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ON THE SHARPNESS OF AN APPROXIMATION CRITERION OF SMOOTHNESS FOR FUNCTIONS ON A SEGMENT

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In this paper we consider some problems related to: the constructive characterization of the approximation of classes of continuous functions on a segment by algebraic polynomials, established by S. M. Nikol'skii, V. K. Dzyadyk, A. F. Timan and Yu. A. Brudnyi (see [3], [5], [6], [9], [12], [13]); the results of S. N. Bernstein [2] and I. I. Ibragimov [8] on the best approximation of the functions x^{α} and $x^{m}\ln x$ by algebraic polynomials; the results of N. K. Bari, S. B. Stechkin and S. M. Lozinskii (see e.g. [1]) on the equivalence of the O- and $\sim$-relations in the constructive characterization of approximation by trigonometric polynomials.

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We introduce the usual notation:

$C^0 :=$ the space of continuous functions $f\colon [0, 1] \to \boldsymbol{R}$,

$C^r := \{f\colon f^{(r)} \in C^0\}$, $r \in \boldsymbol{N}$,

$\mathscr{P}_n :=$ the space of algebraic polynomials of degree $\leqslant n$, $n \in \boldsymbol{N}$,

$\|f\| := \sup\limits_{x\in[0,1]} |f(x)|$,

$E_n(f) := \inf\limits_{p\in\mathscr{P}_n} \|f-p\|$,

$\omega_1(f, t) := \sup\limits_{0\leqslant x_1<x_2\leqslant 1, x_2-x_1\leqslant t} |f(x_2)-f(x_1)|$, the modulus of continuity of $f \in C^0$,

$\operatorname{Lip} 1 := \{f\colon \omega_1(f, t) = O(t)\}$.

Let $\bar{\varepsilon} = \{\varepsilon_n\}$ be a decreasing sequence of positive numbers. Define

$$H[\bar{\varepsilon}] := \{f\colon E_n(f) \leqslant \varepsilon_n \ \forall n \in \boldsymbol{N}\}.$$

E. P. Dolzhenko and E. A. Sevast'yanov ([4], Theorem 7) proved the following theorem.

THEOREM 1 ([4]). *For every $\bar{\varepsilon}$ there is $f \in H[\bar{\varepsilon}]$ such that*

$$\omega_1(f, n^{-2}) \geqslant cn^{-2} \sum_{i=1}^{n} i\varepsilon_i, \quad n \in N, c = \text{const} > 0. \tag{1.1}$$

This implies Theorems 1′ and 1″ below:

THEOREM 1′ ([4]). *If*

$$\sum_{i=1}^{\infty} i\varepsilon_i = \infty,$$

then there exists $f \in H[\bar{\varepsilon}]$ such that $f \notin \mathrm{Lip}\,1$.

THEOREM 1″ ([4], Theorem 8). *If*

$$\sum_{i=1}^{\infty} i\varepsilon_i = \infty,$$

then there exists $f \in H[\bar{\varepsilon}]$ such that $f \notin C^1$.

Later, M. Hasson [7] proved Theorem 1″ under the additional assumption that $i^2 \varepsilon_i$ is decreasing. We present an example to show that this is an essential restriction, in spite of the fact that $\sum_{i=1}^{\infty} i\varepsilon_i < \infty$ implies $i^2 \varepsilon_i \to 0$.

EXAMPLE. Let $\mu(x) := -x \ln^{-1} x$, $x_k := (k!)^{-1}$, $k \in N$. We define a non decreasing function α on $[0, 1]$ by $\alpha(x) = \mu(x_k)$ for $x \in (x_k, x_{k-1}]$ and k even, and $\alpha(x) = \mu(x)$ for $x \in (x_k, x_{k-1}]$ and k odd. Set $\varepsilon_n := \alpha(n^{-2})$, $n \in N$. It is easy to see that

$$\sum_{i=1}^{\infty} i\varepsilon_i = \infty, \qquad \lim_{i \to \infty} i^2 \varepsilon_i = 0.$$

