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EIGENVALUE APPROXIMATION

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Introduction

In this paper the following self-adjoint eigenproblem on a Hilbert space V is considered:

(1) find $\mu \in \boldsymbol{R}$ and $0 \neq u \in V$ such that

$$b(u, v) = \mu \cdot (u, v) \quad \forall\, v \in V.$$

This problem is approximated by a family of eigenproblems on finite dimensional spaces V_h:

(2) find $\mu_h \in \boldsymbol{R}$ and $0 \neq u_h \in V_h$ such that

$$b_h(u_h, v_h) = \mu_h \cdot (u_h, v_h)_h \quad \forall\, v_h \in V_h.$$

There are many papers concerning approximation of the problem (1). Their results are based generally on internal [3] or nonconforming [4] approximation of the space V for primal or dual formulation of the eigenproblem. In this paper a different approach has been applied. We present some sufficient conditions for a convergent approximation of the problem (1), i.e. we give sufficient conditions for convergence of solutions of an arbitrary family of finite dimensional eigenproblems to the solution of the problem (1). The proof of convergence is based on the minimax Poincaré's characterization of eigenvalues [7]. However, on this way we have no error estimation. Some error estimations presented in the paper are based on the Canuto's ideas [1] and obtained by assuming that the considered approximation has some additional properties for eigenfunctions of the problem (1). The Sturm–Liouville problem approximation illustrates the theoretical results of the paper.

1. Formulation of eigenproblem and its approximation

Let V be a complex Hilbert space with a scalar product $(\, , \,)$, and let b: $V \times V \to C$ be a continuous sesquilinear hermitian form. We consider an eigenvalue problem of the following form:

(1.1) find $\mu \in C$, $0 \neq u \in V$ such that

$$b(u, v) = \mu \cdot (u, v) \qquad \forall v \in V.$$

Since $b(u, v)$ and (u, v) are hermitian forms, the eigenvalues of (1.1) are real, i.e. $\mu \in R$.

Let us concentrate ourself on approximation of eigenvalues of (1.1). Suppose that a family $\{V_h\}_{h \in \mathscr{H}}$ of finite dimensional spaces is given. It should be emphasized that V_h is not assumed to be a subspace of V. The scalar product of V_h will be denoted by $(\, , \,)_h$ and the norm by $\| \, \|_h$. Let $\forall h \in \mathscr{H}$ b_h: $V_h \times V_h \to C$ be bounded sesquilinear hermitian form. We consider the following approximate eigenvalue problem:

(1.2) find $0 \neq u_h \in V_h$ and $\mu_h \in R$ such that

$$b_h(u_h, v_h) = \mu_h \cdot (u_h, v_h)_h \qquad \forall v_h \in V_h.$$

We will assume that

H1. $\forall h$ there exists a linear map r_h: $V \xrightarrow{\text{onto}} V_h$ such that

$$\|r_h\| \leqslant r_0 < \infty, \qquad \text{for some constant } r_0;$$

H2. $\forall u \in V$ $\|r_h u\|_h \to \|u\|$ as $h \to 0$;

H3. $\forall u \in V$ $b_h(r_h u, r_h u) \to b(u, u)$ as $h \to 0$;

H4. there exists a topological complement $\mathfrak{M}_h$ of the null space $N(r_h)$ of the operator r_h, i.e. $N(r_h) \oplus \mathfrak{M}_h = V$, such that

(i) $\varepsilon_h \stackrel{\text{df}}{=} \sup\limits_{u \in \mathfrak{M}_h} \dfrac{b_h(r_h u, r_h u) - b(u, u)}{\|u\|^2} \to 0$ as $h \to 0$;

(ii) $(1 - a_h) \stackrel{\text{df}}{=} \sup\limits_{u \in \mathfrak{M}_h} \dfrac{\|u\| - \|r_h u\|_h}{\|u\|} \to 0$ as $h \to 0$.

Let us note that assumption H4 (ii) is equivalent to

$$a_h \stackrel{\text{df}}{=} \inf_{u \in \mathfrak{M}_h} \frac{\|r_h u\|_h}{\|u\|} \to 1 \qquad \text{as} \qquad h \to 0.$$

In the next section of the paper we will prove a convergence of eigenvalues of the problem (1.2) to the suitable eigenvalues of the problem (1.1) under the above assumptions. Now, we present some remarks concerning introduced assumptions and formulation of the eigenproblems.

