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CONTACT STRUCTURES ON $(n-1)$-CONNECTED $(2n+1)$-MANIFOLDS*

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1. Contact and symplectic structures

Let M^{2n+1} be an orientable C^{∞}-manifold. A globally defined 1-form ω is said to be a *contact form* if $\omega \wedge (d\omega)^n \neq 0$. Such a form determines complementary distributions of dimensions $2n$ and 1 in the tangent bundle TM^{2n+1}, the former being called the *contact distribution* D associated to ω,

$$D_x = \{x \in T_x M^{2n+1}:\ \omega_x(X) = 0\}.$$

If the subbundle D is orientable (and this is always the case if n is even), then TM^{2n+1}/D is an orientable real line bundle, and M^{2n+1} admits a nowhere vanishing vector field ξ such that

$$\omega(\xi) = 1 \quad \text{and} \quad d\omega(\xi, X) = 0 \text{ for all vector fields } X \text{ on } M^{2n+1}.$$

ξ is called the *characteristic vector field* of the contact form.

Note that the condition $\omega \wedge (d\omega)^n \neq 0$ implies that D is not integrable, indeed in a certain sense the theory of contact structures is complementary to that of codimension one foliations. Classically contact forms arise as natural structures on constant energy levels of a Hamiltonian system (see Example 1 in Section 2 below).

THEOREM 1.1 (Darboux). *If (M^{2n+1}, ω) is a contact manifold each point $x \in M^{2n+1}$ belongs to a chart $U(x_1, x_2, \ldots, x_{2n+1})$ such that*

$$\omega|U = x_1\,dx_2 + x_3\,dx_4 + \ldots + x_{2n-1}\,dx_{2n} - dx_{2n+1}.$$

For a proof see [Go], VI, Théorème 4.1.

This result suggests an alternative definition of a contact manifold. The local diffeomorphism f of a neighbourhood of the origin $O \in R^{2n+1}$ is said to belong to the contact pseudogroup if $f^*\omega = \lambda\omega$, λ some non-vanishing

* The final version of this paper will appear as part of a monograph on contact geometry.

locally defined C^∞-function and $\omega = \sum_{j=1}^{n} x_{2j-1}\,dx_{2j} - dx_{2n+1}$. Then our orientability assumptions imply that (M^{2n+1}, ω) is a contact manifold if and only if M admits an atlas such that the coordinate transformations belong to the contact pseudogroup. Since in this paper we will be concerned with highly connected manifolds, there is no loss of generality in taking the two definitions to be equivalent.

Darboux' Theorem implies that locally all contact forms are equivalent; globally this is not the case, and we shall give examples below of contact forms on S^{2n+1} which induce distinct almost contact structures on the tangent bundle.

DEFINITION. M^{2n+1} has an *almost contact structure* if the structural group of its tangent bundle may be reduced to $U(n)\oplus 1$.

If a global contact form ω exists, the decomposition of TM^{2n+1} into complementary subbundles of dimensions $2n$ and 1 shows that the structural group reduces from $SO(2n+1)$ to $SO(2n)\oplus 1$. The further reduction to $U(n)\oplus 1$ is obtained by restricting ω to a standard chart, and observing that the coordinate transformations for the subbundle D inside any overlap must be compatible with the matrix $\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$. Therefore such a coordinate transformation has image of the form $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$, and lies in the unitary subgroup of $SO(2n)$.

The existence of such reductions from $SO(2n+1)$ to $U(n)\oplus 1$ is therefore necessary for the existence of a contact form on M^{2n+1}. In general the sufficiency of this condition, that is the compatibility of a family of locally defined forms, is a hard problem in partial differential equations. Our aim in this paper is to provide evidence that, at least for a large class of highly connected manifolds, the condition *is* sufficient. Since our method is topological we are actually proving that for certain manifolds M^{2n+1} there exist *some* reduction of the structural group of TM^{2n+1} for which the integrability conditions for the associated system of partial differential equations are satisfied.

The even dimensional analogue of contact structure is provided by pairs (M^{2n}, Ω) where Ω is a globally defined 2-form such that

(i) $d\Omega = 0$ (Ω is closed) and

(ii) $\Omega^n \neq 0$.

Such a form is called *symplectic*, is locally unique (see Godbillon [Go], op. cit.), and induces an almost complex structure on the tangent bundle TM^{2n}. One obtains an important family of examples as follows: Let $P_n(C)$ be complex projective space with the usual atlas of complex charts

$U_j = \{[\underline{z}]: z_j \neq 0\}$, $j = 0, 1, \ldots, n$, and define real valued functions f_j: $C^n \to \boldsymbol{R}$ by

$$f_j(\underline{w}) = \log\left(1 + \sum_{r=1}^{n} |w_r|^2\right).$$

By composition we obtain a family of functions K_j: $U_j \to \boldsymbol{R}$, $j = 0, \ldots, n$, which by an easy calculation are such that on $U_i \cap U_j \subseteq P_n(\boldsymbol{C})$, the complex 2-form $\tilde{\omega} = -2i\partial\bar{\partial}K_j$ is well defined. If we write $\tilde{\omega}$ in the form $\tilde{\omega} = -2i \sum_{\alpha,\beta=1}^{n} h_{\alpha\beta}\, dz_\alpha \wedge d\bar{z}_\beta$, then $h_{\alpha\bar{\beta}}$ is positive definite Hermitian, $d\tilde{\omega} = 0$, and the real part of $\tilde{\omega}$ defines the (Fubini–Study) symplectic structure on $P_n(\boldsymbol{C})$. One can further show that this structure is integral in the sense that the cohomology class $[\Omega] = [\mathrm{Re}\tilde{\omega}]$ belongs to the image of $H^2(M, \boldsymbol{Z})$ in $H^2_{DR}(M, \boldsymbol{R})$. By naturality such an integral symplectic form is inherited by any projective algebraic variety contained in $P_n(\boldsymbol{C})$.