Let now $\bar{\beta} = \{\beta_n\}$ be any sequence satisfying (a) $i^2 \beta_i$ is decreasing, (b) $0 \leqslant \beta_i \leqslant \varepsilon_i$. Then it is not difficult to check that

$$\sum_{i=1}^{\infty} i\beta_i < \infty.$$

From A. A. Markov's inequality we deduce in the usual way (see e.g. [4], [7]) that if $\sum_{i=1}^{\infty} i^{2r-1} \varepsilon_i < \infty$ $(r \in N)$, then $f \in H[\bar{\varepsilon}] \Rightarrow f \in C^r$. Thus the condition $\sum_{i=1}^{\infty} i\varepsilon_i < \infty$ is necessary and sufficient for the inclusion $H[\bar{\varepsilon}] \subset C^1$ to hold ([4]). M. Hasson [7] assumed the following theorem to be true:

THEOREM 2. *Suppose $i^{2r} \varepsilon_i$ is a decreasing sequence, $r \in N$, $r > 1$. If $\sum_{i=1}^{\infty} i^{2r-1} \varepsilon_i = \infty$, then there exists $f \in H[\bar{\varepsilon}]$ such that $f \notin C^r$.*

Theorem 2 was proved by T. Xie [14] and independently by the author [11]. T. Xie [14] observed that the proof of Theorem 2 yields the following theorem:

THEOREM 2′. *Suppose $i^{2r}\varepsilon_i$ is a decreasing sequence, $r \in N$, $r > 1$. If $\sum_{i=1}^{\infty} i^{2r-1}\varepsilon_i = \infty$, then there exists $f \in H[\bar{\varepsilon}]$ such that $f^{(r-1)} \notin \mathrm{Lip}\,1$.*

If $i^{2r}\varepsilon_i > c > 0$, then the assertions of Theorems 1′, 1″, 2 and 2′ follow from I. I. Ibragimov's results [8] (for details, see Section 3).

In Section 2 we prove Theorem 3 which eliminates the additional assumption of Theorems 2 and 2′. Moreover, Theorem 3 (more precisely: its corollary, Theorem 4) gives a necessary and sufficient condition for the inclusion $H[\bar{\varepsilon}] \subset W^r H_k^{\varphi}$ to hold, where $W^r H_k^{\varphi}$ is the class of functions for which the kth modulus of continuity of the rth derivative, $\omega_k(f^{(r)}, t)$, is bounded by the increasing function $\varphi = \varphi(t)$.

Recall that the kth modulus of continuity of $f \in C^0$ is the function

$$\omega_k(f, t) = \sup_{h \in [0,t]} \sup_{x \in [0, 1-kh]} |\Delta_h^k(f, x)|,$$

where

$$\Delta_h^k(f, x) = \sum_{i=0}^{k}{}' (-1)^{k-i} \binom{k}{i} f(x+ih)$$

is the kth order finite difference of f at x with step h.

THEOREM 3. *Let $k \in N$ and $r+1 \in N$.*

(a) *If*

$$\sum_{i=1}^{\infty} r i^{2r-1}\varepsilon_i = \infty,$$

then there exists $f \in H[\bar{\varepsilon}]$ such that $f \notin C^r$.

(b) *If*

$$\sum_{i=1}^{\infty} r i^{2r-1}\varepsilon_i < \infty,$$

then there exists $f \in H[\bar{\varepsilon}]$ such that $f \in C^r$ but for all $n \in N$

$$\omega_k(f, n^{-2}) \geqslant c\Big(\sum_{i=n+1}^{\infty} r i^{2r-1}\varepsilon_i + n^{-2k}\sum_{i=1}^{n} i^{2(r+k)-1}\varepsilon_i\Big),$$

$$c = c(r, k, \bar{\varepsilon}) = \text{const} > 0.$$

As noted earlier, by using Markov's inequality we deduce in the standard way that Theorem 3 is equivalent to Theorem 4 below. Let us first introduce some notation.