Remark 1. The approximation described above can be expressed in terms of internal approximation of V. $\{\mathfrak{M}_h\}$ defined in H4 is the family of finite dimensional subspaces of V. Moreover,

$$Q_h \overset{\text{df}}{=} r_h|_{\mathfrak{M}_h}^{-1} \cdot r_h$$

is a linear projection of V onto $\mathfrak{M}_h$. If we define $\tilde{b}_h$: $\mathfrak{M}_h \times \mathfrak{M}_h \to \boldsymbol{C}$ as $\tilde{b}_h = b_h \circ r_h$, and $(u, v)_{\mathfrak{M}_h} = (r_h u, r_h v)_h$ for $u, v \in \mathfrak{M}_h$, then the problem (1.2) is replaced by the following one:

find $\mu_h \in \boldsymbol{R}$ and $0 \neq \tilde{u}_h \in \mathfrak{M}_h$ such that

$$\tilde{b}_h(\tilde{u}_h, \tilde{v}_h) = \mu_h \cdot (\tilde{u}_h, \tilde{v}_h)_{\mathfrak{M}_h} \quad \forall\, \tilde{v}_h \in \mathfrak{M}_h.$$

Assumptions H1–H4 can now be formulated as follows:

H1′. $\|Q_h\| \leqslant r_0/a_h < \infty$ (it follows from H1 and H4 (ii));

H2′. $\forall\, u \in V \ \ \|Q_h u\|_{\mathfrak{M}_h} \to \|u\|$;

H3′. $\forall\, u \in V \ \ \tilde{b}_h(Q_h u, Q_h u) \to b(u, u)$;

H4′. (i) $\displaystyle\sup_{u \in \mathfrak{M}_h} \frac{\tilde{b}_h(u, u) - b(u, u)}{\|u\|^2} \to 0$ as $h \to 0$;

(ii) $\displaystyle\sup_{u \in \mathfrak{M}_h} \frac{\|u\| - \|u\|_{\mathfrak{M}_h}}{\|u\|} \to 0$ as $h \to 0$.

In applications the first formulation (without the assumption: $V_h \subset V$) seems to be more convenient.

Remark 2. Assumption H4 is a natural generalization of the assumptions of the first Strang lemma (cf. [2], th. 4.1.1). In the case where $V_h \subset V$ assumption H4 (ii) is equivalent to the following one:

$$\forall\, v_h \in V_h \quad (v_h, v_h)_h \geqslant a_h^2 \|v_h\|^2 \quad \text{and} \quad a_h \to 1.$$

It means that the approximate sesquilinear forms $(\ ,\ )_h$ associated with a family of subspaces V_h are uniformly V_h-elliptic with the constants $a_h \to 1$ as $h \to 0$.

Remark 3. Let A and A_h be operators on V and $\mathfrak{M}_h$, respectively, defined as follows:

$$b(u, v) = (Au, v) \ \forall\, u, v \in V, \quad \tilde{b}_h(u, v) = (A_h u, v)_{\mathfrak{M}_h} \ \forall\, u, v \in \mathfrak{M}_h.$$

If we replace conditions H4 by the stronger ones:

H5. (i) $\displaystyle\tilde{\varepsilon}_h \overset{\text{df}}{=} \sup_{v, u \in \mathfrak{M}_h} \frac{|\tilde{b}_h(u, v) - b(u, v)|}{\|u\| \cdot \|v\|} \to 0,$

(ii) $\displaystyle\tilde{\delta}_h \overset{\text{df}}{=} \sup_{u, v \in \mathfrak{M}_h} \left| \frac{(u, v)_{\mathfrak{M}_h} - (u, v)}{(u, v)} \right| \to 0,$

then it follows from H5 that

$$\sup_{u,v\in\mathfrak{M}_h} \frac{\big((A-A_h)u, v\big)}{\|u\|\cdot\|v\|} \leqslant \tilde{\varepsilon}_h + \tilde{\delta}_h \|A|_{\mathfrak{M}_h}\| \to 0.$$

The similar condition ($\|(A-A_h)|_{\mathfrak{M}_h}\| \to 0$) is analysed in [3].

EXAMPLE 1. Let us consider the space $H_0^1(0,1)$ and let

$$(u, v) = \int_0^1 u'v'\,dt.$$

Let r_h: $H_0^1 \to \boldsymbol{R}_{n-1}$ where $n = 1/h$, and

$$\forall\, u \in H_0^1 \quad r_h(u) = \big[u(h), \ldots, u\big((n-1)h\big)\big].$$

In this case

$$N(r_h) = \{u \in H_0^1 \colon u(ih) = 0,\ i = 1, \ldots, n-1\}.$$

Let

$$\tilde{\mathfrak{M}}_h = \Big\{u \in H_0^1 \cap H^2 \colon \int_0^1 u'v'\,dt = 0\ \forall\, v \in N(r_h)\Big\}.$$

The condition

$$\int_0^1 u'v'\,dt = 0 \qquad \forall\, v \in N(r_h)$$

is equivalent to the following one:

$$\forall\, i = 1, \ldots, n\ \forall\, w \in H_0^1\big((i-1)h, ih\big) \qquad \int_{(i-1)h}^{ih} u''\cdot w\,dt = 0.$$

Hence $u'' = 0$ almost everywhere, and

$$\int_{(i-1)h}^{ih} (u')^2\,dt = \left(\frac{u(ih) - u\big((i-1)h\big)}{h}\right)^2 \cdot h.$$

Thus $\forall\, u \in \tilde{\mathfrak{M}}_h$

$$(u, u) = h \sum_{i=1}^{n} \left(\frac{u(ih) - u\big((i-1)h\big)}{h}\right)^2.$$

If for $\boldsymbol{v}_h = [v_1^h, \ldots, v_{n-1}^h]$ we put

$$(1.3) \qquad (\boldsymbol{v}_h, \boldsymbol{v}_h)_h = h \sum_{i=1}^{n} \left(\frac{v_i^h - v_{i-1}^h}{h}\right)^2, \quad \text{where} \quad v_0^h = v_n^h = 0,$$

then $\forall\, u \in \tilde{\mathfrak{M}}_h$ $\|r_h u\|_h = \|u\|$.