We will next describe one way of relating contact and symplectic forms; if (M^{2n+1}, ω) is contact, then all the Stiefel–Whitney numbers of M^{2n+1} vanish, and $M^{2n+1} = \partial N^{2n+2}$ (see [Gr], Thm. 2.3.2). In a collar neighbourhood of the boundary $M \times [0, 1) \subseteq N$ with normal coordinate t, we may locally extend ω to $(1-t)\,dx_{2n+1} + \sum_{j=1}^{n} x_{2j-1}\,dx_{2j}$. The local exterior derivative of the extended form is closed, takes the form $-dt \wedge dx_{2n+1} + \sum_{j=1}^{n} dx_{2j-1} \wedge dx_{2j}$, and hence satisfies the symplectic condition. Thus any contact manifold is the boundary of some, not necessarily closed, symplectic manifold. If M^{2n+1} is actually the boundary of some compact $(2n+2)$-manifold N such that TN has structural group $U(n+1)$, then by an argument of M. Gromov the symplectic form in the collar extends to $N - D^{2n+2}$; the possibility of extension over the final disc depends on questions of integrability already alluded to. Hence if certain algebraic conditions are fulfilled (for example on the Chern classes of $TM^{2n+1} \oplus 1$) it is possible to regard (M^{2n+1}, ω) as a submanifold of (N^{2n+2}, Ω). Suppose conversely that (N^{2n+2}, Ω) is given, and that M^{2n+1} is a smoothly embedded codimension one (orientable) submanifold such that the restriction of the 2-dimensional class $[\Omega] \in H^2_{DR}(N, \boldsymbol{R})$ vanishes on M, e.g. if M is at least 2-connected. Then in the neighbourhood of M, Ω is the exterior derivative of some 1-form η, which induces a contact form on M, provided that there exists a family of coordinate charts φ_l: $U_l \to \boldsymbol{R}^{2n+2}$, such that the pair $(\varphi_l U_l \subseteq \boldsymbol{R}^{2n+2}, \varphi_l(M \cap U_l)) = (V^{2n+2}, W^{2n+1})$ has the property that no tangent space to W contains the origin of coordinates in $V \subseteq \boldsymbol{R}^{2n+2}$, $(*)$. This is a transversality condition, and it is at very least plausible to suppose that by slightly perturbing the embedding of M^{2n+1} it can always be satisfied. For a proof that the condition (*) is sufficient for the existence of a contact form see [B1], page 8.

Dually to this embedding problem we may try and submerse (M^{2n+1}, ω) in (N^{2n}, Ω). One way round we assume that we are given the pair (N^{2n}, Ω) such that there exists a submersion $p: M^{2n+1} \to N^{2n}$ satisfying the algebraic condition $p^*[\Omega] = 0$ in $H^2_{DR}(M, \boldsymbol{R})$. Then if the complementary line bundle to $p^! TN^{2n}$ in TM^{2n+1} is trivial, that is TM^{2n+1} has structural group $U(n) \oplus 1$ there exists an open subset U of M^{2n+1} admitting a contact form ω such that $p^* \Omega = d\omega$. In one important special case (see Example 2 in Section 2) below this method of construction leads to a contact form defined everywhere on M. Conversely, given (M^{2n+1}, ω) there exists a submersion into a symplectic manifold (N^{2n}, Ω), provided that ω satisfies a suitably strong geometric condition, for example ω is *regular* in the sense that the characteristic field ξ is regular. This means that each point $x \in M^{2n+1}$ has a cubical coordinate neighbourhood U such that each integral curve of ξ passes through U once only as a line segment parallel to the x_{2n+1} axis.

For a more leisurely introduction to the basic ideas of contact geometry we refer the reader to the book of C. Godbillon [Go] and to the notes by D. Blair [Bl]. The former lucidly explains the relation of the geometric ideas to classical mechanics, and the proof of Darboux' Theorem is strongly recommended. Blair's notes are more geometrical in flavour, and also provide an introduction to otherwise rather inaccessible Japanese work on the relation between contact and almost contact structure. This is an aspect of the subject, on which we do not touch in the sections that follow.

2. Methods of construction

In this section we explain a number of constructions, which will be later used to show that a "prime" $(n-1)$-connected $(2n+1)$-manifold satisfying the necessary tangential condition often supports a contact form. We can then use the connected sum theorem (2.3) to combine different methods of construction in order to obtain a theorem valid for composite manifolds.

(1) *The cotangent sphere bundle*

Let T^*M^{n+1} be the total space of the cotangent bundle of an arbitrary $(n+1)$-dimensional manifold with coordinates $(p_1 \dots p_{n+1})$ in the fibre and $(q_1 \dots q_{n+1})$, $q_i = x_i \circ(\text{projection})$ $i = (1, \dots, n+1)$ in the base. The 1-form

$$\beta = \sum_{i=1}^{n+1} p_i \, dq_i$$

restricted to the subbundle of unit cotangent vectors is a contact form, since each fibre is homeomorphic to S^n, and no tangent space to a point in the fibre contains the origin of the fibre in T^*M. Here we have applied condition (*) from Section 1.

