beta_1)$ (resp. $p'_0(\alpha) = p'(\alpha - a\beta_t \dots \beta_1)$). By induction on the number of pairs of weakly commutative paths in (Q, I_1) one obtains a regularly biserial quiver (Q, I) which satisfies (1) and (2). This finishes the proof of our lemma.

A regularly biserial quiver (Q, I) satisfying conditions (1) and (2) of Lemma 4 is said to be *weakly standard.*

LEMMA 5. *A weakly standard quiver (Q, I) contains no two pairs of weakly commutative paths of the form $(\alpha, \beta_n \dots \beta_1)$ and $(\beta_t, \gamma_m \dots \gamma_1)$ for $1 \leqslant t \leqslant n$.*

Proof. Suppose that in a weakly standard quiver (Q, I) there are two pairs of weakly commutative paths of the form $(\alpha, \beta_n \dots \beta_1)$ and $(\beta_t, \gamma_m \dots \gamma_1)$ for some t with $1 \leqslant t \leqslant n$. Then Lemma 2 implies that $1 < t < n$. Thus $\beta_t\beta_{t-1} \notin I$ and $\beta_{t+1}\beta_t \notin I$ imply that either $\beta_t\beta_{t-1} - a\gamma_m \dots \gamma_1\beta_{t-1} \in I$ for some $a \in K^*$, or $\beta_{t+1}\beta_t - b\beta_{t+1}\gamma_m \dots \gamma_1 \in I$ for some $b \in K^*$. In the first case $\beta_{t+1}\gamma_m \dots \gamma_1 \in I$, because (Q, I) is weakly standard. But

$$\beta_n \dots \beta_1 = a\beta_n \dots \beta_{t+1}\gamma_m \dots \gamma_1\beta_{t-1} \dots \beta_1 = 0$$

in KQ/I, so $\beta_n \dots \beta_1 \in I$, contrary to the assumption that $(\alpha, \beta_n \dots \beta_1)$ is a pair of weakly commutative paths. Similarly we obtain a contradiction in the second case and the lemma is proved.

LEMMA 6. *Let (Q, I) be a weakly standard quiver and let $(\alpha, \beta_n \dots \beta_1)$, $(\gamma, \delta_m \dots \delta_1)$ be two different pairs of weakly commutative paths in (Q, I) which have a common arrow. Then either $\delta_i = \beta_n$, $\delta_{i-1} = \beta_{n-1}, \dots, \delta_1 = \beta_j$ for some $i < m$, $j > 1$, or $\beta_i = \delta_m$, $\beta_{i-1} = \delta_{m-1}, \dots, \beta_1 = \delta_j$ for some $i < n$, $j > 1$.*

Proof. Observe that by Lemma 5 we get $\gamma \neq \beta_t$ for any $t = 1, \dots, n$ and $\alpha \neq \delta_l$ for any $l = 1, \dots, m$. Consequently, $\beta_s = \delta_r$ for some $1 \leqslant s \leqslant n$, $1 \leqslant r \leqslant m$. Consider the paths $\beta_{s+1}\beta_s$, $\delta_{r+1}\delta_r \notin I$. If $r < m$ and $s < n$ then

$\delta_{r+1} = \beta_{s+1}$. Indeed, if $\delta_{r+1} \neq \beta_{s+1}$ then, by the weak standardness of (Q, I), either there is a path v such that $(\delta_{r+1}, v\beta_{s+1})$ is a pair of weakly commutative paths in (Q, I), or there is a path u such that $(\beta_{s+1}, u\delta_{r+1})$ is a pair of weakly commutative paths in (Q, I). In both cases we obtain a contradiction to Lemma 5. Similarly, for $r > 1$, $s > 1$, one obtains $\delta_{r-1} = \beta_{s-1}$. Moreover, $\beta_n \ldots \beta_1 \neq \delta_m \ldots \delta_1$, because the pairs are different. If $r = m$ and $s = n$, we again have a contradiction. Indeed, $\alpha = \gamma$, $\beta_n = \delta_m$, $\beta_1 = \delta_1$ and, since our pairs are different, we obtain $\beta_i = \delta_j$ and $\beta_{i+1} \neq \delta_{j+1}$ for some $n > i > 1$, $m > j > 1$. In this case similar arguments applied to $\beta_{i+1}\beta_i$, $\delta_{j+1}\delta_j$ give a contradiction to Lemma 5. Similarly the case $r = s = 1$ is impossible. This finishes the proof.

