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SUBGROUPS AND MODULAR REPRESENTATIONS OF FINITE QUASISIMPLE GROUPS

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This paper is an extended version of the author's lecture at the Semester on Classical Algebraic Structures at the Stefan Banach International Mathematical Center in May 1988. The aim of the paper is to survey the author's recent results on the subject indicated in the title.

After announcing the completion of the classification of finite simple groups (CFSG), emphasis in finite group theory has been shifted to the applications and revision of CFSG and to the study of properties of (known) finite simple groups. It is particularly important to study the closely connected properties of subgroups and representations of finite simple groups.

Let G be a finite group, let K be a field with $\operatorname{char}(K) = p$, and $n \in N$. A (*linear*) *representation of* G *of degree* n *over* K is defined as a homomorphism $G \to \mathrm{GL}_n(K)$. It is particularly difficult to study modular representations of G, i.e. those where p divides the order of G.

Consider the following classical problem.

PROBLEM 1. Describe the finite linear groups of small degree, i.e. finite subgroups in $\mathrm{GL}_n(K)$ for every K and small n.

Beginning with the middle of the past century, this problem attracted attention of many mathematicians. In the seventies of our century it was solved for $K = \mathbf{C}$ and $n \leqslant 9$ in the well-known papers of Jordan, Klein, Valentiner, Blichfeld, Brauer, Lindsey II, Wales, Hoffman, Feit. The case $p > 0$ and $n \leqslant 5$ of Problem 1 was considered before 1982 in the papers by Jordan, Moore, Wiman, Burnside, Dickson, Mitchell, Hartley, Bloom, Mwene, DiMartino, Wagner, Zalesskii, and I. D. Suprunenko (see [26]). The irreducible subgroups of $\mathrm{GL}_n(2)$ are determined by Harada and Yamaki for $n \leqslant 6$ [9] and by the author for $7 \leqslant n \leqslant 10$ [11, 13, 14, 15]. Observe that it is reasonable to restrict

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our attention, when solving Problem 1, to the irreducible subgroups whose composition factors are the known simple groups.

In the last years, intensive investigation began of maximal subgroups in finite groups. One of the main problems in this field is the following.

PROBLEM 2. Determine the maximal subgroups of finite almost simple groups, i.e. of groups G with $S \trianglelefteq G \leqslant \mathrm{Aut}(S)$, where S is a finite nonabelian simple group.

Aschbacher [2] outlined a program based on CFSG of solving this problem. The main idea of this program is to represent every finite almost simple group as the group of automorphisms of some natural object in order to make its subgroup structure transparent.

In [1], Aschbacher essentially reduced the problem of finding the maximal subroups H of a finite almost simple classical group G having the natural projective module V over the field $\mathrm{GF}(q)$ to the case when $S = F^*(H)$ is a nonabelian simple group and V is an absolute irreducible $\mathrm{GF}(q)\,\hat{S}$-module for some covering group $\hat{S}$ of S. A covering group of a simple nonabelian group is also called a quasisimple group. More precisely, a group L is *quasisimple* if $L = [L, L]$ and $L/Z(L)$ is simple. Thus the problem of the subgroup structure of finite almost simple classical groups and, in particular, Problem 1 is reduced to the following problem.

PROBLEM 3. Investigate (modular) representations of finite quasisimple groups.

Of course, Problem 3 is interesting and important independently of Problems 1 and 2 as a problem concerning "external" properties of finite simple groups. New information on representations of quasisimple groups would be useful for revision of CFSG and also for the theory of group geometries.

Now we consider some important problems of the theory of modular representations of finite quasisimple groups which are closely connected to Problems 1 and 2.

Let $p > 0$ and let K be an algebraically closed field. Choose a complete system $x_1, \ldots, x_r$ of representatives of conjugacy classes of p'-elements of the group G. Denote by $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_s\}$ the set of all irreducible complex characters of G, and by $\mathrm{Irr\,Br}_p(G) = \{\varphi_1, \ldots, \varphi_r\}$ the set of all irreducible p-modular Brauer characters of G. Recall that the *Brauer character* β_T of a p-modular representation T: $G \to \mathrm{GL}_n(K)$ of G is defined as follows:

$$\beta_T(g) = \sum_{i=1}^{n} \varepsilon_i(g)^{\mu}, \tag{1}$$

where g is an arbitrary p'-element of G, $\varepsilon_1(g), \ldots, \varepsilon_n(g)$ is the complete system of eigenvalues of the matrix $T(g)$ (i.e.