(2) *Regular contact manifolds*

THEOREM 2.1. *Let N^{2n} be a symplectic manifold such that the cohomology class $[\Omega]$ of the defining form is integral. If M^{2n+1} is the total space of the S^1-bundle over N^{2n} with Chern class equal to $[\Omega]$, M^{2n+1} admits a regular contact form ω such that $p^*\Omega = d\omega$. Here p: $M^{2n+1} \to N^{2n}$ denotes the bundle projection.*

Proof. Cover N^{2n} with a family of open discs U_i, $i \in I$, over each of which the bundle is trivial. On U_i there is a 1-form η_i such that $\Omega|U_i = d\eta_i$, consider the 1-form on $p^{-1}U_i \cong S^1 \times U_i$ given by $dt + p^*\eta_i$. Since $p^*[\Omega] = 0$ by definition of the Chern class these local 1-forms may be chosen to be compatible, that is they combine to give a global 1-form ω on the total space M^{2n+1}. Since $d(dt + p^*\eta_i) = d(p^*\eta_i) = p^*(d_{\eta_i}) = p^*(\Omega|U_i)$, $p^*\Omega = d\omega$. The contact condition is local, and hence may be checked using the form $dt + p^*\eta_i$,

$$(dt + p^*\eta_i) \wedge p^*(d\eta_i)^n = dt \wedge p^*(d\eta_i)^n + p^*\big(\eta_i \wedge (d\eta_i)^n\big).$$

The second term on the right vanishes because N has dimension $2n$, and the first term defines a volume form for the product $S^1 \times U_i$. The form ω is regular, since the flow lines of its characteristic vector field are the fibres S^1 of M^{2n+1}.

The converse to this theorem is also true. If (M^{2n+1}, ω) is regular then some multiple $\lambda\omega$, for some non-vanishing C^∞-function λ, has a characteristic vector field whose flow lines are the orbits of a principal S^1-action. The orbit manifold $M^{2n+1}/S^1 = N^{2n}$ is symplectic, and the (integral) defining form Ω satisfies $p^*\Omega = d\omega$. This result is intuitively clear, for full details see either [Bl], page 14 or [B–W].

In Section 3 below we shall give examples to show that Theorem 2.1 and its converse extends to odd-dimensional manifolds which are Seifert fibered over a suitable symplectic manifold, that is we allow an action of S^1 to have finitely many finite isotropy subgroups. This leads to the definition of a quasi-regular contact form as one whose characteristic vector field has flow lines meeting a suitable cubical coordinate neighbourhood in at most finitely many line segments parallel to the x_{2n+1}-axis.

(3) *3-manifolds as open books*

It is a classical result of Alexander that if M^3 is a closed, connected, orientable 3-manifold then there exists a surface P^2 bounded by single copy of S^1, and a diffeomorphism h: $P^2 \to P^2$ (equal to the identity near ∂P), such that

$$M^3 = (D^2 \times S^1) \bigcup_{S^1 \times S^1} P(h).$$

Here $P(h)$ is the mapping torus of h; the decomposition is called an "open book" with pages the fibres of $P(h)$ and spine the axis $\{0\} \times S^1$ of the solid torus.

THEOREM 2.2. *The manifold just described admits a contact form.*

Proof. Let $d\eta$ be a volume form for the *bounded* surface P. Since the dimension is 2, $d\eta$ is also symplectic. Let $S^1 \times [1, 1+\varepsilon)$ be a collar of the boundary of P with collar coordinate r, and using a partition of unity extend the 1-form $rd\theta$ to all of P in such a way that it vanishes outside the collar. By means of a suitable smooth function λ, and using the convexity of symplectic forms in dimension 2, we may replace η by

$$\eta_1 = (1-\lambda)\eta + \lambda \cdot rd\theta,$$

a 1-form which agrees with $rd\theta$ inside some possibly smaller collar of the boundary and which is still such that $d\eta_1$ is symplectic. Again using convexity there is a 1-form η_2 on the mapping torus which restricts to η_1 on each fibre, and for a suitably large value of K

$$\omega_2 = \eta_2 + Kd\varphi,$$

where φ is the angular coordinate in the base space S^1 of $P(h)$, satisfies the contact condition. This defines the contact form on the union of the pages of the book. Near the spine take $\omega_1 = r^2 d\varphi + d\theta$, and join these two forms by

$$\omega = f_1(r)d\theta + f_2(r)d\varphi,$$

where $f_1(r) = 1$, $f_2(r) = r^2$ $(r \leqslant \varepsilon)$ and $f_1(r) = r$, $f_2(r) = K$ $(r \geqslant 1)$. An easy calculation shows that ω also satisfies the contact condition if $f_1 f_2' - f_1' f_2 \neq 0$, a condition which can clearly be fulfilled.

This argument is due to W. Thurston and E. Winkelnkemper. Although we will not attempt to extend it to higher dimensions in this paper (but see [Th] for an outline of a proof in dimension 5), the number of manifolds which admit "open book" decompositions is large, and it should be regarded as a potentially useful method.

We now come to perhaps the most important result in this section.