Let $k \in N$, $r+1 \in N$, and let $\varphi = \varphi(t)$ be continuous and nondecreasing on $[0, 1]$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t > 0$. Define

$$W^r H_k^{\varphi} := \{f\colon\ \omega_k(f^{(r)}, t) = O(\varphi(t))\},$$

and for $r \neq 0$

$$W^r := W^{r-1} H_1^1 \quad (= W^{r-2} H_2^2 = \ldots = W^0 H_r^r).$$

THEOREM 4. *$H[\varepsilon] \subset W^r H_k^\varphi$ if and only if*

$$\sum_{i=n+1}^{\infty} r i^{2r-1} \varepsilon_i + n^{-2k} \sum_{i=1}^{n} i^{2(r+k)-1} \varepsilon_i = O(\varphi(n^{-2})).$$

Theorem 4 has the following corollaries:

THEOREM 4′. *$H[\varepsilon] \subset C^r$, $r \in N$, if and only if*

$$\sum_{i=1}^{\infty} i^{2r-1} \varepsilon_i < \infty. \tag{1.2}$$

THEOREM 4″. *Let $r \in N$. Condition* (1.2) *is necessary and sufficient for*

$$H[\varepsilon] \subset W^r.$$

Remark 1. "Decreasing" may be replaced by "nonincreasing" everywhere in this paper.

Remark 2. Under the additional assumptions that $\varepsilon_n = n^{-2r} \varphi(n^{-2})$ and φ is a function of the type of the kth modulus of continuity, Theorem 4 was proved in [11].

2. Proof of Theorem 3

In the sequel always $x \in [0, 1]$.

Fix $m \in N$. We will denote by c various positive constants depending on m only.

Observing that for $l \in N$ the function $\sin^2(l \arcsin \sqrt{x})$ is an algebraic polynomial of degree l, we define, for $n \in N$, the algebraic polynomials

$$T_n(x) := \sin^{2(m+2)}\left(\left[\frac{n}{m+2}\right] \arcsin \sqrt{x}\right), \tag{2.1}$$

where $[a]$ is the integer part of a.

Note the following properties of these polynomials:

(a) $\deg T_n \leqslant n$.

(b) For all x

$$0 \leqslant T_n(x) \leqslant 1. \tag{2.2}$$

(c) For $n \leqslant 1/\sqrt{x}$

$$T_n(x) \leqslant c x^{m+2} n^{2(m+2)}. \tag{2.3}$$

(d) For $m+2 \leqslant n \leqslant 1/\sqrt{x}$

$$T_n(x) \geqslant cx^{m+2} n^{2(m+2)}. \tag{2.4}$$

(e) For $j = 0, 1, \ldots, m+1$

$$T_n^{(j)}(0) = 0. \tag{2.5}$$

Let $\alpha = \alpha(x)$ be a continuous increasing function on $[0, 1]$ such that $\alpha(n^{-2}) = \varepsilon_n$ for all $n \in N$. In the sequel, we denote by a various positive constants which depend on α and m only (unlike the constants c, which depend only on m).

We define the function

$$\beta(x) := \sum_{i=1}^{\infty} i^{-3} \varepsilon_i T_i(x) \equiv \sum_{i=m+2}^{\infty} i^{-3} \varepsilon_i T_i(x) \tag{2.6}$$

and the polynomials

$$P_n(x) := \sum_{i=1}^{n} i^{-3} \varepsilon_i T_i(x). \tag{2.7}$$

LEMMA 1. (a) *For all* x

$$0 \leqslant \beta(x) - P_n(x) \leqslant cx\varepsilon_n, \qquad n \in N. \tag{2.8}$$

(b) *For all* x

$$\beta(x) \geqslant ax\alpha(x), \tag{2.9}$$

and for $x \leqslant (m+2)^{-2}$

$$\beta(x) \geqslant cx\alpha(x). \tag{2.10}$$

Proof. (a) Let first $x > (n+1)^{-2}$. Then

$$\beta(x) - P_n(x) = \sum_{i=n+1}^{\infty} i^{-3} \varepsilon_i T_i(x) \leqslant \varepsilon_{n+1} \sum_{i=n+1}^{\infty} i^{-3}$$
$$< \tfrac{1}{2} \varepsilon_{n+1} n^{-2} \leqslant \tfrac{1}{2} \varepsilon_n n^{-2} < 2x\varepsilon_n.$$

