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UNIFORMLY ENCLOSING DISCRETIZATION METHODS FOR SEMILINEAR BOUNDARY VALUE PROBLEMS

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1. Introduction

In the numerical solution of differential equations it is useful to generate lower and upper bounds for the desired solution. The problems under consideration are weakly nonlinear two point boundary value problems. It is possible to introduce special discretization techniques generating bounds by means of monotonicity principles. More precisely, the proposed approach takes advantage of weak maximum principles leading to operators of monotone kind. In contrast to other approaches it is not necessary to use a priori information or correction terms based on estimations of the local error.

In the present paper we summarize the results from [2], [3], [4], [6] supplemented by some new considerations.

2. A first order method

Let us consider the weakly nonlinear boundary value problem

$$(2.1)\qquad \begin{aligned} -u''+g(\cdot, u) &= 0 \quad \text{in } \Omega = (0, 1),\\ u(0) = u(1) &= 0, \end{aligned}$$

under the following assumptions on $g\colon \bar{\Omega}\times R \to R$:

(2.2) (i) $g(x, s) \leqslant g(x, t)$ for all $x \in \bar{\Omega}$ and $s \leqslant t$;
(ii) g is a continuous function.

Let us denote by U the Sobolev space $H_0^1(0, 1)$, by U^* the related dual space and by $\langle \cdot, \cdot\rangle$ the dual pairing between U^* and U.

Introducing the operators L, $G\colon U \to U^*$ defined by

$$\langle Lu, v\rangle := \int_0^1 u' v' \, dx,$$

$$\langle Gu, v\rangle := \int_0^1 g(x, u(x))\, v(x)\, dx$$

we formulate (2.1) as an operator equation: Find $u \in U$ such that

$$Lu + Gu = 0. \tag{2.3}$$

Under the proposed assumptions problem (2.3) admits a unique solution, the operator $L+G$ is strongly monotone and of monotone kind.

Now let be given some grid on the interval [0, 1], i.e.

$$0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = 1.$$

The corresponding step sizes and subintervals we denote by $h_i := x_i - x_{i-1}$, $\Omega_i := (x_{i-1}, x_i)$, $i = 1(1)N$, the mesh width of the grid is characterized by

$$h := \max_{1 \leqslant i \leqslant N} h_i.$$

In the sequel we concentrate ourselves on generating upper solutions. We call an operator G_h: $U \to U^*$ *bounding operator* if the inequality

$$Gv \geqslant G_h v \quad \text{for all } v \in U \tag{2.4}$$

holds. It is possible to construct bounding operators using

$$G_h := P_h G \quad \text{with } w \geqslant P_h w \text{ for all } w \in C(\bar{\Omega}). \tag{2.5}$$

We obtain a first order method by setting, for instance,

$$[P_h w](x) := \min_{\xi \in \bar{\Omega}_i} g(\xi, w(\xi)) \quad \text{for } x \in \Omega_i,\ i = 1(1)N. \tag{2.6}$$

Our approximate problem is defined by: *Find* $u_h \in U$ *such that*

$$Lu_h + G_h u_h = 0. \tag{2.7}$$

If G_h is defined by (2.5), (2.6) one can interpret (2.7) as a C^1-collocation method or as a three point difference scheme. To explain these facts let us introduce the representation

$$u_h(x) = \sum_{i=1}^{N-1} u_i \varphi_i(x) + \sum_{i=1}^{N} w_i \psi_i(x) \tag{2.8}$$

with (u_h is piecewise quadratic because $P_h w$ being piecewise constant)

$$\varphi_i(x) = \begin{cases} (x - x_{i-1})/h_i, & x \in \bar{\Omega}_i, \\ (x_{i+1} - x)/h_{i+1}, & x \in \bar{\Omega}_{i+1}, \\ 0 & \text{otherwise;} \end{cases}$$