PROPOSITION 1. *For any regularly biserial algebra A there exists a weakly standard quiver (Q, I) such that for every pair of weakly commutative paths $(\alpha, \beta_t \ldots \beta_1)$ the following condition is satisfied: if σ (resp. τ) is an arrow in Q such that $\alpha\sigma \notin I$, $\alpha\sigma - b\beta_t \ldots \beta_1\sigma \in I$, $b \in K^*$ (resp. $\tau\alpha \notin I$, $\tau\alpha - a\tau\beta_t \ldots \beta_1 \in I$, $a \in K^*$), then $b = 1$ (resp. $a = 1$).*

Proof. Let (Q, I') be a weakly standard quiver and let $p: KQ \to A$ be a presentation such that $\ker(p) = I'$. Let $(\alpha, \beta_t \ldots \beta_1)$ be a pair of weakly commutative paths such that $\alpha\sigma \notin I'$, $\alpha\sigma - b\beta_t \ldots \beta_1\sigma \in I'$, $b \in K^*$ (resp. $\tau\alpha \notin I'$, $\tau\alpha - a\tau\beta_t \ldots \beta_1 \in I'$, $a \in K^*$). We choose a presentation p_0: $KQ \to A$ in the following way: $p_0(\varkappa) = p(\varkappa)$ for $\varkappa \neq \alpha$ and $p_0(\alpha) = b^{-1}p(\alpha)$ (resp. $p_0(\alpha) = a^{-1}p(\alpha)$). Then $\alpha\sigma - \beta_t \ldots \beta_1\sigma \in \ker(p_0) = I_0$ (resp. $\tau\alpha - \tau\beta_t \ldots \beta_1 \in I_0$). It is clear that (Q, I_0) is a weakly standard quiver too and using Lemmas 5, 6, we may continue this procedure inductively with respect to the number of pairs of weakly commutative paths $(\alpha, \beta_t \ldots \beta_1)$ which do not satisfy the conclusion. In this manner we construct a weakly standard quiver (Q, I) which satisfies the required condition.

A weakly standard quiver satisfying the condition of Proposition 1 is said to be *standard*.

LEMMA 7. *Let $A = KQ/I$, where (Q, I) is a standard regularly biserial quiver. Then, for any arrow $\varepsilon \in Q$, $\varepsilon \cdot A$ and $A \cdot \varepsilon$ are uniserial modules.*

Proof. Let ε be an arrow in Q. Moreover, let $\varepsilon\gamma_1 \ldots \gamma_t$ be the maximal path starting with ε and such that $\varepsilon\gamma_1 \ldots \gamma_t \notin I$. Let i_0 be the smallest index such that the end of γ_{i_0} is the end of an arrow $\tau_{i_0} \neq \gamma_{i_0}$ with $\varepsilon\gamma_1 \ldots \gamma_{i_0-1}\tau_{i_0} \notin I$. Then either there exists a path u such that $\gamma_{i_0-1}\gamma_{i_0} - \gamma_{i_0-1}\tau_{i_0}u \in I$, or there exists a path v such that $\gamma_{i_0-1}\tau_{i_0} - \gamma_{i_0-1}\gamma_{i_0}v \in I$, because (Q, I) is standard.

In the first case by the maximality of $\varepsilon\gamma_1 \ldots \gamma_t$ we find that u is trivial, so $\gamma_{i_0-1}\gamma_{i_0} - \gamma_{i_0-1}\tau_{i_0} \in I$ and $i_0 = t$ by the standardness of (Q, I). Consequently,

$$\varepsilon \cdot A \supset \varepsilon\gamma_1 \cdot A \supset \ldots \supset \varepsilon\gamma_1 \ldots \gamma_{t-1} \cdot A \supset \varepsilon\gamma_1 \ldots \gamma_t \cdot A$$

is the only sequence of submodules of $\varepsilon \cdot A$ and hence $\varepsilon \cdot A$ is uniserial.