Let now $x \leqslant (n+1)^{-2}$. Choose $n_0 \in N$ satisfying $(n_0+1)^{-2} < x \leqslant n_0^{-2}$ and note that by (2.3)

$$\sum_{i=n+1}^{n_0} i^{-3} T_i(x) \leqslant cx^{m+2} \sum_{i=n+1}^{n_0} i^{2m+1} \leqslant cx^{m+2} n_0^{2(m+1)} \leqslant cx.$$

Further,

$$\beta(x) - P_n(x) \leqslant \varepsilon_n \sum_{i=n+1}^{n_0} i^{-3} T_i(x) + \varepsilon_n \sum_{i=n_0+1}^{\infty} i^{-3} T_i(x)$$
$$\leqslant c\varepsilon_n x + \tfrac{1}{2} \varepsilon_n n_0^{-2} \leqslant c\varepsilon_n x.$$

The inequality $\beta(x)-P_n(x)\geqslant 0$ being obvious, the proof of (a) is complete.

(b) We first prove (2.10). Let therefore $x\leqslant(m+2)^{-2}$. We choose $n_0\in N$ satisfying $(n_0+1)^{-2}<x\leqslant n_0^{-2}$ and obtain, by (2.4),

$$\beta(x)\geqslant \varepsilon_{n_0}x^{m+2}\sum_{i=m+2}^{n_0} i^{2m+1}\geqslant c\varepsilon_{n_0}x^{m+2}n_0^{2m+2}\geqslant c\varepsilon_{n_0}x\geqslant cx\alpha(x).$$

This completes the proof of (2.10), and also of (2.9) for $x\leqslant(m+2)^{-2}$. On the other hand, if $x>(m+2)^{-2}$, then

$$(m+2)^3\beta(x)\geqslant \varepsilon_{m+2}T_{m+2}(x)=\varepsilon_{m+2}x^{m+2}\geqslant cx\varepsilon_{m+2}\geqslant ax\alpha(x),$$

which completes the proof of Lemma 1.

Set ([10], [6], p. 168)

$$F(x):=\frac{1}{(m-1)!}\int_x^1 u^{-m-2}\beta(u)x(x-u)^{m-1}\,du. \tag{2.11}$$

LEMMA 2. *For $n\in N$*

$$E_n(F)\leqslant c\varepsilon_n. \tag{2.12}$$

Proof. Define

$$Q_n(x):=\frac{1}{(m-1)!}\int_x^1 u^{-m-2}P_n(u)x(x-u)^{m-1}\,du.$$

By (2.5), Q_n is an algebraic polynomial of degree $\leqslant n-1$ with $Q_n(0)=0$. Now, for $x=0$ we have $|F(0)-Q_n(0)|=0$, and for $x\neq 0$, by (2.8),

$$|F(x)-Q_n(x)|\leqslant c\int_x^1 u^{-m-2}u\varepsilon_n x(u-x)^{m-1}\,du\leqslant cx\varepsilon_n\int_x^1 u^{-2}\,du\leqslant c\varepsilon_n,$$

which finishes the proof.