$$\psi_i(x) = \begin{cases} \frac{1}{2}(x - x_{i-1})(x_i - x), & x \in \bar{\Omega}_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then, our approximate problem (2.7) is equivalent to the C^1-collocation method

$$(2.9)\qquad u_h \in C^1[0, 1], \quad [Lu+Gu](\xi_i) = 0, \quad i = 1(1)N,$$

where the *collocation points* ξ_i are defined by

$$g(\xi_i, u_h(\xi_i)) = \min_{\xi\in\bar{\Omega}_i} g(\xi, w(\xi)).$$

The corresponding three-point difference scheme is characterized by

$$(2.10)\qquad -D_h u_i + g_h(u_{i-1}, u_i, u_{i+1}) = 0$$

with

$$(2.11)\qquad \begin{aligned} &\text{(i)} \quad D_h u_i = \frac{2}{h_i+h_{i+1}}\left(\frac{u_{i-1}}{h_i} - \left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)u_i + \frac{u_{i+1}}{h_{i+1}}\right)\\ &\text{(ii)} \quad g_h = -\frac{h_i}{h_i+h_{i+1}} w_i(u_{i-1}, u_i) - \frac{h_{i+1}}{h_i+h_{i+1}} w_{i+1}(u_i, u_{i+1}) \end{aligned}$$

and

$$w_i(u_{i-1}, u_i) + g\left(\xi_i, u_{i-1}\frac{x_i-\xi_i}{h_i} + u_i\frac{\xi_i - x_{i-1}}{h_i} + w_i\frac{1}{2}(\xi_i - x_{i-1})(x_i-\xi_i)\right) = 0.$$

Now, let us assume

$$(2.12)\qquad |g(x, s)-g(y, t)| \leqslant l(r)(|x-y|+|s-t|) \quad \text{for } |s| \leqslant r,\ |t| \leqslant r,$$

where l: $R^+ \to R^+$ denotes some nondecreasing function. For the method defined by (2.7) with (2.5), (2.6) we proved in [2]

THEOREM 1. *There exists an $h^* > 0$ such that the discrete problem for all $h \in (0, h^*]$ admits a solution, further the relations $u(x) \leqslant u_h(x)$ and $|||u-u_h||| \leqslant Ch$ are valid.*

Here and in the following $|||\cdot|||$ denotes the maximum norm. The proof of Theorem 1 is based on the strong monotonicity and inverse-monotonicity of the operator $L+G$.

The definition of the operator P_h according to (2.6) is connected with some trouble in the implementation of the method. But, of course, it is possible to define P_h in some special cases in a more suitable way taking into account special properties of $g(x, u)$. Let us assume, for instance, that the function g admits the representation

$$(2.13)\qquad g(x, u) = e(x)+f(u)$$

with some Lipschitz-continuous, monotone function $f(\cdot)$.

We denote the finite dimensional space of all functions $v_h(x)$ of the form

(2.14) $$v_h(x) = \sum_{i=1}^{N-1} v_i \varphi_i(x) + \sum_{i=1}^{N} w_i \psi_i(x)$$

by V_h and define a mapping from V_h into the space of piecewise constant functions using

(2.15) $$[Pv_h](x) := e_i + f(\mu_i) \quad \text{for all } x \in \Omega_i$$

with

$$e_i \leqslant e(x) \quad \text{on } \Omega_i, \quad |e(x) - e_i| \leqslant Ch_i \quad \text{for all } x \in \Omega_i$$

$$\mu_i = \min(v_{i-1}, v_i) + \tfrac{1}{8}\min(0, w_i) h_i^2.$$

Further we define a mapping π: $C[0, 1] \to V_h$ by

(2.16) $$[\pi y](x) := \sum_{i=1}^{N-1} v_i \varphi_i(x) + \sum_{i=1}^{N} w_i \psi_i(x)$$

with

$$v_i = y(x_i),$$

$$w_i = \frac{4}{h_i^2}\left(y(x_{i-1/2}) - \frac{y(x_i) + y(x_{i-1})}{2} \right)$$

