

## A NOTE ON THE TRAJECTORIES OF A CLASS OF HEREDITARY SYSTEMS

BOGDAN RZEPECKI

*Institute of Mathematics, A. Mickiewicz University, Poznań, Poland*

### 1. Introduction

A hereditary system is a system whose present state is determined in some way by its past history. M. Kisielewicz [6] considered the problem of the existence of optimal control for a class of hereditary dynamical systems described by a functional-differential equation of the Hale and Cruz ([3]) type

$$\frac{d}{dt} [D(t)H_t x] = f(t, H_t x, u(t)).$$

It is the aim of this note to characterize some properties of the set of admissible trajectories of a class of hereditary systems with the above equation. This class is large enough to include all functional-differential equations of retarded type, Volterra integral equations, difference equations and some types of differential equations of neutral type. For more information and applications we refer the reader to [1]–[6]. See also the references in the papers cited above.

### 2. Preliminaries and assumptions

Let  $X$  be an arbitrary metric space. If  $A, B$  are nonempty subsets of  $X$ , then let

$$\text{Dist}(A, B) = \max[\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)],$$

where  $d(v, Z) = \inf\{\rho(v, z) : z \in Z\}$ . The function  $\text{Dist}$  defines a metric in the family of nonempty bounded closed subsets of  $X$ .

Let us denote by  $\text{comp}(X)$  the metric space of nonempty compact subsets of  $X$  with the distance  $\text{Dist}$ ; by  $\text{Li}_{n \rightarrow \infty} X_n$ ,  $\text{Ls}_{n \rightarrow \infty} X_n$  the lower limit

and the upper limit of a sequence  $(X_n)$  of subsets of  $X$ , respectively. If  $\text{Li } X_n = \text{Ls } X_n = X_0$ , we write  $X_0 = \lim_{n \rightarrow \infty} X_n$ .

For the definitions and the properties of the operations  $\text{Li}$ ,  $\text{Ls}$  and  $\text{Lim}$  we refer to [7]. In particular, it is known that if the space  $X$  is compact, then  $\lim_{n \rightarrow \infty} \text{Dist}(X_n, X_0) = 0$  if and only if  $\lim_{n \rightarrow \infty} X_n = X_0$  and  $\text{comp}(X)$  is also compact.

Throughout this paper  $J = [0, \infty)$ . Moreover, we shall denote by  $\mathbf{R}^n$  the  $n$ -dimensional Euclidean space with the norm  $|\cdot|$  (in particular, let us put  $\mathbf{R} = \mathbf{R}^1$ ), and by  $C_Z (= C(Z, \mathbf{R}))$  the Banach space of continuous functions from any compact  $Z$  of  $\mathbf{R}^n$  to  $\mathbf{R}$  with the usual supremum norm  $\|\cdot\|$ .

Let  $\Omega \in \text{comp}(\mathbf{R})$ . Define

$$\hat{u}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty u(s) \exp(-i\omega s) ds, \quad \omega \in \mathbf{R}$$

to be the Fourier transform of  $u$  in the space  $L_2(J, \mathbf{R}^m)$ . Now let us put

$$U_\Omega = \{u \in L_2(J, \mathbf{R}^m) : \hat{u}(\omega) = 0 \text{ for a.e. } \omega \text{ in } \mathbf{R} \setminus \Omega\}.$$

Then a measurable function  $u: J \rightarrow \mathbf{R}^m$  is said to be an *admissible control* if  $u \in U \subset U_\Omega$  and  $U$  is compact. It is known ([2]) that  $U_\Omega$  together with norm  $\|\cdot\|_L$  (here  $\|u\|_L = (\int_0^\infty |u(t)|^2 dt)^{1/2}$  for  $u$  in  $L_2(J, \mathbf{R}^m)$ ) is a Banach space. (The necessary and sufficient conditions for a subset  $U$  of  $U_\Omega$  to be conditionally compact are that  $U$  is bounded and  $\lim_{s \rightarrow \infty} \int_s^\infty |u(t)|^2 dt = 0$  uniformly with respect to  $u$  in  $U$ .)