Since $\tilde{\mathfrak{M}}_h$ is dense in the orthogonal topological complement $\mathfrak{M}_h$ of $N(r_h)$ in H_0^1, we have

$$1 = \inf_{u \in \tilde{\mathfrak{M}}_h} \frac{\|r_h u\|_h}{\|u\|} = \inf_{u \in \mathfrak{M}_h} \frac{\|r_h u\|_h}{\|u\|}.$$

So, condition H4 (ii) is satisfied with $a_h = 1$.

Remark 4. Let V_0 and W be linear vector spaces and let $A, B\colon V_0 \to W$ be two linear operators. Let us consider the eigenproblem

$$Au = \lambda Bu. \tag{1.4}$$

This problem can be formulated in the form (1.1) under some additional assumptions (cf. [7]).

Assume that there exists a sesquilinear form $f\colon W \times V_0 \to C$ satisfying the condition

C1. if $f(w, v) = 0$ $\forall v \in V_0$ then $w = 0$.

If we define sesquilinear forms $a, b\colon V_0 \times V_0 \to C$ as follows: $a(u, v) = f(Au, v)$, $b(u, v) = f(Bu, v)$ $\forall u, v \in V_0$, then the problem (1.4) is equivalent to

(1.5) find $0 \neq u \in V_0$ and $\lambda \in C$ such that

$$a(u, v) = \lambda b(u, v) \qquad \forall v \in V_0.$$

Suppose that

C2. a and b are hermitian forms;

C3. a is positive definite on V_0;

C4. $\exists c > 0$ $\forall u \in V_0$ $|b(u, u)| \leqslant ca(u, u)$.

Then $\|u\| \overset{\mathrm{df}}{=} \sqrt{a(u, u)}$ is a norm on V_0. Let V be the closure of V_0 in this norm. Under assumptions C2, C3, C4 the problem (1.5) can be extended to (1.1) with the continuous sesquilinear hermitian form $b\colon V \times \times V \to C$ (cf. [7]).

2. Eigenvalues as successive maxima

For the proof of convergence of considered approximation the representation of eigenvalues as successive maxima of respective Rayleigh quotients will be used. Such an approach to eigenvalue approximation is presented for instance in [7]. The Rayleigh quotient for (1.1) is defined by

$$R(v) = \frac{b(v, v)}{(v, v)} \quad \text{for} \quad v \in V. \tag{2.1}$$

The sequence $\{\mu_n\}_{n=1}^{s}$ ($s \leqslant \infty$) of eigenvalues of Rayleigh quotient R is defined by reccurrent formulas.

For $n = 1$:

$$\mu_1 \stackrel{\mathrm{df}}{=} \sup_{v \in V} R(v).$$

If the supremum is attained at u_1, then we define

$$\mu_2 \stackrel{\mathrm{df}}{=} \sup_{\substack{v \in V \\ v \perp u_1}} R(v).$$

In the opposite case the sequence $\{\mu_n\}$ contains one element μ_1 only.

For $n > 1$: if $\mu_1 \geqslant \ldots \geqslant \mu_{n-1}$ are successive suprema attainable at the points $u_1, \ldots, u_{n-1}$, respectively, then

$$\mu_n \stackrel{\mathrm{df}}{=} \sup_{\substack{v \in V \\ v \perp u_1, \ldots, u_{n-1}}} R(v). \tag{2.2}$$

If the supremum μ_n is not attainable, then μ_n is the last element of the sequence $\{\mu_n\}$ and $s \stackrel{\mathrm{df}}{=} n$.

So, the sequence $\{\mu_n\}$ terminates if and only if for some n the supremum is not attained or when V is finite dimensional.

THEOREM 1. *Let $\{\mu_n\}_{n=1}^{s}$ be a sequence of eigenvalues of the Rayleigh quotient* (2.1). *Then $\forall\, n < s$ μ_n is an eigenvalue of* (1.1). *Conversely, there are no eigenvalues of* (1.1) *above μ_1, and if μ is an eigenvalue of* (1.1) *that satisfies $\mu > \mu_n$ for some n, then $\mu = \mu_k$ for some $k < n$. If* $\dim V = \infty$ *and $s < \infty$ then μ_s is the largest member of the essential spectrum of the operator T: $V \to V$ defined by: $b(u, v) = (Tu, v)$ $\forall\, v \in V$.*

The proof of Theorem 1 is contained in [7].

In the same manner we can represent each eigenvalue of the approximate problem (1.2) as an eigenvalue of Rayleigh quotient

$$R_h(v_h) = \frac{b_h(v_h, v_h)}{(v_h, v_h)_h} \tag{2.3}$$

on the space V_h. Since $\dim V_h < \infty$, the sequence $\{\mu_n^h\}_{n=1}^{s_h}$ of the eigenvalues of R_h is identical to the monotonic nonincreasing sequence of all eigenvalues of (1.2) repeated according to their multiplicity.