$$\chi_T(g) = \sum_{i=1}^{n} \varepsilon_i(g)$$

for the character χ_T of the representation T), and μ is some lifting of K^* to $\mathbf{C}$. It is known that

$$\chi_i(x_j) = \sum_{k=1}^{r} d_{ik}\,\varphi_k(x_j) \quad (1 \leqslant j \leqslant r,\ 1 \leqslant i \leqslant s) \tag{2}$$

where $0 \leqslant d_{ik} \in \mathbf{Z}$. Write $D = (d_{ij})_{s\times r}$, $Z = (\chi_i(x_j))_{s\times s}$, and $\Phi = (\varphi_i(x_j))_{r\times r}$. Then the matrices D, Z, Φ are called the (p-modular) *decomposition matrix*, the *character table*, the (p-modular) *Brauer character table* of G, respectively.

Now suppose the character table Z of G is given. The following problems arise.

(a) *The partition of* $\operatorname{Irr}(G)$ *into p-blocks.* The characters $\chi, \psi \in \operatorname{Irr}(G)$ *belong to the same p-block* if and only if there exists a sequence $\chi_{i_1} = \chi, \chi_{i_2}, \ldots, \chi_{i_m} = \psi$ of characters of G such that every pair of consecutive members of this sequence has some common component φ_k in the decomposition (2). If Z is given then problem (a) may be solved by an effective algorithm (see [4]). To any p-block of $\operatorname{Irr}(G)$ there is associated a p-block of $\operatorname{Irr}\operatorname{Br}_p(G)$, namely the set of all components φ_k in the decompositions (2) of the characters from our p-block.

(b) *Finding the matrices D and* Φ. Theoretically the matrix D may be made block-diagonal by permutation of its rows and columns, where each diagonal block corresponds to a p-block of $\operatorname{Irr}(G)$ and the associated p-block of $\operatorname{Irr}\operatorname{Br}_p(G)$. Therefore, the calculation of D is reduced to the calculation of its diagonal blocks. Since the rank of D is equal to r, Φ is uniquely determined by Z and D by solving the system (2) of rs linear equations with r^2 unknowns $\varphi_k(x_j)$ $(1 \leqslant j, k \leqslant r \leqslant s)$.

(c) *Finding the field of definition of a (p-modular) representation* T: $G \to \mathrm{GL}_n(K)$ *with the Brauer character* $\varphi \in \operatorname{Irr}\operatorname{Br}_p(G)$. The *field of definition* of T is defined as the least subfield of K over which T may be realized. It is well known that the field of definition of T is equal to $\mathrm{GF}(p)(\chi_T(x_j) \mid 1 \leqslant j \leqslant r)$. Therefore, having the set $\{\varphi(x_j) \mid 1 \leqslant j \leqslant r\}$ and taking into account the formula (1), we may find the field of definition of T.

The solution of problems (a)–(c) permits one, in particular, to describe the equivalence classes of irreducible representations of a finite group G over every finite subfield of K.

(d) *Counting conjugacy classes and normalizers of absolutely irreducible quasisimple subgroups in* $\mathrm{GL}_n(q)$, *where* $q = p^m$. Let H be a finite quasisimple group, let T: $H \to \mathrm{GL}_n(q)$ be a faithful absolutely irreducible representation of H, and let N be the normalizer of $T(H)$ in $\mathrm{GL}_n(q)$. For any subgroup X of

$\mathrm{GL}_n(q)$, let $\bar{X}$ be the image of X in $\mathrm{PGL}_n(q)$. It is easy to prove that $\overline{T(H)} \trianglelefteq \bar{N} \leqslant \mathrm{Aut}(\overline{T(H)}) \cong \mathrm{Aut}(T(H))$, and the conjugacy classes of absolutely irreducible subgroups of $\mathrm{GL}_n(q)$ isomorphic to H are in one-to-one correspondence with the $\mathrm{Aut}(H)$-orbits of faithful irreducible p-modular Brauer characters of H, and, consequently, $|\mathrm{Aut}(\overline{T(H)})\colon \bar{N}|$ is the order of the $\mathrm{Aut}(H)$-orbits containing β_T. Therefore, the solution of problems (a)–(c) and the conjugacy in $\mathrm{Aut}(H)$ of the faithful irreducible p-modular Brauer characters of H lead to the solution of problem (d).