Let  $Q \in \text{comp}(\mathbf{R})$  and zero is the maximal element of  $Q$ . Suppose that  $\alpha: J \times Q \rightarrow \mathbf{R}$  is a continuous function such that  $\alpha(t, u) \leq \alpha(t, v)$  for  $t \in J$  and  $u, v$  in  $Q$  whenever  $u \leq v$ , and  $\alpha(t, 0) = t$  for  $t \in J$ . Let us put

$$J_0 = \bigcup_{t \geq 0} \overline{\text{conv}}(\{\alpha(t, s) : s \in Q\}) \cap (-\infty, 0]$$

(here  $\overline{\text{conv}}(X)$  denotes the closed convex hull of  $X$ ). Denote by  $\mathcal{X}$  the set of all continuous mappings from  $J_0 \cup [0, \infty)$  to  $\mathbf{R}^n$ . Moreover, for fixed  $t$  in  $J$ , the mapping  $H_t: \mathcal{X} \rightarrow C_Q$  will be defined by

$$(H_t x)(s) = x(\alpha(t, s))$$

for  $s \in Q$ .

The above triple  $(Q, \alpha, H)$  is called a *hereditary structure* [3] if  $H$  is the operator from  $J \times \mathcal{X}$  to  $C_Q$  given by  $(t, x) \mapsto H_t x$ .

The set  $\mathcal{X}$  will be considered as a vector space with the topology of almost uniform convergence. This topology is determined by the sequence  $(p_n)$  of seminorms given by

$$p_n(x) = \sup \{|x(t)| : t \in ([-n, 0] \cap J_0) \cup [0, n]\} \quad (n = 1, 2, \dots)$$

for  $x$  in  $\mathcal{X}$ , and therefore (see [11])  $\mathcal{X}$  is a Fréchet space.

Assumptions (1)–(5) given below are valid throughout the paper and will not be repeated in formulations of particular results. Suppose that:

- (1)  $U$  is a compact subset of  $U_D$ .
- (2)  $(Q, \alpha, H)$  is a hereditary structure.
- (3)  $\varphi: J_0 \rightarrow \mathbf{R}^n$  is a continuous function.

(4)  $f: J \times C_Q \times \mathbf{R}^m \rightarrow \mathbf{R}^n$  is a function with the following properties: it satisfies the Carathéodory condition (i.e., it is a function measurable with respect to the variable  $t \in J$  for all fixed  $(x, v)$  in  $C_Q \times \mathbf{R}^m$ , and continuous with respect to variables  $(x, v) \in C_Q \times \mathbf{R}^m$  for almost all fixed  $t$  in  $J$ ), and

$$|f(t, x, v)| \leq m(t)(1 + \|x\|), \quad |f(t, x, v) - f(t, y, v)| \leq L(t)\|x - y\|$$

for almost every  $t \in J$  and all  $x, y$  in  $C_Q$  and  $v \in \mathbf{R}^m$ , where  $m$  and  $L$  are locally Lebesgue integrable functions of  $J$  into itself.

(5)  $g: J \times C_Q \rightarrow \mathbf{R}^n$  is a continuous function such that  $|g(t, x) - g(t, y)| \leq k\|x - y\|$  for  $t \in J$  and  $x, y$  in  $C_Q$ .

Let us put

$$D(t)y = y(0) - g(t, y) \quad \text{for all } (t, y) \text{ in } J \times C_Q.$$

In this paper we investigate a functional-differential equation of the form

$$(+)\quad \frac{d}{dt}[D(t)H_t x] = f(t, H_t x, u(t)),$$

where  $u$  is a given control function.

Now, we formulate the initial value problem for (+): Given a  $u$  in  $L_2(J, \mathbf{R}^m)$ . We say that a continuous function  $x_u$  from  $J_0 \cup [0, \infty)$  into  $\mathbf{R}^n$  is a solution of (+) with the initial value  $\varphi$  corresponding to the control  $u$  if  $x_u$  coincides with  $\varphi$  on  $J_0$  and

$$D(t)H_t x_u = D(0)H_0 x_u + \int_0^t f(s, H_s x_u, u(s)) ds$$

for  $t$  in  $J$ .

### 3. Some extensions

Let us remark that we can generalize the results given below to an equation of the form

$$(+ +) \quad \frac{d}{dt} [D(t)H_t x] = f_1(t, H_t x, u(t)) + \int_0^t K(t, s) f_2(s, H_s x, u(s)) ds,$$

where  $f_i$  ( $i = 1, 2$ ) satisfy the above assumption (4) and  $K$  is e.g. an  $n \times n$  matrix measurable bounded function defined on  $\{(t, s): 0 \leq s \leq t\}$ . Below we shall treat the case (+) since for (+ +) the proofs are similar and the reader can reproduce them easily.