($x_{i-1/2}$ is the midpoint of Ω_i). Finally, a "bounding operator" is defined by

(2.17) $$G_h = P\pi.$$

The new "bounding operator" (2.17) only fulfills

(2.18) $$Gv_h \geqslant G_h v_h \quad \text{for all } v_h \in V_h,$$

but this property allows us to prove that $u_h(x)$ is an upper bound of the exact solution because from

$$(u_h', v') + (Gu_h, v) = ((G - G_h) u_h, v) \geqslant 0 \quad \text{for all } v \geqslant 0$$

it follows $u_h(x) \geqslant u(x)$.

Similarly, it is possible to transfer all other parts of the proof of Theorem 1 to our new bounding operator, thus the new discrete problem admits a solution, too. An error estimation is based on

$$|||u - u_h||| \leqslant C \, \|Gu_h - g(\cdot, u_h)\|_{L^\infty},$$

and it is not difficult to show that (2.14), (2.15) result in a first order method.

3. The application of the first order method to a singular perturbation problem

Of course, it is also possible to analyze our discretization method, described by (2.7) with (2.5), (2.6), in the context of a finite difference method.

We shall do this for the singularly perturbed boundary value problem

$$-\varepsilon^2 u''+g(\cdot, u) = 0, \qquad u(0) = u(1) = 0 \tag{3.1}$$

additionally assuming $(0 < \varepsilon \ll 1)$)

$$\text{(i)}\quad g\in C^2, \qquad \text{(ii)}\quad 0<\beta^2 \leqslant g_u(x, u) \leqslant B. \tag{3.2}$$

It is well known that now the solution u belongs to C^4 and that near the boundaries boundary layers exist.

For the consistency error we have

$$|r_h u(x_i)| \leqslant K\varepsilon^2 \big(|h_{i+1}-h_i|\,|u'''(x_i)| + \max(h_i^2, h_{i+1}^2) \max_{s\in\bar{\Omega}_i} |u^{(4)}(s)|\big) + K \max(h_i, h_{i+1}) \max_{s\in\Omega_i} |u'(s)|. \tag{3.3}$$

(we denote by K some constant which is independent of ε and h).

In contrast to the usual discretization method

$$-\varepsilon^2 D_h u_i + g(x_i, u_i) = 0 \tag{3.4}$$

the difference scheme (2.10), (2.11) applied to our singular perturbation problem is not stable uniformly in ε. This is due to the fact that the partial derivatives $\partial g_h/\partial u_{i-1}$, $\partial g_h/\partial u_{i-1}$ are not negative and thus, in general, the system (2.10) does not correspond to an M-function. But in the special case

$$h \leqslant \sqrt{2/B}\,\varepsilon \tag{3.5}$$

the theory of M-functions is applicable and the discrete problem is under this restriction uniformly stable with respect to the parameter.

The consistency error is not uniformly bounded in ε because the derivatives of the exact solution grow unboundedly near $x = 0$ and $x = 1$ as ε tends to 0. For the derivatives it holds the estimation

$$|u^{(k)}(x)| \leqslant K\big(1+\varepsilon^{-k}(\exp(-\beta x/\varepsilon)+\exp(-\beta(1-x)/\varepsilon))\big). \tag{3.6}$$

It is possible to bound the consistency error uniformly by using a special discretization mesh. Let us use the grid generating function due to Vulanovic [12]

$$\lambda(t) = \begin{cases} \psi(t) := A\varepsilon t/(q-t), & t\in[0, \tau], \\ \pi(t) := \psi(\tau)+\psi'(\tau)(t-\tau), & t\in[\tau, \tfrac{1}{2}], \\ 1-\lambda(1-t), & t\in[\tfrac{1}{2}, 1], \end{cases} \tag{3.7}$$

where q, A are constants with $q \in (0, \frac{1}{2})$, $A \in (0, q/\varepsilon)$, τ satisfies

$$\tau = \left(q - (Aq\varepsilon(1-2q+2A\varepsilon))^{1/2}\right)/(1+2A\varepsilon) \tag{3.8}$$

and define the special grid by

$$x_i = \lambda(t_i), \quad t_i = ih^*, \quad i = 0(1)N, \; N = 2N_0. \tag{3.9}$$