The following theorem will play the basic role in the proof of convergence of the approximation.

Let V_1 and V_2 be two Hilbert spaces with the scalar products $(\ ,\ )_1$ and $(\ ,\ )_2$, respectively. Let $b_1(u, v)$ and $b_2(u, v)$ be sesquilinear bounded forms on V_1 and V_2, respectively. Let $\mu_1^{(1)} \geqslant \mu_2^{(1)} \geqslant \ldots$ be the eigenvalues of the Rayleigh quotient $b_1(u, u)/(u, u)_1$ on V_1 and let $\mu_1^{(2)} \geqslant \mu_2^{(2)} \geqslant \ldots$ be the eigenvalues of $b_2(v, v)/(v, v)_2$ on V_2.

THEOREM 2 (Mapping principle). *Let $S_1 \subset V_1$ be a subspace containing the eigenfunctions $u_1^1, u_2^1, \ldots, u_n^1$ corresponding to $\mu_1^{(1)}, \ldots, \mu_n^{(1)}$. Let M be* *a linear transformation from S_1 into V_2 satisfying $Mu = 0$ iff $u = 0$. If there exists a nondecreasing function $f(\xi)$ such that the inequality*

$$(2.4) \qquad b_2(Mu, Mu) \geqslant f\left(\frac{b_1(u, u)}{(u, u)_1}\right)(Mu, Mu)_2$$

holds for all nonzero u in S_1, then

$$(2.5) \qquad \mu_n^{(2)} \geqslant f(\mu_n^{(1)}).$$

For the proof see [7], Chapter 3.

3. Convergence

Let $\mu_1 \geqslant \mu_2 \geqslant \ldots$ and $\mu_1^h \geqslant \mu_2^h \geqslant \ldots$ be sequences of eigenvalues of (1.1) and (1.2) respectively, where each eigenvalue is repeated according to its multiplicity.

In this section the distance between lth eigenvalue of the problem (1.1) and lth eigenvalue of the approximate problem (1.2) will be estimated using Theorem 2.

LEMMA 1. *Let assumptions* H1 *and* H2 *be satisfied. If U_n is finite dimensional subspace of V then*

$$\exists\, h_0\ \forall\, h < h_0\ \forall\, 0 \neq u \in U_n \quad r_h u \neq 0.$$

Proof. Suppose that it is not the case. So,

$$\exists\, \{h_j\}\ \exists\, \{u_j\}\ \ 0 \neq u_j \in U_n\ \forall\, j \quad r_{h_j} u_j = 0.$$

Since $\dim U_n < \infty$, the sequences $\{h_j\}$ and $\{u_j\}$ can be choosen in such a way that $\|u_j - u_0\| \to 0$, as $j \to \infty$, for some $u_0 \in U_n$, $\|u_0\| = 1$. From this and from H1 it follows that

$$\|r_{h_j} u_j\|_h \geqslant \|r_{h_j} u_0\|_h - \|r_{h_j}(u_j - u_0)\|_h \geqslant \|r_{h_j} u_0\|_h - a\|u_j - u_0\|.$$

The right-hand side tends to 1, by H2. Hence

$$\exists\, j_0\ \forall\, j \geqslant j_0 \quad \|r_{h_j} u_j\|_h > 0,$$

what contradicts the definition of u_j.

THEOREM 3. *Let assumptions* H1–H3 *be satisfied. If μ_l is l-th eigenvalue of* (1.1) *and $l < s$ then*

$$\forall\, \varepsilon > 0\ \exists\, h_\varepsilon\ \forall\, h < h_\varepsilon \quad \mu_l^h \geqslant \mu_l \cdot \frac{1}{1+\varepsilon} - \varepsilon.$$

Proof. Let U_l denote the subspace of V spanned by the first l eigenvectors $u_1, \ldots, u_l$ of the problem (1.1) corresponding to the eigenvalues

$\mu_1, \ldots, \mu_l$. For the proof we shall apply Theorem 2 for $V_1 = V$, $V_2 = V_h$, $S = U_l$ and $M = r_h$. Since U_l is finite dimensional, it follows from Lemma 1 that

$$\forall\, 0 \neq u \in U_l \quad r_h u \neq 0 \quad \text{for} \quad h < h_0.$$

Moreover, by H2 and H3

$$\forall\, \varepsilon\ \exists\, h(\varepsilon)\ \forall\, h < h(\varepsilon)\ \forall\, u \in U_l$$

$$|(r_h u, r_h u)_h - (u, u)| \leqslant \varepsilon(u, u), \qquad |b_h(r_h u, r_h u) - b(u, u)| \leqslant \varepsilon(u, u).$$

Thus

$$(r_h u, r_h u)_h \leqslant (1+\varepsilon)(u, u), \qquad b_h(r_h u, r_h u) \geqslant b(u, u) - \varepsilon(u, u)$$

and

$$\frac{b_h(r_h u, r_h u)}{(r_h u, r_h u)_h} \geqslant \frac{b(u, u)}{(u, u)}\, \frac{1}{(1+\varepsilon)} - \frac{\varepsilon}{1+\varepsilon}.$$

Thus according to Theorem 2

$$\mu_l^h \geqslant \mu_l \cdot \frac{1}{1+\varepsilon} - \frac{\varepsilon}{1+\varepsilon}.$$

The theorem above does not yet give the convergence of μ_l^h to μ_l in the general case. However, it implies this convergence in the special cases.