THEOREM 2.3 (C. Meckert). *If (M^{2n+1}, ω), (M'^{2n+1}, ω') are contact manifolds the connected sum $M \# M'$ admits a contact form ω'' equal to ω on $M - U$ and to ω' on $M' - U'$, where U and U' are sufficiently small open sets containing the connecting sphere S^{2n}.*

Sketch proof, for the *full* details see [M]. In order to form the connected sum delete the interiors of n-discs $D^{2n+1} \subseteq M^{2n+1}$, $D'^{2n+1} \subseteq M'^{2n+1}$ and identify the resulting copies of S^{2n} by means of an orientation reversing diffeomorphism h.

We may suppose that $D \subseteq U$, $D' \subseteq U'$, where U and U' are so small that

$$\omega|U = \sum_{j=1}^{n} (x_{2j-1}\,dx_{2j} - \tfrac{1}{4}x_{2j}\,dx_{2j-1}) + dx_{2n+1},$$

$$\omega'|U = \sum_{j=1}^{n} y_{2j-1}\,dy_{2j} - dy_{2n+1}.$$

The first form being obtained by a modification of the proof of Darboux' Theorem.

Write $r = (x_1^2 + \ldots + x_{2n+1}^2)^{1/2}$, $\lambda(r) = r^4$, and define the glueing diffeomorphism $h \in \mathrm{Diff}(\boldsymbol{R}^{2n+1} \setminus \{0\})$ by

$$h(x_i) = y_i, \qquad \text{with } y_i = \frac{x_i}{r^2},\ i = 1, 2, \ldots, 2n+1.$$

Consider the sequence of spherical shells in $\boldsymbol{R}^{2n+1}$ defined by the sequence of radial distances

$$r_0 = \tfrac{3}{8} < \tfrac{1}{2} < \tfrac{3}{4} < 3 < \tfrac{10}{3} < \tfrac{11}{3} < r_1 = 4.$$

We shall show that on the union of these shells, bounded by the spheres of radii $\frac{3}{8}$ and 4, there is a contact form equal to $r^4 h^* \omega'$ near S_0 and to ω near S_1. This will be enough for the construction of ω'' on $M \# M'$.

(o) $\frac{3}{8} \leqslant r \leqslant \frac{1}{2}$, $\omega_0 = r^4 h^* \omega' = \sum_{j=1}^{n} \left(x_{2j-1}\,dx_{2j} - \frac{2}{r} x_{2j-1} x_{2j}\,dr \right)$

$$-r^2\,dx_{2n+1} + 2x_{2n+1}\,r\,dr.$$

(i) $\frac{1}{2} \leqslant r \leqslant \frac{3}{4}$, $\omega_1 = \sum_{j=1}^{n} [x_{2j-1}\,dx_{2j} - (r-\frac{1}{2})\,x_{2j}\,dx_{2j-1}]$

$$-r^2\,dx_{2n+1} + 2\left(x_{2n+1} - \frac{1}{r^2} \sum_{j=1}^{n} x_{2j-1} x_{2j} \right) r\,dr.$$

(ii) $\frac{3}{4} \leqslant r \leqslant 3$, $\omega_2 = \sum_{j=1}^{n} (x_{2j-1}\,dx_{2j} - \frac{1}{4} x_{2j}\,dx_{2j-1})$

$$-\tfrac{4}{9}(3-r)\left[r^2\,dx_{2n+1} + \frac{2}{r^2} \sum_{j=1}^{n} x_{2j-1} x_{2j}\,r\,dr \right] + 2x_{2n+1}\,r\,dr.$$

(iii) $3 \leqslant r \leqslant \frac{10}{3}$, $\omega_3 = \sum_{j=1}^{n} (x_{2j-1}\,dx_{2j} - \frac{1}{4} x_{2j}\,dx_{2j-1}) + 2x_{2n+1}\,r\,dr$

$$+3(r-3)\,dx_{2n+1}.$$

(iv) $\frac{10}{3} \leqslant r \leqslant \frac{11}{3}$, $\omega_4 = \sum_{j=1}^{n} (x_{2j-1}\,dx_{2j} - \frac{1}{4} x_{2j}\,dx_{2j-1}) + dx_{2n+1}$

$$+6(\tfrac{11}{3} - r)\, x_{2n+1}\, r\, dr.$$

(v) $\frac{11}{3} \leqslant r \leqslant 4$, $\omega_5 = \sum_{j=1}^{n} (x_{2j-1}\,dx_{2j} - \frac{1}{4} x_{2j}\,dx_{2j-1}) + dx_{2n+1}$.

It is easy to check that $\omega_i = \omega_{i+1}$ for $r = r_{i+1}$, $0 \leqslant i \leqslant 4$, and slightly harder to check that each form ω_i does satisfy the contact condition throughout its range of definition. This done, one has constructed ω'' as a C^0-form on $M \# M'$, and it remains to smooth ω'' at the spheres separating the various shells. In order to do this one defines a suitable smooth function $\varphi(r)$, $0 \leqslant \varphi \leqslant 1$, and considers the form

$$\varphi\omega_i + (1-\varphi)\,\omega_{i+1}.$$

Since the set of contact forms is not convex in dimensions $\geqslant 3$ it is far from obvious that this intermediate form defines a volume element. That this is however the case for the forms we have defined (note that $\omega_{i+1} - \omega_i = (r - a_i)\eta$, $a_i = 1/2, \ldots, 11/3$) follows by explicit calculation (see [M], "lemme de lissage").