In the second case if $v \neq \gamma_{i_0+1} \dots \gamma_t$, then let i_1 be the minimal index such that $i_1 > i_0$ and γ_{i_1} does not lie on the path v. Since $\varepsilon\gamma_1 \dots \gamma_t$ is maximal, the standardness of (Q, I) and Lemmas 5, 6 imply that $v = \gamma_{i_0+1} \dots \gamma_{i_1-1}$. Again using the standardness of (Q, I) shows that $i_1 - 1 = t$ and

$$\varepsilon \cdot A \supset \varepsilon\gamma_1 \cdot A \supset \dots \supset \varepsilon\gamma_1 \dots \gamma_t \cdot A$$

is the only sequence of submodules of $\varepsilon \cdot A$, and so $\varepsilon \cdot A$ is uniserial. If v is trivial then as above we see that $\varepsilon \cdot A$ is also uniserial. The proof that $A \cdot \varepsilon$ is uniserial is similar.

COROLLARY 1. *Every regularly biserial algebra is biserial.*

Proof. Let $A = KQ/I$ be a regularly biserial algebra. Lemma 4 and Proposition 1 imply that the quiver (Q, I) may be chosen standard. We now show that any indecomposable projective A-module $x \cdot A$ satisfies: $\mathrm{rad}_A(x \cdot A)$ is a sum of at most two uniserial submodules whose intersection is zero or simple. Here x is a vertex of Q. If x is a sink of at most one arrow δ then $\mathrm{rad}_A(x \cdot A) = \delta \cdot A$ is uniserial by Lemma 7. If x is a sink of two different arrows γ, δ then $\mathrm{rad}_A(x \cdot A) = \gamma \cdot A + \delta \cdot A$, where $\gamma \cdot A$, $\delta \cdot A$ are uniserial by Lemma 7. If $\gamma \cdot A \cap \delta \cdot A$ is zero or simple, then $x \cdot A$ is biserial.

Suppose that $Y = \gamma \cdot A \cap \delta \cdot A$ has length $\geqslant 2$. It is clear that Y is uniserial. If $y \cdot A$ is the projective cover of the uniserial module Y then y is a source of two different arrows τ, ε such that there are paths $\delta w_1 \tau$, $\gamma w_2 \varepsilon$. Moreover, Y not being simple implies that there exists an arrow λ with sink y and $\varepsilon\lambda \notin I$, $\tau\lambda \notin I$. Then the standardness of (Q, I) implies either the existence of a path u for which $\varepsilon\lambda - u\tau\lambda \in I$, or the existence of a path v for which $\tau\lambda - v\varepsilon\lambda \in I$. In the first case the uniseriality of $\gamma \cdot A$ implies that $\varepsilon = \gamma$ and $u = \delta u_1$; but then $\gamma \cdot A/Y$ is a simple module, $\gamma \cdot A + \delta \cdot A \approx \gamma \cdot A/Y \oplus \delta \cdot A$ and $x \cdot A$ is biserial. In the second case we find similarly that $\delta \cdot A/Y$ is simple, $\gamma \cdot A + \delta \cdot A \approx \gamma \cdot A \oplus \delta \cdot A/Y$, and $x \cdot A$ is biserial.

The proof that $A \cdot x$ is biserial is similar.

3. Universal coverings

Let A be a locally bounded K-category and $A \approx KQ_A/I_A$. Assume that the quiver Q_A is connected. Let x_0 be a fixed vertex of Q_A and let W be the topological universal covering of Q_A with the base point x_0. From [12] there is a natural map q: $W \to Q_A$ given by the action of the fundamental group $\Pi_1(Q_A, x_0)$. For the bound quiver (Q_A, I_A) a *minimal relation* in I_A is an element $\varrho = \sum_{i=1}^{n} \lambda_i u_i \in I_A(x, y)$, where $\lambda_i \in K^*$ and u_i is a path from x to y, such that $n \geqslant 2$ and for every nonempty proper subset $T \subset \{1, \dots, n\}$, $\sum_{i \in T} \lambda_i u_i \notin I_A(x, y)$ [11].