To appreciate the generality of (+ +), let us consider some special cases of this equation. If  $g = 0$ , then (+ +) reduces to the equation

$$x'(t) = f_1(t, H_t x, u(t)) + \int_0^t K(t, s) f_2(s, H_s x, u(s)) ds$$

which is usually referred to as an equation of retarded type

$$x'(t) = f_1(t, x_t, u(t)) + \int_0^t K(t, s) f_2(s, x_s, u(s)) ds$$

(here we employed the conventional notation  $x_t(\theta) = x(t+\theta)$  for  $-r \leq \theta \leq 0$  with  $r \geq 0$ ). On the other hand, (+ +) includes the general class of equations

$$\frac{d}{dt} [D(t)x_t] = f_1(t, x_t, u(t)) + \int_0^t K(t, s) f_2(s, x_s, u(s)) ds$$

and the equation

$$x'(t) = h(t, x_t) + \int_0^t K(t, s) f(s, x(s), u(s)) ds$$

investigated by Ahmed [1].

### 4. Existence of the unique solution

The method used here is based on a type of the Banach Contraction Principle in a Fréchet space (see [9], [10]).

Assume that  $E$  is a Fréchet space [11] and denote by  $P = \{p_n: n = 1, 2, \dots\}$  a saturated family of seminorms which generates the topology of  $E$ . Let us state the fixed-point result in the following form:

PROPOSITION. Let  $X$  be a nonempty subset of  $E$ . Suppose that  $T$  and  $F$  are mappings from  $X$  into  $E$  satisfying the following conditions:

(1)  $T$  is one-to-one and  $T[X]$  is a closed set, (2)  $F[X] \subset T[X]$ , and (3)  $p(Fx - Fy) \leq q \cdot p(Tx - Ty)$  for  $p$  in  $P$  and  $x, y$  in  $X$  and with a constant  $q < 1$ . Then there exists a unique point  $x_0$  in  $X$  such that  $Fx_0 = Tx_0$ .

THEOREM 1. For an arbitrary  $u$  there exists a unique solution  $x_u$  of (+) with the initial value  $\varphi$  corresponding to the control  $u$ .

Proof. Let  $X$  be the set of all  $x$  in  $\mathcal{X}$  which are identically equal to the function  $\varphi$  on  $J_0$ . Define mappings  $T, F$  as follows:

$$(Tx)(t) = p^{-1} \cdot \exp \left( -r \int_0^t L(s) ds \right) \cdot x(t),$$

$$(Fx)(t) = \begin{cases} p^{-1} \cdot \exp \left( -r \int_0^t L(s) ds \right) \cdot \varphi(t) & \text{if } t \in J_0, \\ p^{-1} \cdot \exp \left( -r \int_0^t L(s) ds \right) \cdot \left( \varphi(0) + g(t, H_t x) - \right. \\ \left. - g(0, H_0 x) + \int_0^t f(s, H_s x, u(s)) ds \right) & \text{if } t \geq 0 \end{cases}$$

for  $x$  in  $X$ , where  $p > 3k$  and  $r > 3$  are constants. Evidently,  $T$  is a one-to-one mapping such that  $F[X] \subset T[X]$  and  $T[X]$  is a closed subset of  $\mathcal{X}$ .

Let  $x, y \in X$  and let  $n$  be a positive integer. For  $0 \leq t \leq n$ , we obtain

$$\begin{aligned} |g(t, H_t x) - g(t, H_t y)| &\leq k \|H_t x - H_t y\| \\ &= k \cdot \sup_{\tau \in Q} |x(a(t, \tau)) - y(a(t, \tau))| \\ &\leq k \cdot \exp \left( r \int_0^t L(s) ds \right) \cdot \sup_{\tau \in Q} |(Tx)(a(t, \tau)) - \\ &\quad - (Ty)(a(t, \tau))| \leq k \cdot \exp \left( r \int_0^t L(s) ds \right) \cdot \\ &\quad \cdot \sup_{0 \leq s \leq n} |(Tx)(s) - (Ty)(s)|, \\ &\int_0^t |f(s, H_s x, u(s)) - f(s, H_s y, u(s))| ds \\ &\leq \int_0^t L(s) \|H_s x - H_s y\| ds \end{aligned}$$