In [12] it was shown that the consistency error for the approximation of $u''(x_i)$ admits the order 2 uniformly with respect to ε and that the estimate.

$$\max(h_i, h_{i+1})\varepsilon^{-1} \leqslant Kh^*$$

holds. Thus, we have

THEOREM 2. *Let us apply the first order enclosing discretization technique described by* (2.7) *with* (2.5), (2.6) *to the singularly perturbed boundary value problem* (3.1) *under the additional assumptions* (2.3). *Let us use the special grid defined by* (3.7), (3.8) *and assume that* ε *is not too small, that means,* $h \leqslant \sqrt{2/B}\,\varepsilon$. *Then, it holds*

$$|||u - u_h||| \leqslant Kh^*$$

where K *does not depend on the parameter* ε.

4. Higher order methods

To generate higher order methods it is necessary to improve the approximation properties of the operator G_h (respectively P_h, if G_h is defined by (2.5)). In [4] we described such bounding operators G_h defined by (2.5) using piecewise interpolation and shifting to define P_h.

Let us choose some natural number $k \geqslant 1$ and define

$$\mathscr{P}_k = \{v \in L_2(0, 1)\colon v|_{\Omega_i} \text{ polynomial of degree } \leqslant k\}.$$

We use an equidistant auxiliary grid on every subinterval $\bar{\Omega}_i$ given by

$$\sigma_j^i := x_{i-1} + \frac{j}{k}h_i, \quad j = 0(1)k,$$

and define a piecewise interpolation operator $S_k\colon C[0, 1] \to \Omega_k$ by $S_k u|_{\Omega_i}$:= interpolation polynomial of u with knots σ_j^i $(j = 0(1)k)$. Further we introduce the operator $p\colon C[0, 1] \to \mathscr{P}_0$ by

$$[pw](x) := \min_{\xi \in \bar{\Omega}_i} w(\xi) \quad \text{for all } x \in \Omega_i \tag{4.1}$$

and define

$$P_h := S_k + p(I - S_k) \tag{4.2}$$

(I is the identity).

It is not so easy to prove that the generated discrete problem (2.7) admits a solution. In our proof we used the theory of pseudomonotone operators [4] and an auxiliary variational inequality [5], respectively. The convergence proof works like the convergence proof for finite element methods. Analogously as for collocation methods, we obtain the following result:

THEOREM 3. *There exist an $h^* > 0$ such that the discrete problem for all $h \in (0, h^*]$ admits a solution. If all partial derivatives of order less or equal to $k+1$ are continuous on Ω_i for $i = 1(1)N$ then it holds the error estimate*

$$|||u-u_h||| \leqslant Ch^{k+1}.$$

The numerical experiments meet the theoretical expectation [4].

5. Grid generation via enclosing discretization

In this section we propose a principle for the generation of grids using information from the lower and upper solution. The basic idea is to subdivide these intervals where the difference between the upper and lower solutions is relatively large.

Let some initial grid $Z^1 = \{x_i^1 \colon i = 0, \ldots, N_1\}$ be given. The related subintervals we denote by Ω_i^1 and assume $h(Z^1)$ to be small enough that the discrete problems possess a solution. The algorithm under consideration generates a sequence (Z^k) of grids. The bounding operators on the grids Z^k we denote by G^k.

Algorithm

Step 1. Let some initial grid Z^1 be given. Select some $\varrho \in (0, 1)$ and set $k = 1$.

Step 2. Determine $\underline{u}^k(x)$, $\bar{u}^k(x)$ by solving the corresponding discrete problems (2.7) (with $\underline{G}^k$, $\bar{G}^k$ instead of G_h).