Let N be the normal subgroup of $\Pi_1(Q_A, x_0)$ generated by all elements of the form $[\gamma^{-1} u^{-1} v \gamma]$, where γ is a walk from x_0 to x and u and v are paths from x to y such that there is a minimal relation in I_A of the form $\varrho = \sum_{i=1}^{n} \lambda_i w_i$ such that $w_1 = u$, $w_2 = v$. Then the group $\Pi(Q_A, I_A) \approx \Pi_1(Q_A, x_0)/N$ is called the *fundamental group* of the bound quiver (Q_A, I_A) [11]. We define $\tilde{Q}_A$ as the orbit quiver W/N and the map $\pi\colon \tilde{Q}_A \to Q_A$ is given by the action of $\Pi(Q_A, I_A)$ on $\tilde{Q}_A$. The map π gives a Galois covering $\pi\colon K\tilde{Q}_A \to KQ_A$ of path categories and consequently we obtain a Galois covering $F\colon K\tilde{Q}_A/\tilde{I}_A \to KQ_A/I_A$ with the group $\Pi(Q_A, I_A)$, where $\tilde{I}_A$ is the ideal in $K\tilde{Q}_A$ generated by all elements u such that $\pi(u) \in I_A$. Then $\tilde{A} = K\tilde{Q}_A/\tilde{I}_A$ is called the *universal cover* of A [4] determined by the presentation $KQ_A/I_A \xrightarrow{\sim} A$.

A locally bounded K-category is said to be *simply connected* [1] if A is triangular and for any presentation $p\colon KQ_A \to A$ the fundamental group $\Pi(Q_A, I_p)$ is trivial, where the triangularity of A means that its bound quiver does not contain oriented cycles.

An algebra A is said to be *standard* [16] if there is a Galois covering $F\colon \tilde{A} \to A$ such that $\tilde{A}$ is simply connected.

Now we are able to prove the following theorem.

THEOREM 1. *Let A be a basic connected reduced regularly biserial algebra. Then there exists a standard quiver (Q, I) such that:*

(i) $A \approx KQ/I$.

(ii) *$\Pi(Q, I)$ is a free nonabelian group.*

(iii) *$R \approx K\tilde{Q}/\tilde{I}$ is a simply connected regularly biserial K-category.*

In particular, A is standard.

Proof. From Proposition 1, $A \approx KQ/I$ for a standard quiver (Q, I). Let $(\hat{Q}, \hat{I})$ be the bound quiver which is obtained from (Q, I) by removing all arrows α for which there is a pair of weakly commutative paths (α, v) in (Q, I) and by removing all relations involving these arrows. Moreover if, in the pair (α, v), v is also an arrow then we remove only one of the arrows α, v (no matter which one). Thus $(\hat{Q}, \hat{I})$ is a special biserial bound quiver. Moreover, the construction of the universal covering implies $(\tilde{\hat{Q}}, \tilde{\hat{I}}) = (\hat{\tilde{Q}}, \hat{\tilde{I}})$. Since A is reduced, $(\hat{Q}, \hat{I})$ is a bound quiver with $\hat{I}$ generated by paths. This implies that $(\tilde{\hat{Q}}, \tilde{\hat{I}})$ is a tree and $\tilde{\hat{I}}$ is generated by paths. So the fact that $(\tilde{\hat{Q}}, \tilde{\hat{I}}) = (\hat{\tilde{Q}}, \hat{\tilde{I}})$ and Lemmas 2, 4, 5 and 6 imply that $K\tilde{Q}/\tilde{I}$ is a simply connected K-category and consequently (iii) is proved. Hence A is also standard.

It remains to prove (ii), because (i) is obvious by the choice of (Q, I). Observe that $\Pi(\hat{Q}, \hat{I})$ is a free nonabelian group and $\Pi(Q, I) \approx \Pi(\hat{Q}, \hat{I})$, so $\Pi(Q, I)$ is free nonabelian. Our theorem is proved.

4. Main result

The main aim of this section is to outline the proof of the following theorem.