3. The classification of $(n-1)$-connected $(2n+1)$-manifolds

In this section we first summarise the main results from C. T. C. Wall's paper [Wa], and then show how certain prime manifolds, that is manifolds which cannot be non-trivially decomposed as connected sums, may be represented as Brieskorn varieties. On such a variety it is an easy matter to write down an explicit contact form.

Let $n \geqslant 4$, $n \neq 7$, and for technical reasons suppose that $n \neq 0, 1 \pmod 8$. If M^{2n+1} is closed and $(n-1)$-connected, M admits a handle decomposition with one 0-handle, k n-handles, k $(n+1)$-handles and one $(2n+1)$-handle for some suitable value of k. (If the $(2n+1)$-handle is missing, we shall say that the resulting bounded manifold is almost-closed and write $\bar{M}$.) Hence M is the union of two handlebodies, one of which, N say, is obtained from D^{2n+1} by attaching k copies of $D^n \times D^{n+1}$ (corresponding to the generators of $\pi_n(M)$ to the boundary along $S^{n-1} \times D^{n+1}$. The complement N' consisting of $(n+1)$- and $(2n+1)$-handles is diffeomorphic to N, and hence the problem of classification has two parts:

(i) classify the handlebodies N, and

(ii) determine how automorphisms of the boundary of N give rise to different closed manifolds M. Looking more closely at this problem one arrives at the following list of invariants:

A. $H_n(M, Z)$ together with its quadratic structure. This consists of a nonsingular bilinear map b defined on the torsion subgroup taking values in $\boldsymbol{Q}/\boldsymbol{Z}$ with $b(x, x) \equiv 0 (n = \text{even})$, or of a quadratic map with associated bilinear map $2b (n = \text{odd})$.

B. Tangential invariants $\begin{cases} \alpha \in \operatorname{Hom}(H_n(M, Z), \pi_{n-1}(SO)) \\ \hat{\beta} \in H^{n+1}(M, \pi_n(SO)) = H_n \otimes \pi_n(SO) \\ \hat{\varphi} \in H^{n+1}(M, Z/2) \cong H_n \otimes Z/2 \ (n = \text{even}, n \neq 4, 8). \end{cases}$

The embedding of S_i^{n-1} in ∂D^{2n+1} associated with the generators of $H_n(M, Z) \cong \pi_n(M)$, $i = 1, \ldots, k$, has a canonically trivialised normal bundle. The map α compares this trivialisation with that inherited from $f_i(S^{n-1} \times D^{n+1})$. The invariant $\hat{\beta}$ similarly describes the *stable* framing of the $(n+1)$-handles; alternatively $\hat{\beta}$ arises in the discussion of C^∞-automorphisms of N (part (ii) of the general classification programme). These two invariants together determine the stable tangent bundle of the manifold M, since the Atiyah–Hirzebruch spectral sequence reduces to the short exact sequence

$$0 \to H^{n+1}(M, \pi_n(SO)) \to KO(M) \to H^n(M, \pi_{n-1}(SO)) \to 0.$$

Indeed the table below shows that, since we exclude values of $n = 0, 1 (\text{mod } 8)$ either α or $\hat{\beta}$ vanishes.

The third tangential invariant $\hat{\varphi}$ arises, because the framing of the $(n+1)$-handles actually involves a map taking values in the non-stable group $\pi_n(SO_n)$; the suspension map is bijective for odd values of n (we exclude $n = 7$) or if $n = 4, 8$. For other even values of n the kernel splits off as a direct summand of order 2, for many of these calculations see the paper of M. Kervaire [Ke].

The possible values for α and $\hat{\beta}$ lie in the groups listed in Table I below:

Table I

n	α	$\hat{\beta}$
2	$kZ/2$	0
3	0	kZ
4	kZ	0
5	0	0
6	0	0
7	0	kZ

We can now state the main result of [Wa], Theorem 7, page 284.

THEOREM 3.1. *If $n \geq 4$, $n \neq 7$, $n \neq 0, 1$ (8), the diffeomorphism classes of almost-closed $(n-1)$-connected $(2n+1)$-manifolds $\bar{M}^{2n+1}$ are in (1-1) correspondence with the sets of invariants* **A** *and* **B** *defined above. Moreover, if*

$\bar{M}$ and $\bar{M}'$ are two manifolds of the type considered the invariants of the (boundary) connected sum are the direct sum of those of $\bar{M}$, $\bar{M}'$.

One obtains a theorem for closed, as opposed to almost-closed manifolds by setting $\alpha = 0$; this amounts to confining attention to boundaries of the $(2n+2)$-disc with $(n+1)$-handles attached (see [Wa], Theorem 8). This is a restriction only if $n = 2, 4$.

Remarks: **1.** If $n \equiv 2, 4 \pmod 8$, α distinguishes between different S^{n+1} bundles over S^n, if $\alpha = 0$ we have $S^{n+1} \times S^n$.

2. If $n \equiv 3, 7 \pmod 8$, $\hat{\beta}$ is expressible in terms of the non-vanishing Pontrjagin class $p_{(n+1)/4}$.

3. The non-stable invariant $\hat{\varphi}$ distinguishes between the product $S^{n+1} \times S^n$ and the non-trivial S^n-bundle over S^{n+1}, which is the (co) tangent sphere bundle. The vanishing of $\hat{\varphi}$ corresponds to the triviality of $TS^{n+1} \oplus 1$.

If $n = 2$ the argument must be modified, see [Ba]. We include this special case, because it leads to a particularly elegant application of our methods.