As we see, the calculation of D gives a lot of information on modular representations of a group and on subgroups of $\mathrm{GL}_n(q)$. An algorithm of calculation of decomposition matrices is unknown even in the relatively simple case of p-blocks with cyclic defect group. Neither is it known whether D (up to permutation of rows and columns) or even the set $\{\varphi_i(1) | 1 \leqslant i \leqslant r\}$ is uniquely determined by Z (a problem of Feit [7]).

There are only a few complete results of the calculation of D for known quasisimple groups. This is done only for the alternating groups of small degree, for some Chevalley groups of small Lie rank, and for some sporadic groups. The "Atlas of finite groups" [6] contains the character tables of the covering groups for every sporadic simple group and for some other "small" simple groups. Certainly, it is necessary to have a similar atlas of Brauer character tables. Recently the activity of investigations on Problem 3 has increased. In particular, the author knows that Parker, using a computer, studies intensively the modular representations of the groups whose character tables appear in [6].

Our results on Problem 3 are obtained independently of Parker's work. In [17], [18] we calculated the decomposition matrices for $\hat{J}_2$ and $\mathrm{Aut}(J_2)$ and all p. The proof is based on the Brauer theory of modular representations of finite groups and is of combinatorial nature.

The progress in solving Problem 3 allows us to make the next step in solving Problem 1, namely to describe the absolutely irreducible quasisimple subgroups and its normalizers in $\mathrm{GL}_6(q)$, where q is a prime power. In particular, we have the following.

THEOREM 1 [16]. *Let H be an absolutely irreducible quasisimple subgroup of $\mathrm{GL}_6(q)$, where $q = p^n$ for some prime p and some integer $n \geqslant 1$. Suppose that the nonabelian composition factor in H is a known simple group. Then H is isomorphic to one of the following groups*: $\mathrm{SL}_2(p^m)$, $p \geqslant 3$, $m|n$, *and* $p \geqslant 7$ *for* $m = 1$; $\mathrm{SL}_3(p^m)$, $p \geqslant 3$, $m|n$; $\mathrm{SU}_3(p^m)$, $p \geqslant 3$, $2m|n$; $\Omega_6^{\pm}(p^m)$, $m|n$; $\mathrm{Sp}_6(p^m)$, $m|n$; $\mathrm{SU}_6(p^m)$, $2m|n$; $\mathrm{SL}_6(p^m)$, $m|n$; $G_2(p^m)'$, $p = 2$, $m|n$; $\mathrm{SL}_2(5)$, $2 \neq p \neq 5$; $L_2(7)$, $2 \neq p \neq 7$; $\mathrm{SL}_2(7)$, $2 \neq p \neq 7$, *and* $2|n$ *for* $p \equiv \pm 3 \pmod 8$; $3.A_6$, $p \geqslant 5$, $q \equiv 1 \pmod 3$; $6.A_6$, $p \geqslant 5$, *and* $2|n$ *for* $p \not\equiv 1, 7 \pmod{24}$; A_7, $p \neq 7$; $2.A_7$, $p = 3$, $2|n$; $3.A_7$, $q \equiv 1 \pmod 3$; $6.A_7$, $p \geqslant 5$, *and* $2|n$ *for* $p \not\equiv 1, 7 \pmod{24}$; $\mathrm{SL}_2(11)$, $2 \neq p \neq 11$, *and* $2|n$ *for* $p \equiv 2, 6, 7, 8, 10 \pmod{11}$; $L_2(13)$, $p = 2$, $2|n$; $\mathrm{SL}_2(13)$, $2 \neq p \neq 13$, *and* $2|n$ *for* $p \equiv 2, 5, 6, 7, 8, 11 \pmod{13}$; $U_3(3)$, $p \geqslant 7$; $2.L_3(4)$,

$p = 3$, $6.L_3(4)$, $p \geqslant 5$, $q \equiv 1 \pmod 3$; $U_4(2)$, $p \geqslant 5$; $3.U_4(3)$, $p = 2$, $2|n$; $6.U_4(3)$, $p \geqslant 5$, *and* $q \equiv 1 \pmod 3$; $2.M_{12}$, $p = 3$; J_2, $p = 2$, $2|n$; $2.J_2$, $p \geqslant 3$, *and* $2|n$ *for* $p \equiv 2, 3 \pmod 5$; $3.M_{22}$, $p = 2$, $2|n$. *(Here $a.X$ denotes an extension of a cyclic group of order a by a group X). In addition, the group* $\mathrm{GL}_6(q)$ *has exactly one conjugacy class of absolutely irreducible quasisimple subgroups for each of the above-indicated types of groups, with the exception of* $6.A_7$ *(two classes),* $\mathrm{SL}_2(7)$ *(two classes for $p \neq 7$), and* $\mathrm{SL}_2(p^m)$ *($m-1$ classes for $p \in \{3, 5\}$, and m classes for $p \geqslant 7$).*