THEOREM 2. *Let A be a finite-dimensional regularly biserial K-algebra. Then A is tame and $\beta(A) \leqslant 2$.*

A complete proof of this result contained in [13] is technical and tedious. In this proof we apply Galois covering techniques developed for representation-infinite algebras by P. Dowbor and A. Skowroński in [6]. In fact, we need a more general version of the main result of [6] which we present below.

Let R be a locally bounded K-category, G a group of K-linear automorphisms of R acting freely on the objects of R, R/G the quotient category [10] whose objects are the G-orbits of objects of R and F: $R \to R/G$ the covering functor attaching to each object x of R its G-orbit $G \cdot x$. F induces a functor $F_{\bullet}$: $\mathrm{MOD}\, R/G \to \mathrm{MOD}\, R$ attaching the module $N \circ F^{\mathrm{op}}$ to the R/G-module N. Moreover, there is [8, 6] a functor F_λ: $\mathrm{MOD}\, R \to \mathrm{MOD}\, R/G$ which is left adjoint to $F_{\bullet}$ and acts on the objects as follows: for any $M \in \mathrm{MOD}\, R$ and any object r of R/G,

$$(F_\lambda M)(r) = \bigoplus_{Fx=r} M(x);$$

if $r_1 \xrightarrow{\alpha} r_2$ is a morphism in R/G then the map $(F_\lambda M)(\alpha)$: $(F_\lambda M)(r_2) \to (F_\lambda M)(r_1)$ attaches to $(m_x) \in \bigoplus_{Fx=r_2} M(x)$ the element

$$\Big(\sum_x M({}_x\bar{\alpha}_y)(m_x)\Big) \in \bigoplus_{Fy=r_1} M(y),$$

where ${}_x\bar{\alpha}_y$ is determined by the formula $\sum_{Fy=r_1} F({}_x\bar{\alpha}_y) = \alpha$.

For every full subcategory L of R we denote by G_L the stabilizer $\{g \in G;\ gL = L\}$ of L. By GL we denote the full subcategory of R consisting of G-orbits of all objects of L. The group G acts on $\mathrm{MOD}\, R$ by ${}^g(-)$ such that ${}^gM = M \circ g^{-1}$ for every $M \in \mathrm{MOD}\, R$. For every $M \in \mathrm{MOD}\, R$ we denote by G_M the stabilizer $\{g \in G;\ {}^gM \approx M\}$. Moreover, we assume that G acts freely on $(\mathrm{ind}\, R)/\approx$, the set of isoclasses of objects in $\mathrm{ind}\, R$. Observe that, if G is torsion-free and acts freely on R, then G acts freely on $(\mathrm{ind}\, R)/\approx$.

An *R-action* of G on an R-module M [8] is given by K-linear maps $v(g, x)$: $M(x) \to M(gx)$ which are defined for every object x of R and every element g of G and satisfy the following conditions:

(a) $v(1, x) = 1_{M(x)}$.

(b) $v(h, gx)v(g, x) = v(hg, x)$ for all $x \in R$ and $g, h \in G$.

(c) $v(g, x)M(\alpha) = M(g\alpha)v(g, y)$ for every $\alpha \in R(x, y)$ and every $g \in G$.

By $\mathrm{MOD}^G R$ we denote the category whose objects are pairs (M, v), where v is an R-action of G on $M \in \mathrm{MOD}\, R$, and the morphism set from (M, v) to (M', v') consists of all R-homomorphisms from M to M' which are compatible with the R-actions of G. We denote this set by $\mathrm{Hom}_R^G(M, M')$. By $\mathrm{Mod}_f^G R$ we denote the full subcategory of $\mathrm{MOD}^G R$ consisting of all $(M, v) \in \mathrm{MOD}^G R$ such that $M \in \mathrm{Mod}\, R$ and $\mathrm{supp}(M)$ is contained in a finite number of G-orbits of R. From [8] we have

LEMMA 8. *Under the above notation the functor $F_{\bullet}$ induces an equivalence $\mathrm{mod}\, R/G \to \mathrm{Mod}_f^G R$. Moreover, F_λ induces an injection from the set $((\mathrm{ind}\, R)/\approx)/G$ of G-orbits in $(\mathrm{ind}\, R)/\approx$ into the set $(\mathrm{ind}\, R/G)/\approx$.*