Remark 5. Let us consider the case $V_h \subset V$. Then according to Poincaré's principle (cf. [7], Th. 5.1, Chap. 3)

$$\mu_l = \sup_{\substack{S \subset V \\ \dim S = l}} \min_{v \in S} R(v) \geqslant \sup_{\substack{S \subset V_h \\ \dim S = l}} \min_{v \in S} R(v).$$

From this it follows that

$$\mu_l^h - \mu_l \leqslant \sup_{\substack{S \subset V_h \\ \dim S = l}} \min_{v \in S} R_h(v) - \sup_{\substack{S \subset V_h \\ \dim S = l}} \min_{v \in S} R(v).$$

If we assume that $R_h(v) = R(v)$ on V_h (as in the Galerkin method) then $\mu_l^h - \mu_l \leqslant 0$. This inequality together with Theorem 3 gives convergence $\mu_l^h \to \mu_l$ as $h \to 0$. It is a know result (cf. [5], [7]).

THEOREM 4. *Let assumptions* H4 *be satisfied. If* μ_l *is* l*-th eigenvalue of* (1.1) *and* $l \leqslant s$, *then*

$$\mu_l^h \leqslant \frac{1}{a_h^2} (\mu_l + \varepsilon_h).$$

Proof. According to H4 $V = N(r_h) \oplus \mathfrak{M}_h$. Let for any h

$$M_h \overset{\mathrm{df}}{=} (r_h|_{\mathfrak{M}_h})^{-1}.$$

So, M_h is a linear one-to-one transformation from V_h onto $\mathfrak{M}_h$. By assumption H4 (i)

$$\forall\, u \in \mathfrak{M}_h \quad b(u, u) \geqslant b_h(r_h u, r_h u) - \varepsilon_h(u, u).$$

Thus, for $u \in \mathfrak{M}_h$,

$$\frac{b(u, u)}{(u, u)} \geqslant \frac{b_h(r_h u, r_h u)}{(r_h u, r_h u)_h} \cdot \frac{(r_h u, r_h u)_h}{(u, u)} - \varepsilon_h \geqslant a_h^2 R_h(r_h u) - \varepsilon_h.$$

Applying now Theorem 2 for $V_1 = V_h$, $V_2 = V$, $S_1 = V_h$ and $M = M_h$ we obtain

$$\mu_l \geqslant a_h^2 \cdot \mu_l^h - \varepsilon_h$$

what proves Theorem 4.

Corollary 1. *If assumptions* H1–H4 *are satisfied then*

$\forall\, \varepsilon\ \exists\, h_\varepsilon\ \forall\, h < h_\varepsilon$

$$-\left(\frac{\varepsilon}{2}\mu_l + \varepsilon\right) \leqslant \mu_l^h - \mu_l \leqslant \left(\frac{1}{a_h^2} - 1\right)\mu_l + \frac{\varepsilon_h}{a_h^2}.$$

Corollary 2. *If* $\lambda_l = 1/\mu_l$, $\lambda_l^h = 1/\mu_l^h$ *then under assumptions* H1–H4

$\forall\, \varepsilon\ \exists\, h_\varepsilon\ \forall\, h < h_\varepsilon$

$$\lambda_l \frac{(a_h - 1) - \varepsilon_h \mu_l}{1 + \varepsilon_h \mu_l} \leqslant \lambda_l^h - \lambda_l \leqslant \frac{\varepsilon(1 + 2\lambda_l)}{1 - 2\varepsilon\lambda_l}\lambda_l.$$

4. Error estimates

In this section we assume additionally that b_h are uniformly bounded, i.e. $\exists\, \gamma > 0\ \forall\, u_h \in V_h$

$$|b_h(u_h, u_h)| \leqslant \gamma \|u_h\|_h^2.$$

Let $u_l \in V$, $\|u_l\| = 1$ be an eigenvector of (1.1) corresponding to μ_l. Moreover, let $u_l^h \in V_h$, $\|u_l^h\|_h = 1$ be an eigenvector of (1.2) corresponding to μ_l^h.