LEMMA 3. *Let $k\in N$, $k\leqslant m$. Write $r:=m-k$.*

(a) *If $\int_0^1 ru^{-r-2}\beta(u)\,du<\infty$, then $F\in C^r$, but for $h\in(0,1/k]$*

$$\begin{aligned}\Delta_h^k(F^{(r)},0)=mh\int_0^h\dots\int_0^h\int_{h+u_1+\dots+u_{k-1}}^{kh}\beta(u_k)u_k^{-m-2}\,du_k\,du_{k-1}\dots du_1\\ +rh\int_0^h\dots\int_0^h\int_h^{h+u_1+\dots+u_{k-1}}(h+u_1+\dots+u_{k-1})^{-k}\\ \times\beta(u_k)u_k^{-r-2}\,du_k\,du_{k-1}\dots du_1\end{aligned} \tag{2.13}$$

$$+mh^k \int_{kh}^{1} \beta(u)u^{-m-2}\,du + \frac{1}{k!}\int_0^h r\beta(u)u^{-r-2}\,du$$

$$\geqslant mh^k \int_{kh}^{1} \beta(u)u^{-m-2}\,du + \frac{1}{k!}\int_0^h r\beta(u)u^{-r-2}\,du.$$

(b) *If*

$$\int_0^1 r\beta(u)u^{-r-2}\,du = \infty, \tag{2.14}$$

then $F \notin C^r$.

Proof. (a) Let

$$\int_0^1 r\beta(u)u^{-r-2}\,du < \infty. \tag{2.15}$$

If $r = 0$, then $F \in C^0$ by Lemma 2. If $r \neq 0$, then

$$\lim_{x\to 0} F^{(r)}(x) = \frac{(-1)^k}{k!}\int_0^1 r\beta(u)u^{-r-2}\,du = F^{(r)}(0). \tag{2.16}$$

To see this, note first that

$$\lim_{x\to 0} x\int_x^1 \beta(u)u^{-r-3}\,du = 0. \tag{2.17}$$

Indeed, for any $\varepsilon > 0$ choose $\delta_1 > 0$ so that $\int_0^{\delta_1} \beta(u)u^{-r-2}\,du < \varepsilon$. Now choose $\delta \in (0, \delta_1]$ so that $\delta\int_{\delta_1}^1 \beta(u)u^{-r-3}\,du < \varepsilon/2$. This gives for $x \in (0, \delta)$

$$x\int_x^1 \beta(u)u^{-r-3}\,du = x\int_x^{\delta_1} \frac{1}{u}\beta(u)u^{-r-2}\,du + x\int_{\delta_1}^1 \beta(u)u^{-r-3}\,du$$

$$\leqslant \frac{x}{x}\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Now (2.16) follows from (2.17), the inequality $x \leqslant u$ and the identity

$$F^{(r)}(x) = \frac{1}{k!}\int_x^1 (x-u)^{k-1}(mx-ru)\beta(u)u^{-m-2}\,du, \qquad x \neq 0. \tag{2.18}$$

We now prove (2.13). To do this, we introduce the functions

$$F_1(x) := \frac{m}{k!}\int_x^{kh} x(x-u)^{k-1}\beta(u)u^{-m-2}\,du, \qquad F_2(x) := x^{-1}F_1(x),$$

$$F_3(x) := -\frac{r}{k!}\int_x^h (x-u)^{k-1}\beta(u)u^{-m-1}\,du, \qquad F_4(x) := x^{-1}F_3(x), \qquad x \neq 0.$$

We extend F_1 and F_2 by continuity to $x = 0$ and note that $F_1(0) = 0$, $F_3(0) = F^{(r)}(0)$. Now, since $\Delta_h^k(x^j, 0) = 0$ for $j = 1, \ldots, k-1$ and $\Delta_h^k(x^k, 0) = k!\,h^k$, we obtain

$$\Delta_h^k(F^{(r)}, 0) = \Delta_h^k(F_1, 0) + mh^k \int_{kh}^{1} \beta(u)\, u^{-m-2}\, du + \Delta_h^k(F_3, 0).$$

Furthermore,

$$\begin{aligned}
\Delta_h^k(F_1, 0) &= (-1)^k F_1(0) + kh\Delta_h^{k-1}(F_2, h)\\
&= kh\Delta_h^{k-1}(F_2, h)\\
&= kh \int_0^h \ldots \int_0^h F_2^{(k-1)}(h+u_1+\ldots+u_{k-1})\, du_{k-1}\ldots du_1\\
&= mh \int_0^h \ldots \int_0^h \int_{h+u_1+\ldots+u_{k-1}}^{kh} \beta(u_k)\, u_k^{-m-2}\, du_k\, du_{k-1}\ldots du_1 \geqslant 0.
\end{aligned}$$