The proof of Theorem 1 uses the Brauer theory of modular representations of finite groups together with the representation theory of algebraic groups. In the course of proof we calculate the decomposition matrices of some "small" quasisimple groups, including $\hat{J}_2$. In particular, Theorem 1 together with the above-mentioned reduction result of Aschbacher solves (modulo CFSG) problem 8.39 b) from [21]. Kleidman (see [10]) described (modulo CFSG) all maximal subgroups in finite classical almost simple groups of dimension $\leqslant 12$. Theorem 1 was obtained by the author independently of this result of Kleidman.

Aschbacher [3] classified all irreducible FL-modules for every quasisimple group L of Lie type over a finite or algebraically closed field F. He uses this result in the investigation of subgroups in the groups $E_6(F)$, ${}^2E_6(F)$, and $F_4(F)$. The last groups are considered as the isometry groups of a symmetric 3-linear form on the 27-dimensional module. From this the special role of the number 27 is clear.

We obtain a complete classification of absolutely irreducible p-modular representations of degree $\leqslant 27$ of quasisimple groups of Lie type defined over a finite field of characteristic $\neq p$. The proof consists in calculating the decomposition matrices of quasisimple groups of Lie type from a list obtained by using a result of Landazuri and Seitz [22]. In particular, we have the following.

THEOREM 2. *A finite nonabelian simple group G of Lie type defined over a field of characteristic p has an absolutely irreducible projective representation of degree $\leqslant 27$ over a field of characteristic $\neq p$ if and only if G is isomorphic to one of the following groups::* $L_2(q)$ *for* $4 \leqslant q \leqslant 53$ *and* $q \neq 32$, $L_3(q)$ *for* $q \in \{3, 4\}$, $U_3(q)$ *for* $q \in \{3, 4, 5\}$, $L_4(2)$, $L_4(3)$, $\mathrm{PSp}_4(q)$ *for* $q \in \{3, 4, 5, 7\}$, $U_4(3)$, $U_5(2)$, $\mathrm{Sz}(8)$, $\mathrm{PSp}_6(2)$, $\mathrm{PSp}_6(3)$, $U_6(2)$, $\Omega_8^+(2)$, $P\Omega_7(3)$, $G_2(3)$, $G_2(4)$, ${}^3D_4(2)$, ${}^2F_4(2)'$.

Note that large groups from Theorem 2 belong to the second half of the list of simple groups which appears in [6], and their orders reach milliards. The results on decomposition matrices of quasisimple groups investigated in Theorem 2 may be considered as a contribution to the future atlas of Brauer character tables of "small" quasisimple groups, and also may be useful for the classification of maximal subgroups in almost simple classical groups of dimension $\leqslant 27$ (it remains to treat the quasisimple groups of alternating and sporadic type), and in the exceptional groups $F_4(q)$, $E_6(q)$, ${}^2E_6(q)$.

The use of CFSG reduces many questions in finite group theory to the examination of the corresponding properties for known finite simple groups, which is often difficult. We now consider some such results concerning subgroups of finite simple groups.

Menegazzo [23] introduced the class of IM-groups, i.e. groups in which every proper subgroup is the intersection of some maximal subgroups. He also determined all finite solvable IM-groups. Further Migliorini [24] and Bianchi and Tamburini [5] obtained criteria of nonsimplicity of finite IM-groups, and, in particular, proved in [5] that any minimal finite nonsolvable IM-group is simple. The question of existence of finite nonsolvable IM-groups remained open. Yu. N. Mukhin drawed our attention to this question in connection with the study of the subgroup lattices in topological groups. We have the following theorem.

THEOREM 3 [20]. *A finite* IM-*group whose simple sections are known simple groups is solvable.*

As a corollary we obtain the solvability of profinite IM-groups.

Also we have the following theorem.

THEOREM 4 [19]. *A finite group G whose composition factors are known simple groups is 2-nilpotent if and only if the normalizer of every Sylow subgroup has odd index in G.*

Theorem 4 gives a negative answer to question 5.37 from [21]).