$$\begin{aligned}
&= p \cdot \int_0^t L(s) \sup_{\tau \in Q} |(Tx)(\alpha(s, \tau)) - \\
&\quad - (Ty)(\alpha(s, \tau))| \exp \left( r \cdot \int_0^{\alpha(s, \tau)} L(\sigma) d\sigma \right) ds \\
&< p \cdot \sup_{0 \leq s \leq n} |(Tx)(s) - (Ty)(s)| \cdot \int_0^t L(s) \exp \left( r \int_0^s L(\sigma) d\sigma \right) ds \\
&< p \cdot r^{-1} \cdot \exp \left( r \cdot \int_0^t L(s) ds \right) \cdot \sup_{0 \leq s \leq n} |(Tx)(s) - (Ty)(s)|;
\end{aligned}$$

hence

$$\begin{aligned}
|(Fx)(t) - (Fy)(t)| &\leq p^{-1} \cdot \exp \left( -r \int_0^t L(s) ds \right) \cdot (|g(0, H_0 x) - g(0, H_0 y)| + \\
&\quad + |g(t, H_t x) - g(t, H_t y)| + \\
&\quad + \int_0^t |f(s, H_s x, u(s)) - f(s, H_s y, u(s))| ds) \\
&\leq (p^{-1} k + r^{-1}) \cdot \sup_{0 \leq s \leq n} |(Tx)(s) - (Ty)(s)|
\end{aligned}$$

and it follows that  $p_n(Fx - Fy) \leq (p^{-1}k + r^{-1}) \cdot p_n(Tx - Ty)$ . Therefore our fixed-point proposition is applicable to the mappings  $T$ ,  $F$  and space  $\mathcal{X}$  with the subset  $X$ , and we are done.

## 5. Main results

Let us denote by  $x_u$  the solution of (+) with the initial value  $\varphi$  corresponding to the admissible control  $u$ . By the set of admissible trajectories of (+) we understand the family of all  $x_u$  where  $u$  ranges over  $U$ . It was shown in [6] (Lemma 12) that this set is a compact subset of  $\mathcal{X}$ . Next, we use the following notations:

$$P = [0, T] \quad \text{with} \quad T \geq 0,$$

$$S(U) = \{x_u: u \in U\},$$

$$S(t, U) = \{x(t): x \in S(U)\} \quad \text{for} \quad t \in J,$$

$$S(I, U) = \bigcup \{S(t, U): t \in I\} \quad \text{for} \quad I \subset P.$$

By the compactness of  $S(U)$ , we easily conclude that  $S(I, U)$  is compact whenever  $I$  is a closed subset of  $P$ . Now, we prove

LEMMA 1. Let  $(I_n)$  be a sequence of nonempty subsets of  $P$  such that  $\text{Ls}_{n \rightarrow \infty} I_n$  is nonempty, and let  $\text{Ls}_{n \rightarrow \infty} I_n \subset I_0$ . Then

$$\sup \{d(q, S(I_0, U)) : q \in \overline{S(I_n, U)}\} \rightarrow 0$$

as  $n \rightarrow \infty$ .

*Proof.* Assume the existence of  $\varepsilon > 0$  and a subsequence  $(I_i)$  of the sequence  $(I_n)$  such that

$$\sup \{d(q, S(I_0, U)) : q \in \overline{S(I_i, U)}\} \geq \varepsilon \quad \text{for } i \geq 1.$$

Fix an index  $i$ . Then there exists a sequence  $(q_m^{(i)})$  with  $q_m^{(i)}$  in the compact  $\overline{S(I_i, U)}$  and  $\varepsilon - 1/m < d(q_m^{(i)}, S(I_0, U))$  for  $m = 1, 2, \dots$ . Without loss of generality we may suppose that  $q_m^{(i)} \rightarrow q_i$  as  $m \rightarrow \infty$ . Consequently,  $d(q_i, S(I_0, U)) \geq \varepsilon$  with  $q_i \in \overline{S(I_i, U)}$  for  $i = 1, 2, \dots$ . Similarly ( $S(P, U)$  is compact) we obtain  $q_i \rightarrow q$  as  $i \rightarrow \infty$ . From the above follows  $d(q, S(I_0, U)) \geq \varepsilon$ , whence  $q \notin \overline{S(I_0, U)}$ .