Following [6] we denote by $\operatorname{mod}_1 R/G$ the full subcategory of $\operatorname{mod} R/G$ consisting of all modules of the form $F_\lambda M$, where $M \in \operatorname{mod} R$. Modules from $\operatorname{mod}_1 R/G$ will be called modules *of the first kind.* Denote by $\operatorname{mod}_2 R/G$ the full subcategory of $\operatorname{mod} R/G$ consisting of all modules which do not have direct summands from $\operatorname{mod}_1 R/G$. These modules are called modules *of the second kind.*

A module $M \in \operatorname{Ind} R$ is called *weakly G-periodic* if $\operatorname{supp}(M)$ is infinite and $\operatorname{supp}(M)/G_M$ is finite.

Let H be a group of K-linear automorphisms of a locally bounded K-category C which acts freely on $\mathcal{A}(\operatorname{ind} C)/\approx$. Denote by $\operatorname{Mod}^H_{f1} C$ (resp. $\operatorname{Mod}^H_{f2} C$) the full subcategory of $\operatorname{Mod}^H_f C$ consisting of all $M \in \operatorname{Mod}^H_f C$ such that $M = \bigoplus_{i \in I} Z_i$, where $Z_i \in \operatorname{ind} C$ (resp. $Z_i \in \operatorname{Ind} C$ and $Z_i \notin \operatorname{mod} C$) for every $i \in I$.

Let R be a locally bounded K-category and G a group of K-linear automorphisms of R acting freely on $(\operatorname{ind} R)/\approx$. Let $\mathcal{S}$ be a family of locally bounded K-categories. A family of functors e^S_λ: $\operatorname{Mod} S \to \operatorname{Mod} R$, $S \in \mathcal{S}$, is said to be *G-separating* if the following conditions are satisfied:

(a) For any $S \in \mathcal{S}$, the functor e^S_λ admits a left and right adjoint functor $e^S_\bullet$: $\operatorname{Mod} R \to \operatorname{Mod} S$.

(b) For any $S \in \mathcal{S}$, there exists a subgroup G_S of G such that G_S acts freely on S and ${}^g(-)e^S_\lambda \approx e^S_\lambda\, {}^g(-)$, ${}^g(-)e^S_\bullet \approx e^S_\bullet\, {}^g(-)$ for any $g \in G_S$.

(c) For any $S \in \mathcal{S}$, the class of weakly G_S-periodic S-modules is nonempty and coincides with the class of sincere S-modules in $\operatorname{Ind} S$.

(d) For any weakly G-periodic R-module M, there exists exactly one $S \in \mathcal{S}$ such that $M \approx e^S_\lambda(X)$ for some weakly G_S-periodic S-module X.

(e) $e^S_\lambda(Y) \in \operatorname{mod} R$ for any $Y \in \operatorname{mod} S$, $S \in \mathcal{S}$.

(f) $e^S_\bullet(N) \in \operatorname{Mod}^{G_S}_{f1} S$ for any $N \in \operatorname{mod} R$, $S \in \mathcal{S}$.

(g) For any $S \in \mathcal{S}$ and any G-orbit O in R, the intersection $O \cap R_S$ is contained in a finite number of G_S-orbits, where R_S denotes the support of any weakly G_S-periodic R-module of the form $e^S_\lambda(X)$ for any weakly G_S-periodic S-module X.

(h) For any S, $S' \in \mathcal{S}$ and $Z \in \operatorname{Mod} S$, $e^{S'}_\bullet e^S_\lambda(Z)$ is a direct sum of finite-dimensional S'-modules.

(i) For any $y \in R$, $r \geqslant 1$, there are only a finite number of $S \in \mathcal{S}$ such that $|S/G_S| \leqslant r$ and $y \in R_S$.

Denote by $\mathcal{S}_0$ a fixed set of categories from $\mathcal{S}$ such that the subcategories R_S, $S \in \mathcal{S}_0$, form a set of representatives of G-orbits in the set of all categories R_S, $S \in \mathcal{S}$.