THEOREM 3.2. *Two simply-connected 5-manifolds are diffeomorphic if and only if they have the same 2-dimensional **Z**-homology groups (plus bilinear structure). Such a manifold decomposes uniquely as a connected sum of prime manifolds M_k $\{1 \leqslant k \leqslant \infty\}$, which bound parallelisable manifolds, with possibly one extra summand X_j $\{j = -1$ or $1 \leqslant j \leqslant \infty\}$ with $w_2(X_j) \neq 0$.*

We shall produce models for the manifolds M_k below. Among the manifolds X_j only one has a $U(2) \oplus 1$ structure on its tangent bundle ($\delta w_2 = 0$ in $H^3(X_j, \mathbf{Z})$). This is X_∞ which has an open book decomposition with typical page equal to $P_2(\mathbf{C}) - D^4$ and identification map h equal to the identity.

In order to produce examples of highly connected manifolds we consider smooth actions of S^1 on M^{2n+1}, which are almost free in the sense that there is a non-trivial cyclic isotropy subgroup $\mathbf{Z}/\alpha$. We shall suppose further that $M^{2n+1}/S^1 = P_n(\mathbf{C})$ and that the projection is the composition of a principal bundle map and a branched covering map. Let the image of the exceptional orbits be a hypersurface L of degree δ in $P_n(\mathbf{C})$, at a point x of such an orbit the slice representation of the isotropy subgroup $\mathbf{Z}/\alpha$ is described by the matrix

$$\left(\begin{array}{c|c} e^{2\pi i v/\alpha} & 0 \\ \hline 0 & 1_{2n-2} \end{array}\right)$$

and we may define β uniquely by

$$0 < \beta < \alpha, \quad (\alpha, \beta) = 1 \quad \text{and} \quad v\beta \equiv 1 \pmod \alpha.$$

By a generalisation of the familiar argument in dimension 3 (see [O–W], Theorem 3.15). M^{2n+1} is determined up to orientation preserving equivariant diffeomorphism by the integral orbit invariants $(\delta; \alpha, \beta)$ with the proviso that for n even, $(\delta; \alpha, \beta) \sim (\delta; \alpha, \alpha-\beta)$. The total space M^{2n+1} is 1-connected if and only if $1 = \pm(\alpha+\delta\beta)$, and in this case the first non-zero homology group is

$$H_n(M, \mathbf{Z}) \cong \underbrace{\mathbf{Z}/\alpha \times \dots \times \mathbf{Z}/\alpha}_{\varkappa(n)}$$

where $\varkappa(\delta, n) = [(\delta-1)^{n+1}-(-1)^{n+1}]/\delta+(-1)^{n+1}$ ([O–W], Theorem 4.9). For small values of δ we have the following table:

Table II

δ \ n	n = odd	n = even	$n = 2$
1	0	0	0
2	1	0	0
3	$\frac{1}{3}(2^{n+1}-1)+1$	$\frac{1}{3}(2^{n+1}+1)-1$	2

Thus M^{2n+1} is a homotopy sphere if and only if $\delta = 1$ ($\delta = 1$ or 2) for n odd (n even).

Now consider the Brieskorn variety

$$V(a_0, \dots, a_{n+1}) = \{\mathbf{z} \in \mathbf{C}^{n+2}: f(\mathbf{z}) = z_0^{a_0} + \dots + z_{n+1}^{a_{n+1}} = 0\} \cap S^{2n+3}.$$

There is an S^1-action on V induced by $t(z_0, \dots, z_{n+1}) = (t^{d/a_0} z_0, \dots, t^{d/a_{n+1}} z_{n+1})$, for $t \in \mathbf{C}^\times$, $d =$ l.c.m. $(a_0 \dots a_{n+1})$, which gives V the structure of a Seifert fibration. Note that for unrestricted values of $a_0 \dots a_{n+1}$ there may be more than one non-principal orbit type.

It is easy to verify that the $(2n+1)$-dimensional C^∞-manifold $V(a_0, \dots, a_{n+1})$ admits the contact form

$$\omega = \tfrac{1}{2} i \left[\sum_{j=0}^{n+1} \frac{1}{a_j} (z_j \, d\bar{z}_j - \bar{z}_j \, dz_j) \right] \quad \text{(see [L–M]).}$$

Now consider the special case $a_0 = 2k+1$, $a_1 = a_2 = \dots = a_{n+1} = 2$, for which the orbit space is known to be $P_n(\mathbf{C})$ (compare the argument on page 158 of [O–W]), and for which the S^1-action has a single non-trivial isotropy subgroup $\mathbf{Z}/2k+1$. The non-principal orbits project to the points of a quadric hypersurface, that is $\delta = 2$. It follows from Table II that if n is even $V(2k+1, 2, \dots, 2)$ is a homotopy sphere (bounding a parallelisable $2n+2$ manifold). We have proved

THEOREM 3.3. *The homotopy sphere* $\Sigma^{2n+1} = V(2k+1, 2, \dots, 2)$ *admits a contact form. If* $(2k+1) \equiv \pm 1 \pmod 8$, *we obtain the standard sphere, if* $(2k+1) \equiv \pm 3 \pmod 8$ *the Kervaire sphere.*

More careful analysis shows that the forms on the standard sphere may be distinguished by the reductions to $U(n)\oplus 1$ which they induce on the tangent bundle. Recall that the complexified tangent bundle defines a class in $\pi_{2n+1}(BU_n)$ which is cyclic of order $n!$

If n is odd, $V(2k+1, 2, \ldots, 2)$ has nth homology group isomorphic to a single copy of $Z/2k+1$ (see Table II again). Using the language of the classification theorem (3.1) we distinguish between two cases:

(i) L_{2k+1}, $H_n(L_{2k+1}) \cong Z/2k+1$ and $1\oplus 1 \mapsto c/2k+1$, c a quadratic residue modulo $(2k+1)$, and

(ii) L'_{2k+1}, $H_n(L_{2k+1}) \cong Z/2k+1$ and $1\oplus 1 \mapsto c/2k+1$, c a non-residue.