For each eigenvector u_l we define $P_h u_l \in V_h$ as follows: $P_h u_l$ is the solution of the equation

$$(4.1) \qquad (P_h u_l, v_h)_h = \frac{1}{\mu_l} b_h(r_h u_l, v_h) \quad \forall\, v_h \in V_h.$$

The term $\|P_h u_l - r_h u_l\|_h$ will be applied to estimate a distance between the eigenvectors u_l and u_l^h in the discrete norm $\|u_l^h - r_h u_l\|_h$ and next for an estimation of $|\mu_l - \mu_l^h|$. Let us set

$$\varepsilon_l^h = \sup_{v \in \mathfrak{M}_h} \frac{b_h(r_h u_l, r_h v) - b(u_l, v)}{\|v\|}, \tag{4.2}$$

$$\eta_l^h = \sup_{v \in \mathfrak{M}_h} \frac{(u_l, v) - (r_h u_l, r_h v)_h}{\|v\|}. \tag{4.3}$$

LEMMA 2. *If* H4 (ii) *is satisfied, then*

$$\|P_h u_l - r_h u_l\|_h \leqslant \frac{1}{a_h} \left(\eta_l^h + \frac{1}{\mu_l} \varepsilon_l^h \right).$$

Proof. Let

$$w_l^h \stackrel{\mathrm{df}}{=} (r_h|_{\mathfrak{M}_h})^{-1} (P_h u_l - r_h u_l).$$

From the definition of P_h it follows that

$$\begin{aligned}
&(P_h u_l - r_h u_l, P_h u_l - r_h u_l)_h \\
&\qquad = (u_l, w_l^h) - (r_h u_l, r_h w_l^h)_h + (P_h u_l, r_h w_l^h)_h - (u_l, w_l^h) \\
&\qquad \leqslant \eta_l^h \|w_l^h\| + \frac{1}{\mu_l} \big(b_h(r_h u_l, r_h w_l^h) - b(u_l, w_l^h) \big) \\
&\qquad \leqslant \left(\eta_l^h + \frac{1}{\mu_l} \varepsilon_l^h \right) \|w_l^h\|.
\end{aligned}$$

From assumption H4 (ii)

$$\forall\, w \in \mathfrak{M}_h \qquad \|w\| \leqslant \frac{1}{a_h} \|r_h w\|_h.$$

Thus

$$\|w_l^h\| \leqslant \frac{1}{a_h} \|P_h u_l - r_h u_l\|_h.$$

Finally we have

$$\|P_h u_l - r_h u_l\|_h^2 \leqslant \frac{1}{a_h} \left(\eta_l^h + \frac{1}{\mu_l} \varepsilon_l^h \right) \|P_h u_l - r_h u_l\|_h$$

what proves Lemma 2.

For the sake of simplicity we shall only consider the case of simple eigenvalues. The idea of proofs of following theorems is the same as in [1] and based on the hypothesis that there are no eigenvalues $\lambda_k^h = 1/\mu_k^h$ for $k \neq l$ and $h < h_0$ in the sufficiently small neighbourhood of $\lambda_l = 1/\mu_l$.

In the same manner as in [1] the results can be extended to the case of eigenvalues of multiplicity greater than 1.

THEOREM 5. *Let μ_l be a simple eigenvalue of* (1.1) *and $l \leqslant s$. If assumptions* H1–H4 *are satisfied, then*

$$\|r_h u_l - u_l^h\|_h \leqslant \left(2 + \frac{2}{\mu_l d_l}\right) \|r_h u_l - P_h u_l\|_h + \big|\|u_l\| - \|r_h u_l\|_h\big|, \tag{4.4}$$

where $0 < d_l \leqslant |\lambda_l - \lambda_j^h|$ $\forall\, j \neq l$.

Proof. The idea of the proof is the same as in [1]. Since (1.2) is a self-adjoint eigenproblem on finite dimensional space V_h, the orthonormal sequence $\{u_i^h\}_{i=1}^{r(h)}$ ($r(h) = \dim V_h$) of eigenvectors of (1.2) is a base of V_h. Thus

$$P_h u_l = \sum_{i=1}^{r(h)} (P_h u_l, u_i^h)_h \cdot u_i^h. \tag{4.5}$$

Let us write $a_l^h \stackrel{\mathrm{df}}{=} (P_h u_l, u_l^h)_h$. The suitable choice of the sign of u_l induces $a_l^h > 0$.

According to definition of P_h,

$$(P_h u_l, u_i^h)_h = \frac{1}{\mu_l} b_h(r_h u_l, u_i^h).$$

Since

$$b_h(r_h u_l, u_i^h) = \mu_i^h (r_h u_l, u_i^h)_h,$$

we get

$$\lambda_i^h (P_h u_l, u_i^h)_h = \lambda_l (r_h u_l, u_i^h)_h.$$

After subtraction of the term $\lambda_l (P_h u_l, u_i^h)_h$ from the both sides we obtain

$$(\lambda_i^h - \lambda_l)(P_h u_l, u_i^h)_h = \lambda_l (r_h u_l - P_h u_l, u_i^h)_h.$$

So, by the above relation it follows from (4.5) that

$$\|P_h u_l - a_l^h u_l^h\|_h^2 = \sum_{\substack{i=1 \\ i \neq l}}^{r(h)} \left(\frac{\lambda_l}{\lambda_i^h - \lambda_l}\right)^2 (r_h u_l - P_h u_l, u_i^h)_h^2.$$