Analogously,

$$\begin{aligned}
\Delta_h^k(F_3, 0) &= (-1)^k F_3(0) + kh\Delta_h^{k-1}(F_4, h)\\
&= \frac{1}{k!} \int_0^h r\beta(u)\, u^{-r-2}\, du\\
&\quad + rh \int_0^h \ldots \int_0^h \int_h^{h+u_1+\ldots+u_{k-1}} (h+u_1+\ldots+u_{k-1})^{-k}\\
&\qquad\qquad \times \beta(u_k)\, u_k^{-r-2}\, du_k\ldots du_1,
\end{aligned}$$

which completes the proof of (a).

(b) Assume (2.14) holds (of course $r \neq 0$). Suppose to the contrary that $F \in c^r$. Then for $h \in (0, 1/2r]$

$$|\Delta_h^r(F, 0)| \leqslant ah^r,$$

but

$$\begin{aligned}
|\Delta_h^r(F, 0)| &= rh \int_0^h \ldots \int_0^h \int_{h+u_1+\ldots+u_{r-1}}^{1} \frac{(u_r-h-u_1-\ldots-u_{r-1})^{m-r}\beta(u_r)}{(m-r)!\, u_r^{m+2}}\, du_r\ldots du_1\\
&\geqslant ch^r \int_{2rh}^{1} \beta(u)\, u^{-r-2}\, du.
\end{aligned}$$

This completes the proof of Lemma 3.

Finally, Theorem 3 follows from Lemmas 2 and 3, from the estimate (2.9) and from the equiconvergence of the integrals

$$\int_0^1 r\beta(u)\, u^{-r-2}\, du \quad \text{and} \quad \int_0^1 r\alpha(u)\, u^{-r-1}\, du.$$

3

We will denote by c_i positive constants independent of n and t. We write $A(n, t) \sim B(n, t)$ if $c_1 A(n, t) \leqslant B(n, t) \leqslant c_2 A(n, t)$. Set

$$\tilde{W}^r H_k^\varphi := \{f\colon\ \omega_k(f^{(r)}, t) \sim \varphi(t)\}, \qquad k \in N,\ r+1 \in N.$$

Apart from the monotonicity and continuity of the function φ, we clearly have to assume here that $t^{-k}\varphi(t)$ is nonincreasing for $t > 0$.

Let $\alpha = r+\beta$, where $r+1 \in N$, $\beta \in (0, 1)$. S. N. Bernstein [2] proved that

$$E_n(x^\alpha) \sim (1/n^2)^\alpha. \tag{3.1}$$

Note that $x^\alpha \in \tilde{W}^r H_1^{t^\beta} = \ldots = \tilde{W}^0 H_{r+1}^{t^\alpha}$.

I. I. Ibragimov [8] proved that

$$E_n(x^m \ln x) \sim (1/n^2)^m, \qquad m \in N. \tag{3.2}$$

Note that $x^m \ln x \in \tilde{W}^{m-1} H_2^t = \ldots = \tilde{W}^0 H_{m+1}^{t^m}$. Ibragimov [8] also obtained such relations for functions of a more general type, belonging to other classes $\tilde{W}^r H_k^\varphi$.

We will write $\varphi \in S(r, k)$ (see e.g. [1]) if

$$\int_0^t r u^{-1} \varphi(u)\, du + t^k \int_t^1 u^{-k-1} \varphi(u)\, du \leqslant c_3 \varphi(t), \qquad t \in (0, \tfrac{1}{2}]. \tag{3.3}$$

The following result generalizes (3.1) and (3.2).