Step 3. With the notation

$$d_i^k := \max_{x \in \Omega_i^k} (\bar{u}^k(x) - \underline{u}^k(x)) \qquad (i = 1, \ldots, N_k),$$

$$D^k := \max_{1 \leqslant i \leqslant N_k} d_i^k,$$

$$I_k := \{i \in \{1, \ldots, N_k\} \colon d_i^k \geqslant \varrho D^k\},$$

$$y_i^k := \tfrac{1}{2}(x_{i-1}^k + x_i^k) \qquad (i \in I_k)$$

define a new grid by

$$Z^{k+1} = Z^k \{ y_i^k \colon i \in I_k \}.$$

Set

$$N_{k+1} = N_k + \operatorname{card} I_k$$

and denote the grid points contained in Z^{k+1} by x_i^{k+1} ($i = 0, 1, \ldots, N_{k+1}$). Reset

$$k^{\text{new}} := k^{\text{old}} + 1$$

and go to step 2.

In [3] we succeeded in proving the following

THEOREM 4. *Let us suppose that in step 2 of the algorithm above the first order method characterized by the bounding operator defined by* (2.5), (2.6) (*and the corresponding operator for lower solutions*) *is used. Then, the functions* $\underline{u}^k(x)$, $\bar{u}^k(x)$ *generated have the properties*

$$\underline{u}^k(x) \leqslant \underline{u}^{k+1}(x) \leqslant u(x) \leqslant \bar{u}^{k+1}(x) \leqslant \bar{u}^k(x) \quad \textit{for all } x \in [0, 1],$$

$$\lim_{k\to\infty} |||\bar{u}^k(x) - \underline{u}^k(x)||| = 0.$$

In numerical experiments we obtained improved error bounds using the grid generation algorithm in connection with higher order methods in step 2, too. However, the theoretical foundation of the feedback grid generation cannot be carried out similarly to the first-order technique. The proof of Theorem 4 is based on the following two statements. First, for the first-order method it holds: From v, $w \in C^1[0, 1]$ and

$$Lv + G_h v \leqslant Lw + G_h w,$$

$$v(0) \leqslant w(0), \qquad v(1) \leqslant w(1),$$

it follows $v(x) \leqslant w(x)$ for sufficiently small h. And, second, for the first order method an enclosure $\underline{u}_Z \leqslant u \leqslant \bar{u}_Z$ is sharpened using a finer grid. It is an open problem whether or not higher order methods preserve these properties.

6. A second approach to enclosing discretizations

As in Section 2 we start from the formulation of problem (2.1) as an operator equation: Find some $u \in U$ such that

$$Lu + Gu = 0. \tag{6.1}$$

In contrast to the discretization technique described above we now shall formulate an iteration process of the form

$$\begin{aligned} &\text{(i)} \quad (L+D)\,u^{k+1} = s^k; \\ &\text{(ii)} \quad s^{k+1} = \min(s^k, Qu^{k+1}) \end{aligned} \tag{6.2}$$

using some auxiliary operators D, Q such that $\{u^k\}$ converges monotone to an upper (lower) solution of the original problem (6.1). This iteration process is similar to the technique of Sattinger [10]

$$(L + \varrho I)\,\hat{u}^{k+1} = (\varrho I - G)\,\hat{u}^k$$

generating a monotone sequence but we propose a modification which is numerically constructive.

First, we define the operator D by

$$[Dw](x) := d_i w(x) \quad \text{for all } x \in \Omega_i \tag{6.3}$$

where the nonnegative constants d_i are to specify in such a manner that the required property of $D-G$ is fulfilled. Now we choose some finite-dimensional space W_h of the type

$$W_h = \{w \in L_2(0, 1)\colon w|_{\Omega_i} \text{ polynomial of degree } \leqslant l\} \tag{6.4}$$

and define

$$V_h = (L+D)^{-1} W_h. \tag{6.5}$$

We choose $s^0 \in W_h$, $u^0 \in V_h$ and define Q to be a mapping $Q\colon V_h \to W_h$ which satisfies

