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A COLLECTION OF REFERENCES ON HYSTERESIS

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§ 1. Definition of hysteresis

Ferromagnetism, ferroelectricity, plasticity, supercooling and superheating effects are classical examples of hysteresis phenomena occurring in physics. Hysteresis appears also in porous media filtration, in biology, in chemistry, and in other natural sciences. Hysteresis effects can be found also in economics, in politics, and so on. Actually, hysteresis is also a feature of human behaviour, as some classical tests in experimental psychology have shown. Hysteresis is a memory effect, and can be used for storing information.

Let us consider a constitutive relation between two real variables $u \mapsto w$ in a time interval $[0, T]$. Hysteresis appears when the *output* $w(t)$ is not uniquely determined by the *input* $u(t)$ at the same instant $t \in]0, T]$, but instead $w(t)$ depends on the evolution of u in $[0, t]$:

$$w(t) = [F(u)](t), \tag{1}$$

where F is a *causal operator*. In order to exclude other memory effects like viscosity, we require that F be *rate independent*, i.e. that w depends just on the range of u in $[0, t]$ and on the order in which these values are taken, not on its velocity.

Consider for instance the hysteresis loop of Fig. 1. The couple (u, w) follows the exterior path ABCD only if u increases from values $u < u_1$ to values $u > u_2$, and then decreases from $u > u_2$ to $u < u_1$. However, if u inverts its direction when $u_1 < u < u_2$, then the couple (u, w) moves into the interior of the loop.

Notice that this graphic representation of hysteresis is made possible by the rate independence property; if the latter did not hold, then the path of the couple (u, w) would depend on the velocity of u at any t.

We then propose the following two definitions:

DEFINITION 1: *Hysteresis = Rate independent Memory Effect.*

Definition 2: An operator acting between two spaces of (either scalar- or vector-valued) functions of time is named a *hysteresis operator* if and only if it is causal and rate independent.

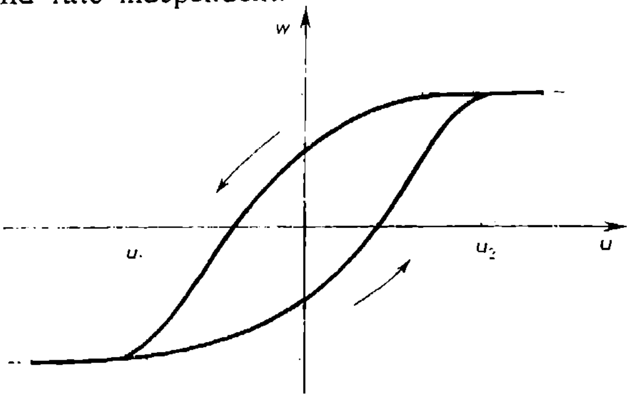

Fig. 1

The output w depends also on the initial conditions of the system; in (1) this information is already contained in the operator F, however it seems more convenient to display it. The initial state of the system can be represented in several ways; in the most simple case, it is characterized just by the initial values of u and w, so that (1) can be replaced by

(2) $$w(t) = [F(u, w^0)](t),$$

with the condition that

(3) $$[F(u, w^0)](0) = w^0.$$

Notice that the information on $u(0)$ is already contained in the argument u.

It is also reasonable to require some form of *transition* (*or semi-group*) *property*; for instance, for F as in (1), one can expect that

(4) $$\{F(u(\tau+\cdot), [F(u(\cdot), w^0)](\tau))\}(t) = [F(u(\cdot), w^0)](\tau+t).$$

However, this property can also have other forms, depending how the information on the initial conditions is represented.

The rate independence property excludes the possibility of representing F by means of a time convolution, like in the standard model of viscosity effects. Of course, by stating that a phenomenon exhibits a hysteresis effect, we do not exclude the presence of other memory effects. On the contrary, aside hysteresis one often encounters also rate dependent effects, typically with fading memory. Thus, more generally, (1) can be replaced by

(5) $$w(t) = [F(u)](t) + [G(u)](t),$$

with G rate dependent memory operator.

Notice that in the limit case of no memory the hysteresis operator degenerates into a Nemytskii operator, namely

(6) $$[F(u)](t) = \varphi(u(t)),$$

with φ real function.

§ 2. Mathematical literature on hysteresis

Although engineers, physicists and other scientists have been dealing with hysteresis for a long time [10, 14, 15, 24–26, 28, 66, 80, 81, 94], it seems that mathematicians started to study it just more recently.

Plasticity is an exception to this statement; this also is a rate independent memory effect, namely a hysteresis phenomenon according to our previous definition. However, it is a peculiar example, in that in many cases it can be treated without using memory operators, e.g. by means of variational inequalities. Thus plasticity was tackled by mathematicians much before other hysteresis phenomena, and there is a large mathematical literature about it, [27, 37, 43, 73, 84, 92, 93] e.g.

A systematic research on hysteresis was performed in the last twenty years by a group of mathematicians led by Krasnosel'skii in Moscow. Several of their papers also appeared in English translation; the main results then formed the nucleus of the monograph [56] of Krasnosel'skii and Pokrovskii. Since its very beginning [47], this research was based on the concept of *hysteresis operator*, this led to establishing the mathematical foundations of hysteresis effects appearing in natural phenomena and in engineering. On the background of this analysis, one can see such topics as system theory, robotics, control problems, etc.

Since the beginning of the years 1980's, mathematical studies on hysteresis were started also by several western mathematicians. Here we do not find a unique school, the largest group being apparently that of Augsburg (5 people). Accordingly these researches follow several directions.

Some of them concern *ferromagnetic* hysteresis and its physical foundations [11–13, 21, 23, 29, 34, 42, 97–100]. An especially interesting model for (one-dimensional) ferromagnetism had been proposed by the physicist Preisach already in 1935 [94] and was extensively used by physicists and engineers [26, 28, 80, 81]. It also has interesting mathematical properties [7–9, 19, 22, 38, 53, 55, 68–72, 103, 108].

Other hysteresis phenomena which are under current study are *shape memory* effects exhibited by certain alloys [1–3, 5, 30, 31, 78, 79, 82, 83, 110], *porous media filtration* [65, 67, 74–76, 87–91], and certain *chemical* and *biological pattern formation* phenomena like Liesegang rings [4, 39, 40, 44, 95, 102]. Other researches concern *control problems* for systems exhibiting hysteresis effects [16–18, 33, 35, 36, 96]. Another field of study is the analysis of partial differential equations with hysteresis operators [4, 16–18, 32, 33, 44, 57–63, 67, 95–105]. We also point out the surveys [45, 46, 56, 107–109].

Many other modelling and mathematical problems concerning hysteresis are still unsolved, or even unaddressed. Among them, even the archetypal problem of giving a macroscopic vectorial model of ferromagnetism, capable of being coupled with the Maxwell equations.

§ 3. References on hysteresis

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