Now we define the notion of the prime graph of a finite group, which first appeared in connection with some cohomological studies (see [8]). Let G be a finite group. The *prime graph* $\Gamma(G)$ of G is constructed as follows: the vertices are all primes dividing the order of G, and two vertices p, q are joined by an edge if and only if G contains an element of order pq. Denote by $t(G)$ the number of connected components of the graph $\Gamma(G)$.

It turns out that the nonconnectivity of $\Gamma(G)$ is closely connected with the decomposition of the augmentation ideal of G as a right module (see [8]). This fact aroused interest in the study of finite groups G with $t(G) > 1$. In an unpublished work of Gruenberg and Kegel the structure of a finite group G with $t(G) > 1$ has been restricted, and, in particular, the solvable groups with this property were completely determined. This result implies that if G is a nonsolvable group with $t(G) > 1$ not isomorphic to a Frobenius group then G has a nonabelian composition factor X with $t(G) \leqslant t(X)$. Therefore, the study of a finite nonsolvable group G with $t(G) > 1$ is reduced to the case of a simple group. Williams [25] obtained an explicity description of the connected components of the prime graph for every known finite simple group, with the exception of finite simple groups of Lie type in even characteristic. We give such a description in the remaining case. That solves Problem 9.16 from [21] (see also Problem (3.8) from [12]). In particular, we have the following.

THEOREM 5. *Let G be a finite simple group of Lie type in even characteristic. Then the prime graph of G is connected except for the following cases*:

(1) *two components*: $A_2(q)$ $(q > 4)$, $A_{p-1}(q)$ $(p \geqslant 5)$, $A_p(q)$ $(q-1 \mid p+1)$, ${}^2A_{p-1}(q)$, ${}^2A_p(q)$ $(q+1 \mid p+1,\ (p, q) \neq (5, 2))$, ${}^2A_3(2)$, $C_p(2)$, $C_n(q)$ $(n = 2^m,\ m \geqslant 1)$, $D_p(2)$ $(p \geqslant 5)$, ${}^2D_{p+1}(2)$ $(p \neq 2^m - 1)$, ${}^2D_n(q)$ $(n = 2^m,\ m \geqslant 2,\ (n, q) \neq (p+1, 2))$, ${}^3D_4(q)$, $G_2(q)'$, ${}^2F_4(2)'$, $E_6(q)$, ${}^2E_6(q)$ $(q > 2)$;

(2) *three components*: $A_1(q)$ $(q > 2)$, $A_2(2)$, ${}^2A_5(2)$, ${}^2D_{p+1}(2)$ $(p = 2^m - 1,\ m \geqslant 2)$, $F_4(q)$, ${}^2F_4(q)$ $(q > 2)$, $E_7(2)$;

(3) *four components*: $A_2(4)$, ${}^2B_2(q)$ $(q > 2)$, ${}^2E_6(2)$, $E_8(q)$ $(q \equiv 2, 3 \pmod 5)$;

(4) *five components*: $E_8(q)$ $(q \equiv 1, 4 \pmod 5)$. *Here p denotes an odd prime, and q denotes a power of* 2.

Besides the application to the augementation ideal of a finite group the notion of prime graph is also useful in other respects. If π is any component of the prime graph of a finite nonsolvable group G and π does not contain the prime 2, then G contains a nilpotent π-Hall subgroup H which is isolated in G. A subgroup H of G is called *isolated* if $H \cap H^g = 1$ or H for any element g of G, and for all h in $H - \{1\}$, $C_G(h) \leqslant H$. Conversely, if G contains an isolated subgroup H then the prime divisors of H form a connected component of $\Gamma(G)$.

It is possible to use this description to find all those finite simple groups of Lie type in even characteristic which are $C\pi\pi$-groups (π is a set of primes). A finite group is called a *$C\pi\pi$-group* if the centralizers of nonunit π-elements in this group are π-groups. In particular, as an illustration we have the following theorems.

THEOREM 6. *If G is a finite simple group of Lie type in even characteristic and* 5 *divides the order of G, then G is a C55-group if and only if G is one of the following*: $L_2(4)$, $L_3(4)$, $\mathrm{Sp}_4(2)' \cong A_6$, *or* $U_4(2)$.

THEOREM 7. *If G is a finite simple group of Lie type in even characteristic and G contains an element of order* 6, *then G is a $C\pi\pi$-group for $\pi = \{2, 3\}$ if and only if G is isomorphic to* $U_4(2)$, *or to* $G_2(2)' \cong U_3(3)$.

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