$$Qw \geqslant (D-G)w \quad \text{for all } w \in V_h. \tag{6.6}$$

More precisely, we need initial values $\underline{s}^0$, $\underline{u}^0$ and $\bar{s}^0$, $\bar{u}^0$ for the iteration process generating lower and upper solutions, respectively. Let us suppose

$$\begin{array}{ll} \text{(i)} & \underline{u}^0 \leqslant \bar{u}^0; \\ \text{(ii)} & (D-G)w_1 \leqslant (D-G)w_2 \text{ for all } w_{1,2} \text{ with } \underline{u}^0 \leqslant w_1 \leqslant w_2 \leqslant \bar{u}^0; \\ \text{(iii)} & (D-G)\bar{u}^0 \leqslant \bar{s}^0,\ (D-G)\underline{u}^0 \geqslant \underline{s}^0; \\ \text{(iv)} & (L+D)\bar{u}^0 \geqslant \bar{s}^0,\ (L+D)\underline{u}^0 \leqslant \underline{s}^0. \end{array} \tag{6.7}$$

As above we only state the following theorem from [6] concerning the sequence $\{\bar{u}_k\}$ which tends to an upper solution. We omit the bar in the notation again.

THEOREM 5. *Under the assumptions* (6.6), (6.7) *the iteration process* (6.2) *generates a sequence* $\{u_k\}$ *with the following properties*:

(i) $u_{k+1} \leqslant u_k \leqslant \ldots \leqslant u_0$;

(ii) $\{u_k\}$ *tends to some* $\{\tilde{u}\}$ *in* $H^1(0, 1)$ *and* $C[0, 1]$;

(iii) *For the limit function* $\tilde{u}$ *it holds* $\tilde{u} \geqslant u$ *and*

$$|||\tilde{u}-u||| \leqslant C\,\|Q\tilde{u}-(D-G)\tilde{u}\|_{L^\infty}. \tag{6.8}$$

The realization of this second approach to enclosing discretizations strongly depends on the concrete definition of the space W_h and the operator Q. One expects a first order method in choosing W_h to be the space of piecewise constant functions, a second order method by choosing W_h as the space of piecewise linear functions and defining Q in an adequate way and so on.

In the first case it is obvious to define

$$[Qw](x) := \max_{\xi \in \bar{\Omega}_i} \big(d_i w(\xi) - g(\xi, w(\xi))\big) \quad \text{for all } x \in \Omega_i. \tag{6.9}$$

Similarly as in Section 2 it is possible to make the method more practicable under special assumptions on g. Let us assume again

$$g(x, u) = e(x) + f(u) \tag{6.10}$$

(compare (2.13) and the following statements). Now we have the advantage that it is only necessary to define the operator Q on V_h. All functions from V_h admit a representation

$$v_h(x) = \sum_{i=1}^{N-1} v_i \varphi_i^*(x) + \sum_{i=1}^{N} w_i \psi_i^*(x) \tag{6.11}$$

with

$$\text{(6.12)}\quad \begin{aligned} &\text{(i)} \quad \varphi_i^*(x) = \begin{cases} \sinh(\sqrt{d_i}(x - x_{i-1}))/\sinh(\sqrt{d_i}\, h_i) & \text{for } x \in \bar{\Omega}_i, \\ \sinh(\sqrt{d_{i+1}}(x_{i+1} - x_i))/\sinh(\sqrt{d_{i+1}}\, h_{i+1}) & \text{for } x \in \bar{\Omega}_{i+1}, \\ 0 & \text{otherwise;} \end{cases} \\ &\text{(ii)} \quad \varphi_i^*(x) = \begin{cases} (1 - \varphi_{i-1}^*(x) - \varphi_i^*(x))/d_i & \text{for } x \in \bar{\Omega}_i, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Then, it is adequate to define