Let $S \in \mathcal{S}_0$ and let $E^S_\bullet$: $\operatorname{Mod}^G_f R \to \operatorname{Mod}^{G_S}_f S$ be the functor induced by $e^S_\bullet$: $\operatorname{Mod} R \to \operatorname{Mod} S$. Then $E^S_\bullet$ admits a left and right adjoint functor E^S_λ: $\operatorname{Mod}^{G_S}_f S \to \operatorname{Mod}^G_f R$ given by

$$E^S_\lambda(X) = \bigoplus_{g \in U_S} {}^g(e^S_\lambda(X)) = \prod_{g \in U_S} {}^g(e^S_\lambda(X)),$$

where $X \in \mathrm{Mod}_f^{G_S} S$ and U_S is a fixed set of representatives of the left cosets G modulo G_S. The module $e_\lambda^S(X)$ admits an R-action of G_S which is induced by the given S-action of G_S on X.

Now we are able to state the following generalization of Theorem 3.1 from [6].

THEOREM 3. *Let R be a locally bounded K-category and G a group of K-linear automorphisms of R acting freely on* $(\operatorname{ind} R)/\approx$. *Let $\{e_\lambda^S\}_{S\in\mathcal{S}}$ be a G-separating family of functors. Then there exists an equivalence of categories*

$$E\colon \coprod_{S\in\mathcal{S}_0} (\operatorname{mod} S/G_S)/[\operatorname{mod}_1 S/G_S] \to (\operatorname{mod} R/G)/[\operatorname{mod}_1 R/G].$$

Moreover, the Auslander–Reiten quiver $\Gamma_{R/G}$ of R/G is isomorphic to the disjoint union of quivers

$$\Gamma_R/G \amalg \Big(\coprod_{S\in\mathcal{S}_0} (\Gamma_{S/G_S})\Big)_2,$$

where $(\Gamma_{S/G_S})_2$ is the union of connected components of Γ_{S/G_S} whose points are isoclasses of S/G_S-modules of the second kind.

Now let A be a K-category. Then every contravariant functor Q: $R \to \mathrm{MOD}\, A^{\mathrm{op}}$ will be called an *A-R-bimodule*. Every A-R-bimodule Q induces a functor $-\otimes_A Q\colon \mathrm{MOD}\, A \to \mathrm{MOD}\, R$, where $(V\otimes_A Q)(x) = V\otimes_A Q(x)$ for any $V \in \mathrm{MOD}\, A$ and $x \in R$.

Let B be a weakly G-periodic R-module with an R-action v of the group G_B on B. Then $F_\lambda B$ admits a structure of KG_B-R-bimodule, where KG_B is the group algebra of the group G_B over K. More precisely, for every $Gx \in R/G$, $(F_\lambda B)(Gx)$ is a left free KG_B-module of rank $\sum_{y\in W_x} \dim_K B(y)$, where W_x is a set of representatives of G_B-orbits in the set Gx. If G_B is an infinite cyclic group then $F_\lambda B$ is a $K[T, T^{-1}]$-R/G-bimodule and we obtain a functor $-\otimes_{K[T,T^{-1}]} F_\lambda(B)\colon \operatorname{mod} K[T, T^{-1}] \to \operatorname{mod} R/G$.

Let $\{e_\lambda^S\}_{S\in\mathcal{S}}$ be a G-separating family of functors such that any $S\in\mathcal{S}$ is a line, that is, Q_S is a linear quiver whose underlying graph $\bar{Q}_S$ is of the form ${}_\infty A_\infty$. Let $B_S = e_\lambda^S(X_S)$ for some weakly G_S-periodic S-module X_S and

$$\Phi^S = -\otimes_{K[T,T^{-1}]} F_\lambda(B_S)\colon \operatorname{mod} K[T, T^{-1}] \to \operatorname{mod} R/G.$$

Under the above notation we have the following generalization of Theorem 3.6 from [6].