Since for $i \neq l$ $|\lambda_i^h - \lambda_l| \geqslant d_l$, we have

$$\|P_h u_l - a_l^h u_l^h\|_h^2 \leqslant \frac{\lambda_l^2}{d_l^2} \sum_{\substack{i=1 \\ i \neq l}}^{r(h)} (r_h u_l - P_h u_l, u_i^h)_h^2$$

and by the Parseval equality

$$\|P_h u_l - a_l^h u_l^h\|_h^2 \leqslant \left(\frac{\lambda_l}{d_l}\right)^2 \|r_h u_l - P_h u_l\|_h^2. \tag{4.6}$$

Now, in order to obtain (4.4) we use the inequality

$$\|r_h u_l - u_l^h\|_h \leqslant \|r_h u_l - a_l^h u_l^h\|_h + |a_l^h - 1|.$$

Since

$$|a_l^h - 1| = \big|\|a_l^h u_l^h\|_h - \|r_h u_l\|_h + \|r_h u_l\|_h - 1\big|$$
$$\leqslant \|a_l^h u_l^h - r_h u_l\|_h + \big|\|r_h u_l\|_h - 1\big|,$$

we obtain

$$\|r_h u_l - u_l^h\|_h \leqslant 2\,\|r_h u_l - P_h u_l\|_h + 2\,\|P_h u_l - a_l^h u_l^h\|_h + \big|\|r_h u_l\|_h - 1\big|$$

and by (4.6) it follows from this that (4.4) holds.

THEOREM 6. *If μ_l is a simple eigenvalue of* (1.1) *and assumptions* H1–H4 *are satisfied, then*

$$(4.7)\qquad |\mu_l - \mu_l^h| \leqslant \frac{1}{1 - \|r_h u_l - u_l^h\|_h}\left\{\alpha\,\|r_h u_l - P_h u_l\|_h + \beta\,\big|\|u_l\| - \|r_h u_l\|_h\big|\right\},$$

where α, β are constants independent of h.

Proof. Since

$$\mu_l (P_h u_l, r_h u_l)_h = b_h(r_h u_l, r_h u_l)$$

and

$$\mu_l^h (u_l^h, r_h u_l)_h = b_h(u_l^h, r_h u_l),$$

we have

$$b_h(r_h u_l - u_l^h, r_h u_l) = \mu_l (P_h u_l - u_l^h, r_h u_l)_h + (\mu_l - \mu_l^h)(u_l^h, r_h u_l)_h.$$

Assuming that $(u_l^h, r_h u_l)_h \neq 0$ we get

$$\mu_l - \mu_l^h = \frac{1}{(u_l^h, r_h u_l)_h}\left[b_h(r_h u_l - u_l^h, r_h u_l) - \mu_l (P_h u_l - u_l^h, r_h u_l)_h\right].$$

Since

$$|(u_l^h, r_h u_l)_h| = |1 + (u_l^h, r_h u_l - u_l^h)_h| \geqslant 1 - \|r_h u_l - u_l^h\|_h$$

and b_h are uniformly bounded, the following inequality holds:

$$|\mu_l - \mu_l^h| \leqslant \frac{1}{1 - \|r_h u_l - u_l^h\|_h}\left[(\gamma + \mu_l)\|r_h u_l - u_l^h\|_h + \mu_l \|P_h u_l - r_h u_l\|_h\right].$$

Now, applying (4.4) we obtain (4.7).

5. Example

Let us consider the Sturm–Liouville problem

$$-(pu')' = \lambda u,$$
$$u(0) = 0, \quad u(1) = 0.$$

The variational formulation leads to the following problem:

(5.1) find $\mu \in \boldsymbol{R}$ and $0 \neq u \in H_0^1$ such that

$$b(u, v) = \mu \cdot a(u, v) \quad \forall\, v \in H_0^1$$

where

$$a(u, v) = \int_0^1 p(t)u'(t)v'(t)dt, \quad b(u, v) = \int_0^1 u(t)v(t)dt.$$

If $0 < \bar{\alpha} \leqslant p(t) \leqslant \bar{\beta} < \infty$ and $p \in H^1(0,1)$, then $a(u, v)$ is a scalar product on H_0^1 which implies the norm equivalent to the norm of H_0^1. The form b is bounded on H_0^1.

Let us consider the approximation of H_0^1 described in Example 1.

Let scalar product $(\ ,\ )_h$ on $\boldsymbol{R}^{n-1}$ $(n = 1/h)$ and form b_h be defined as follows: for

$$\boldsymbol{u} = (u_1, \ldots, u_{n-1}), \quad \boldsymbol{v} = (v_1, \ldots, v_{n-1}), \quad \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{R}^{n-1}$$

we put $u_0 = u_n = v_0 = v_n = 0$ and

$$(\boldsymbol{u}, \boldsymbol{v})_h = h \sum_{i=1}^{n} p_i \frac{u_i - u_{i-1}}{h} \cdot \frac{v_i - v_{i-1}}{h};$$

$$b_h(\boldsymbol{u}, \boldsymbol{v}) = h \sum_{i=1}^{n} u_i v_i,$$

where $p_i = p(x_i)$.