Furthermore we have the relation $L_{2k+1} \# L_{2k+1} \cong L'_{2k+1} \# L'_{2k+1}$. By examining the cup product in N^{n+1} of the complex hypersurface defining V one sees that $1 \otimes 1 = -((-1)/2k+1)^{n+1/2}$. From this one concludes that if $(2k+1)$ equals the odd prime power p^r, then

(iii) $n = 1 \pmod 4$, $L(p^r) \cong V(p^r, 2, \ldots, 2)$, and

(iv) $n = 3 \pmod 4$, $L'(p^r) \cong V(p^r, 2, \ldots, 2)$ $(p = 3 \pmod 4)$, and $L(p^r) \cong V(p^r, 2, \ldots, 2)$ $(p = 1 \pmod 4)$.

These assertions follow from the fact that for $p = 1 \pmod 4$, -1 is a quadratic residue, and for $p = 3 \pmod 4$ a non-residue. In the former case the parity of $\frac{1}{2}(n+1)$ is irrelevant.

Remark. The manifolds L_{2k+1}, L'_{2k+1} are only well defined up to homeomorphism, since we can form the connected sum with a homotopy sphere without changing the homology structure.

4. The Main Theorems

In order to use the classification theorem (3.1) for closed manifolds we shall assume that the tangential invariant $\alpha = 0$. In the special case $n = 2$ (Theorem 3.2) this implies that $w_2(M) = 0$, so we exclude the manifold X_∞. Moreover since the result in this limiting dimension motivates our entire argument we state it first.

THEOREM 4.1. *Let M^5 be a 1-connected 5-manifold such that $w_2(M) = 0$ and $H_2(M, Z)$ contains no element of order 3. Then M^5 admits a contact form.*

Proof. Up to diffeomorphism M^5 is classified by $H_2(M, Z)$, hence it is enough to construct a manifold carrying a contact form with the same homology group. Since S^3 is a group, its cotangent sphere bundle equals $S^2 \times S^3$ and is contact by section 2 (1). Inspection of Table II shows that $V(p^r, 3, 3, 3)$ has Z/p^r as non-trivial isotropy group, that the non-principal orbits map to a cubic curve in $CP(2)$, and hence that

$$H_2(V(p^r, 3, 3, 3), Z) = Z/p^r \times Z/p^r, \quad p \neq 3.$$

Since the pairing $b(\cdot,\cdot)$ is skew-symmetric, if Z/p^r occurs in $H_2(M, Z)$ it must occur an even number of times. Hence $V(p^r, 3, 3, 3)$ represents the prime manifold M_{p^r}. By hypothesis M_{3^r} does not occur in the prime decomposition, and the result follows from the connected sum theorem (2.3).

One can say something about the prime 3. The Orlik–Wagreich formula for $\varkappa(\delta, n)$ shows that $\varkappa(4, 2) = 6$, and hence that

$$V(3^r, 4, 4, 4) \cong 3M_{3^r}.$$

This suggests that there is no essential obstruction to the existence of a contact form on any 1-connected M^5 with $w_2(M) = 0$.

It is clear that the same method will work in higher dimensions. The following result is the easiest to state, although it is a long way short of the most general one can obtain.

THEOREM 4.2. (i) *Let $n = 5 (\mathrm{mod}\, 8)$ and let M^{2n+1} be an $(n-1)$-connected odd torsion manifold. Then M^{2n+1} is homeomorphic to a smooth manifold admitting a contact form, provided that each prime manifold of the form L'_{p^r} occurs an even number of times.*

(ii) *Let $n = 6 (\mathrm{mod}\, 8)$ and let M^{2n+1} be an $(n-1)$-connected manifold, such that $H_n(M, \mathbf{Z})$ is torsion free. Then M^{2n+1} is homeomorphic to a smooth manifold admitting a contact form.*

Proof. The congruence class of n is such that $\alpha = \hat{\beta} = 0$. The invariants in case (i) reduce to the odd torsion group $H_n(M, \mathbf{Z})$ plus its quadratic structure, and we can realise M up to homeomorphism as a connected sum of Brieskorn varieties. Now argue as for $n = 2$. In case (ii) the invariants reduce to the torsion free group $H_n(M, \mathbf{Z})$ and the non-stable tangential invariant $\hat{\varphi}$. Again we may take the connected sum of Brieskorn varieties; since n is even, we have

$V^{2n+1}(2, 2, \ldots, 2) \cong$ (co)tangent S^n-bundle to S^{n+1},

$V^{2n+1}(8, 2, \ldots, 2) \cong S^n \times S^{n+1}$.

These identifications are due to L. Kaufmann [Ka]. We must restrict to homeomorphism in both cases, since taking the connected sum with a homotopy sphere does not alter the algebraic invariants.