THEOREM 5. *For every $k \in N$, every r with $r+1 \in N$ and every $\varphi \in S(r, k)$ there exists $f \in \tilde{W}^r H_k^\varphi$ such that $E_n(f) \sim n^{-2r}\varphi(n^{-2})$.*

Proof. In [11] we constructed a function $f \in \tilde{W}^r H_k^\varphi$ such that

$$E_n(f) \leqslant n^{-2r}\varphi(n^{-2}) \qquad \text{for all } n \in N.$$

Therefore it suffices to show that if $\varphi \in S(r, k)$, then

$$E_n(f) \geqslant c_4 n^{-2r}\varphi(n^{-2}). \tag{3.4}$$

Let $\alpha(t)$ denote a nondecreasing continuous function on $[0, 1]$ such that $\alpha(n^{-2}) = E_n(f)$. Obviously, $\alpha(t) \leqslant c_5 t^r \varphi(t)$; to simplify the writing we assume $c_5 = 1$. It follows from Markov's inequality that

$$\varphi(t) \leqslant c_6 \left(\int_0^t \frac{r\alpha(u)}{u^{r+1}}\, du + t^k \int_t^1 \frac{\alpha(u)}{u^{r+k+1}}\, du \right), \qquad t \in (0, \tfrac{1}{2}].$$

Reasoning as N. K. Bari and S. B. Stechkin [1] we will show that if $\alpha(t) = a^{r+k} t^r \varphi(t)$ at some point $t \in (0, \frac{1}{2}]$ (where $a = \text{const}$, $0 < a < 1$), then for

$t_* := \sqrt{a}\,t$ we have

$$\varphi(t_*) \leqslant c_6\Big(\int_0^{t_*} r u^{-r-1}\alpha(u)\,du + t_*^k \int_{t_*}^1 u^{-r-k-1}\alpha(u)\,du\Big) \tag{3.5}$$
$$\leqslant 8c_6 c_3^2 \varphi(t_*)\ln^{-1}(1/a).$$

Indeed, by (3.3),

$$r\varphi(\sqrt{a}\,t_*)\ln\frac{1}{\sqrt{a}} \leqslant \int_{\sqrt{a}t_*}^{t_*} r u^{-1}\varphi(u)\,du \leqslant c_3\,\varphi(t_*),$$

$$(t_*/t)^k\varphi(t)\ln\frac{1}{\sqrt{a}} \leqslant t_*^k \int_{t_*}^{t} u^{-k-1}\varphi(u)\,du \leqslant c_3\,\varphi(t_*),$$

whence

$$r\varphi(\sqrt{a}\,t_*) \leqslant 2c_3\,\varphi(t_*)\ln^{-1}(1/a), \qquad \varphi(t) \leqslant 2c_3\,\varphi(t_*)a^{-k/2}\ln^{-1}(1/a).$$

Therefore

$$\int_0^{t_*} r u^{-r-1}\alpha(u)\,du + t_*^k\int_{t_*}^1 u^{-r-k-1}\alpha(u)\,du$$
$$\leqslant \int_0^{\sqrt{a}t_*} r u^{-1}\varphi(u)\,du + \int_{\sqrt{a}t_*}^{t_*} r u^{-r-1} a^{r+k} t^r \varphi(t)\,du$$
$$+ t_*^k\int_{t_*}^{t} u^{-r-k-1} a^{r+k} t^r\varphi(t)\,du + t_*^k\int_t^1 u^{-k-1}\varphi(u)\,du$$
$$\leqslant rc_3\,\varphi(\sqrt{a}\,t_*) + (t/t_*)^r a^{-r/2} a^{r+k}\varphi(t) + (t/t_*)^r a^{r+k}\varphi(t)$$
$$+ c_3(t/t_*)^k\varphi(t)$$
$$\leqslant 8c_3^2\,\varphi(t_*)\ln^{-1}(1/a).$$

We have thus proved (3.5), and hence also (3.4), with the constant $c_4 = \exp(-8(r+k)c_3^2 c_6)$. This completes the proof of Theorem 5.

Remark. For $k = 1$ and $r = 0$, Theorem 5 was obtained by E. P. Dolzhenko and E. A. Sevast'yanov [4].

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