The approximate problem has the form

(5.2) find $\mu_h \in \boldsymbol{R}$ and $0 \neq \boldsymbol{u} \in \boldsymbol{R}^{n-1}$ such that

$$b_h(\boldsymbol{u}, \boldsymbol{v}) = \mu_h(\boldsymbol{u}, \boldsymbol{v})_h \quad \forall\, \boldsymbol{v} \in \boldsymbol{R}^{n-1}.$$

It is easy to verify that assumptions H1–H4 are satisfied. Let us note that $b_h(r_h u, r_h u)$ converges to $b(u, u)$ uniformly on H_0^1. We have

$$\left| h \sum_{i=1}^{n} u_i^2 - \int_0^1 u^2 dt \right| \leqslant ch\|u\|^2 \quad \text{where} \quad u_i = u(x_i).$$

So, assumption H4 (i) is satisfied with the constant $\varepsilon_h = O(h)$.

Let $\tilde{\mathfrak{M}}_h$ be the subspace of H_0^1 defined in Example 1, and let $\mathfrak{M}_h$ be the closure of $\tilde{\mathfrak{M}}_h$ in H^1. In Example 1 it was stated that if $p \equiv 1$, then assumption H4 (ii) is satisfied with $a_h = 1$. Let $p \not\equiv 1$. For $u \in \tilde{\mathfrak{M}}_h$ (cf. Ex. 1)

$$\|u\|^2 = \sum_{i=1}^{n} \left(\frac{u(ih) - u\big((i-1)h\big)}{h} \right)^2 \int_{(i-1)h}^{ih} p(x)\,dx.$$

Thus $\forall\, u \in \tilde{\mathfrak{M}}_h$

$$\frac{\|r_h u\|_h^2}{\|u\|^2} = 1 - c_h(u),$$

where

$$c_h(u) = \frac{\sum_{i=1}^{n} \left(\frac{u(ih) - u\big((i-1)h\big)}{h}\right)^2 \int_{(i-1)h}^{ih} \big(p(x) - p(ih)\big)\,dx}{\|u\|^2}.$$

We have an estimation

$$\sup_{u \in \tilde{\mathfrak{M}}_h} |c_h(u)| \leqslant \sqrt{h}\, \frac{\|p'\|_{L^2}}{\bar{a}}.$$

Since $\mathfrak{M}_h$ is the closure of $\tilde{\mathfrak{M}}_h$, it follows from the above that $\forall\, u \in \mathfrak{M}_h$

$$1 - \sqrt{h}\, \frac{\|p'\|_{L^2}}{\bar{a}} \leqslant \frac{\|r_h u\|_h}{\|u\|} \leqslant 1 + \sqrt{h}\, \frac{\|p'\|_{L^2}}{\bar{a}}.$$

Thus we obtain condition H4 (ii) with the constant $a_h = 1 + O(\sqrt{h})$. Here the rate of convergence a_h to 1 depends on regularity of p.

It remains to find ε_i^h and η_i^h (cf. (4.2), (4.3)) for our example.

$$\Big|\int_0^1 uv\,dt - h \sum_{i=1}^{n} u_i v_i\Big| \leqslant \Big|\sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} (u - u_i) v\,dt\Big| + \Big|\sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} u_i (v - v_i)\,dt\Big|.$$

After integration by parts of the first term and using the equality

$$u_i = h \sum_{j=1}^{i-1} \frac{u_{j+1} - u_j}{h}$$

in the second, we obtain

$$|b(u, v) - b_h(r_h u, r_h v)| \leqslant \Big|\sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} v(\xi)\,d\xi \int_{x_i}^{x_{i+1}} u'(\xi)\,d\xi\Big| +$$

$$+ \Big|\sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} \int_{x_i}^{t} v(\xi)\,d\xi\, u'(t)\,dt\Big| + \Big|\sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} h \sum_{j=1}^{i-1} \frac{u_{j+1} - u_j}{h} \int_{x_i}^{t} v'(\xi)\,d\xi\,dt\Big|.$$

Now, applying the Schwartz inequality we get

$$|b(u, v) - b_h(r_h u, r_h v)| \leqslant c h^{3/2} \|r_h u\|_h \|v\|_{H^1}.$$

Thus according to definition (4.2)

$$\varepsilon_l^h \leqslant c\|r_h u_l\|_h h^{3/2}.$$

Now, for simplicity, let $p \equiv 1$. Let $v, w \in \mathfrak{M}_h$. Then $(w, v) = (r_h w, r_h v)_h$. Thus, taking into account that $Q_h = r_h|_{\mathfrak{M}_h}^{-1}$, r_h is orthogonal (in this case) projection onto $\mathfrak{M}_h$ and $r_h Q_h u = r_h u$, we obtain for $v \in \tilde{\mathfrak{M}}_h$

$$\begin{aligned} |(u_l, v) - (r_h u_l, r_h v)_h| &\leqslant |(u_l - Q_h u_l, v)| + |(Q_h u_l, v) - (r_h u_l, r_h v)_h| \\ &= |(u_l - Q_h u_l, v)| + |(r_h Q_h u_l, r_h v)_h - (r_h u_l, r_h v)_h| \\ &\leqslant c\|u_l - Q_h u_l\|_{H^1}\|v\|. \end{aligned}$$

Thus

$$\eta_l^h \leqslant c\,\mathrm{dist}(u_l, \mathfrak{M}_h).$$

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