If $n = 6 \ (\mathrm{mod}\, 8)$ we can pick up some torsion manifolds. Thus if $p \neq 3$, $V(p^r, 3, \ldots, 3)$ has $H_n(V, \mathbf{Z})$ isomorphic to $\frac{1}{3}(2^{n+1}+1)-1$ copies of $\mathbf{Z}/p^r$, and as in the case of dimension 5 by taking $a_1 = \ldots = a_{n+1} = 4$ we can also derive some information at the prime 3. From the point of view of integrability in general it is perhaps important that for each prime number *some* torsion manifold admits a contact form.

5. Concluding Remarks

It should be clear that the theorems stated in the previous section are provisional in that they provide evidence for the conjecture that all $(n-1)$-connected $(2n+1)$-almost contact manifolds admit a globally defined form.

The first task is to complete the theory in the case $\alpha = \hat{\beta} = 0$ (and $H_n(M, \boldsymbol{Z})$ contains no elements of order 2). For this we need a good geometric model for $S^n \times S^{n+1}$ (n = odd) corresponding to $V(8, 2, \ldots, 2)$ (n = even), and some sort of converse to Theorem 2.3 to enable us to split a composite odd torsion manifold in such a way that the Brieskorn form induces contact forms on each component of the connected sum.

If the stable tangential invariants are non-zero, the theory of almost contact manifolds seems to go as follows. The obstructions to reducing the structural group of the tangent bundle belong to $H^s\left(\;, \pi_{s-1}\left(\dfrac{SO(2n+1)}{U(n)\oplus 1}\right)\right)$, $s = n$ or $n+1$, and since both homotopy groups are stable we can use the Bott isomorphism (see [Bt], p. 315), $\pi_{s-1}\left(\dfrac{O}{U}\right) \cong \pi_s(O)$. Modulo 8 for the homotopy groups of O we have:

Table III

	1	3	5	7
π_n	$Z/2$	Z	0	Z
π_{n+1}	0	0	0	$Z/2$

n = odd

	0	2	4	6
π_n	$Z/2$	0	0	0
π_{n+1}	$Z/2$	Z	0	Z

n = even

Again neglecting 2-torsion the obstructions vanish for $n = 0, 1, 4, 5 \pmod 8$, and comparing Tables I and III for $n = 6 \pmod 8$ also. For $n = 2 \pmod 8$ we need to interpret an element in $H^{n+1}(\;, \boldsymbol{Z})$ and for $n \equiv 3, 7 \pmod 8$ an element in $H^n(\;, \boldsymbol{Z})$. Recall from the discussion in Section 3 that $KO(M)$ is defined by the short exact sequence

$$0 \to H^{n+1}(M, \pi_n(SO)) \to KO(M) \to H^n(M, \pi_{n-1}(SO)) \to 0$$

with a similar sequence for $KU(M)$ (replace SO by U). Also in the non-exceptional case, $n \neq 0, 1 \pmod 8$, at least one of the invariants α or $\hat{\beta}$ vanishes, and we need consider only the left or right hand term. We have

$n = 2 \pmod 8$

$$\begin{tikzcd} KU(M) \arrow[r] \arrow[d, "\wr"] & KO(M) \arrow[d, "\wr"] & \\ H^n(M, \boldsymbol{Z}) \arrow[r] & H^n(M, \boldsymbol{Z}/2) \arrow[r, "\delta"] & H^{n+1}(M, \boldsymbol{Z}) \end{tikzcd}$$

and

$n = 3, 7 \pmod 8$

$$\begin{tikzcd} H^{n+1}(M, \boldsymbol{Z}) \arrow[r, "\cong"] \arrow[d] & KU(M) \arrow[d] \\ \hat{\beta} \in H^{n+1}(M, \boldsymbol{Z}) \arrow[r, "\cong"] & KO(M) \end{tikzcd}$$

where δ is the Bockstein map associated to the coefficient sequence $0 \to \mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \to \mathbf{Z}/2 \to 0$. Recall also that $\hat{\beta}$ is expressible in terms of the Pantrjagin class $p_{(n+1)/4}(TM)$, and that the Pontrjagin classes of the real bundle underlying a complex bundle are even. We now have enough information to deduce that

$n = 2 \pmod 8$: *M^{2n+1} is almost contact if* $0 = \delta w_n \in H^{n+1}(M, \mathbf{Z})$ (this is well known if $n = 2$, compare [Gr] and Theorem 3.2 above), and

$n = 3, 7 \pmod 8$: *M^{2n+1} is almost contact if the tangential invariant $\hat{\beta}$ is even.*

I do not know whether this is always the case, it is so if $n = 3, 7$, but these are cases not included in the general classification.

The problem now is to produce models for the prime manifolds with non-trivial (stable) tangential structure. One possibility here is to describe such manifolds as open books—this works well in dimension 5, see [A'C], and the construction of Thurston–Winkelnkemper in dimension 3 at least suggests a procedure for constructing a contact form.

The third task is to handle 2-torsion. In the non-exceptional case the programme outlined is probably enough, but for $n = 0, 1 \pmod 8$ some new idea may be needed.

A final remark about non-simply-connected manifolds: here in dimensions greater than 3 hardly anything is known. One can produce examples of contact forms on S^1-bundles over integral symplectic manifolds, and R. Lutz has a method of construction for principal bundles fibered by tori, including T^5 (see [L]).

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