$$[Qv_h](x) := -e_i + d_i \mu_i^* f(\mu_i^*) \quad \text{for all } x \in \Omega_i,\ v_h \in V_h \tag{6.13}$$

with

$$\mu_i^* := \max(v_{i-1}, v_i) + \tfrac{1}{8} \max(0, w_i) h_i^2.$$

The proof of Theorem 5 shows the validity of the estimate

$$s^k \geqslant (D - G) u^{k+1}. \tag{6.14}$$

Therefore, we could formulate the iteration process (6.2) with (6.9) in the form

$$\text{(6.15)}\quad \begin{aligned} &\text{(i)}\ (L + D) u^{k+1} = s^k; \\ &\text{(ii)}\ s_i^{k+1} = s_i^k - \min_{\xi \in \bar{\Omega}_i} \big(s_i^k - \big(d_i u^{k+1}(\xi) - g(\xi, u^{k+1}(\xi))\big)\big). \end{aligned}$$

(s_i^k denotes the restriction of s^k to Ω_i).

For generating a second order method we choose W_h as the space of piecewise linear functions and use again the idea of the reformulation of the iteration process in the form (6.15). Let us define ξ_i by

$$\begin{aligned} s_i^k(\xi_i) - [d_i u^{k+1}(\xi_i) - g(\xi_i, u^{k+1}(\xi_i))] \\ = \min_{\xi \in \bar{\Omega}_i} \{ s_i^k(\xi) - [d_i u^{k-1}(\xi) - g(\xi, u^{k+1}(\xi))] \}. \end{aligned}$$

Further we denote by $t_u^{i,1}(x)$ the linear interpolant of $d_i u - g(\cdot, u(\cdot))$ with respect to the gridpoints x_{i-1} and x_i and by $t_u^{i,2}(x)$ the linear function on

$[x_{i-1}, x_i]$ which passes $d_i u - g(\cdot, u(\cdot))$ in x_{i-1} or x_i (if $\xi_i = x_{i-1}$ or $\xi_i = x_i$), majorizes $d_i u - g(\cdot, u(\cdot))$ over the interval $[x_{i-1}, x_i]$ and is tangent to $d_i u - g(\cdot, u(\cdot))$ in some point. Using

$$t_u^i(x) = \begin{cases} t_u^{i,1}(x) & \text{if } t_u^{i,1}(x) \text{ majorizes } d_i u - g(\cdot, u(\cdot)) \\ & \text{over the interval } [x_{i-1}, x_i], \\ t_u^{i,2}(x) & \text{otherwise,} \end{cases}$$

we define the iteration process

$$\text{(6.16)} \quad \begin{aligned} &\text{(i)} \ (L+D)u^{k+1} = s^k \\ &\text{(ii)} \ s_u^{k+1}(x) = \begin{cases} s_i^k(x) - \min\limits_{\xi \in \bar{\Omega}_i} \left[s_i^k(x) - \left(d_i u^{k+1}(\xi) - g(\xi, u^{k+1}(\xi)) \right) \right] \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \xi_i \in \Omega_i, \\ t_{u^{k+1}}^i(x) \qquad\qquad \text{if } \xi_i = x_{i-1} \text{ or } \xi_i = x_i. \end{cases} \end{aligned}$$

Now, again Theorem 5 holds and from (6.8) we obtain the second order accuracy of the approximation of $\tilde{u}$ to the exact solution with respect to h. A more practicable version of a second order method is described in [6].

7. Concluding remarks

In our paper we did not speak about the numerical solution of the auxiliary problems generated by the enclosing discretization principle of Section 2 and 4. Special iteration methods have been developed in [5].

The extension of enclosing discretization techniques to partial differential equations leads to significant troubles because no explicit representation of $L^{-1} W_h$ for the function spaces W_h used above is available in the multidimensional case.

Some progress has been achieved in extending our enclosing discretization methods to $1-D$ linear parabolic problems [7], [9] using a modification of the Rothe-method for the discretization in time.

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