THEOREM 4. *Let R be a locally bounded K-category and let G be a group of K-linear automorphisms of R acting freely on* $(\operatorname{ind} R)/\approx$. *Moreover, let $\{e_\lambda^S\}_{S\in\mathcal{S}}$ be a G-separating family of functors such that any $S\in\mathcal{S}$ is a line. Then the family $\{\Phi^S\}_{S\in\mathcal{S}_0}$ induces an equivalence of categories*

$$\Phi\colon \coprod_{S\in\mathcal{S}_0} \operatorname{mod} K[T, T^{-1}] \to (\operatorname{mod} R/G)/[\operatorname{mod}_1 R/G].$$

Moreover,

$$(\Gamma_{R/G})_2 = \coprod_{\mathscr{S}_0} \Gamma_{K[T,T^{-1}]},$$

where $\Gamma_{K[T,T^{-1}]}$ is the Auslander–Reiten quiver of the category of finite-dimensional $K[T, T^{-1}]$-modules. Moreover, R/G is tame iff R is tame.

The proofs of Theorems 3 and 4 are straightforward generalizations of the the proofs of [6, Theorem 3.1] and [6, Theorem 3.6].

Now we can sketch the main steps of the proof of Theorem 2, contained in [13]. Let A be a regularly biserial algebra. From Lemma 1 we may assume that A is reduced. By Theorem 1, A admits a simply connected Galois covering $\tilde{A} \to A$ with a free nonabelian group G. Then using the vector space category methods [14, 15] one classifies all finite-dimensional indecomposable $\tilde{A}$-modules and proves that $\tilde{A}$ is tame and $\alpha(\tilde{A}) \leqslant 2$. Further, we construct a G-separating family of functors satisfying the conditions of Theorem 4. Applying Theorem 4 we conclude that A is tame and $\beta(A) \leqslant 2$.

References

[1] I. Assem and A. Skowroński, *On some classes of simply connected algebras*, Proc. London Math. Soc. (3) (56) (1988), 417–450.

[2] M. Auslander and I. Reiten, *Representation theory of artin algebras IV. Invariants given by almost split sequences*, Comm. Algebra 5 (1977), 443–518.

[3] —, —, *Uniserial functors*, in: Proc. ICRA II, Ottawa 1979, Lecture Notes in Math. 832, Springer, 1980, 1–47.

[4] R. Bautista, P. Gabriel, A. V. Roiter and L. Salmerón, *Representation-finite algebras and multiplicative bases*, Invent. Math. 81 (1985), 217–285.

[5] P. Dowbor and A. Skowroński, *On the representation type of locally bounded categories*, Tsukuba J. Math. 10 (1) (1986), 63–72.

[6] —, —, *Galois coverings of representation-infinite algebras*, Comment. Math. Helv. 62 (1987), 311–337.

[7] Yu. A. Drozd, *On tame and wild matrix problems*, in: Matrix Problems, Kiev 1977, 104–114.

[8] P. Gabriel, *The universal cover of a representation-finite algebra*, in: Proc. ICRA III, Puebla 1980, Lecture Notes in Math. 903, Springer, 1981, 68–105.

[9] D. Happel and C. M. Ringel, *The derived category of a tubular algebra*, in: Proc. ICRA IV, Ottawa 1984, Lecture Notes in Math. 1177, Springer, 1986, 156–180.

[10] S. MacLane, *Categories for the Working Mathematician*, Springer, 1971.

[11] R. Martínez-Villa and J. A. de la Peña, *The universal cover of a quiver with relations*, J. Pure Appl. Algebra 30 (1983), 277–292.

[12] W. Massey, *Algebraic Topology: An Introduction*, Graduate Texts in Math. 56, Springer, 1976.

[13] Z. Pogorzały, Doctoral Thesis, Toruń 1987 (in Polish).

[14] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math. 1099, Springer, 1984.

[15] D. Simson, *Representations of partially ordered sets, vector space categories and socle projective modules*, preprint, Paderborn 1983.
[16] A. Skowroński, *Selfinjective algebras of polynomial growth*, Math. Ann. 285 (1989), 177–199.
[17] A. Skowroński and J. Waschbüsch, *Representation-finite biserial algebras*, J. Reine Angew. Math. 345 (1983), 172–181.
[18] B. Wald and J. Waschbüsch, *Tame biserial algebras*, J. Algebra 95 (1985), 480–500.