Now, let  $(t_i)$  and  $(x_i)$  be sequences with  $t_i \in \bar{I}_i$ ,  $x_i \in S(U)$ ,  $q_i = x_i(t_i)$  such that  $t_i \rightarrow t_0$  and  $x_i \rightarrow x_0$  as  $i \rightarrow \infty$ . Consider the restrictions of  $x_i$  and  $x_0$  to the compact  $P$ . Then  $(x_i(t_i))$  converges to  $x_0(t_0)$ , and therefore  $q = x_0(t_0)$ . We have  $x_0 \in S(U)$  and  $t_0 \in \text{Ls}_{n \rightarrow \infty} \bar{I}_n$ , that means  $q \in \overline{S(I_0, U)}$ . This contradiction concludes the proof.

LEMMA 2. Let  $(I_n)$  be a sequence such as in Lemma 1 and, moreover, let  $S(I_0, U) \subset \text{Li}_{n \rightarrow \infty} S(I_n, U)$ . Under these assumptions

$$\text{Dist}(\overline{S(I_n, U)}, \overline{S(I_0, U)}) \rightarrow 0$$

as  $n \rightarrow \infty$ .

*Proof.* To show that  $\text{Lim}_{n \rightarrow \infty} \overline{S(I_n, U)} = \overline{S(I_0, U)}$  (comp  $\{S(P, U)\}$  is a compact space) it suffices to note that

$$\text{Ls}_{n \rightarrow \infty} S(I_n, U) \subset \overline{S(I_0, U)} \quad \text{and} \quad \overline{S(I_0, U)} \subset \text{Li}_{n \rightarrow \infty} S(I_n, U).$$

We shall only show the first inclusion.

Let  $p \in \text{Ls}_{n \rightarrow \infty} S(I_n, U)$ . Then there exists indices  $k_1 < k_2 < \dots$  and points  $p_{k_n}$  in  $S(I_{k_n}, U)$  such that  $p_{k_n} \rightarrow p$  as  $n \rightarrow \infty$ . We have

$$d(p_{k_n}, \overline{S(I_0, U)}) \leq \sup \{d(q, \overline{S(I_0, U)}) : q \in \overline{S(I_{k_n}, U)}\} \quad \text{for } n = 1, 2, \dots$$

On account of Lemma 1 it follows that

$$\sup \{d(q, \overline{S(I_0, U)}): q \in \overline{S(I_n, U)}\} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This implies  $d(p, \overline{S(I_0, U)}) = 0$ , and we get our inclusion.

Now, let  $(I_n)$  be a sequence from  $\text{comp}(P)$  such that  $\text{Dist}(I_n, I_0) \rightarrow 0$  as  $n \rightarrow \infty$ . Then  $\text{Ls}_{n \rightarrow \infty} I_n = I_0 = \text{Li}_{n \rightarrow \infty} I_n$  and it is easily verified that  $S(I_0, U) \subset \text{Li}_{n \rightarrow \infty} S(I_n, U)$ . Consequently, Lemma 2 is applicable to our sequence and therefore

$$\text{Dist}(S(I_n, U), S(I_0, U)) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

So we have proved the following

**THEOREM 2.** *The transformation  $I \mapsto S(I, U)$  maps continuously  $\text{comp}(P)$  into  $\text{comp}(\mathbf{R}^n)$ .*

The restriction of  $I \mapsto S(I, U)$  to  $P$  ( $P$  is regarded as a subspace of  $\text{comp}(P)$ ) defines another mapping from  $P$  onto  $\{S(t, U): t \in P\}$ . Therefore, by Theorem 2, we deduce that  $t \mapsto S(t, U)$  is continuous on  $J$  (cf. [6], Lemma 14).

Finally, by means of the function  $t \mapsto \{(t, x(t)): x \in S(U)\}$  we can generalize the above result. More precisely, we have

**THEOREM 3.** *Denote by  $S_I(U)$  (here  $I$  is a subset of  $P$ ) the set of all points  $(t, x(t))$  in  $I \times \mathbf{R}^n$  such that  $t \in I$  and  $x \in S(U)$ . Then the map  $I \mapsto S_I(U)$  is a homeomorphism between  $\text{comp}(P)$  and  $\{S_I(U): I \in \text{comp}(P)\}$  ( $\subset \text{comp}(P \times \mathbf{R}^n)$ ).*

*Proof.* Modifying the proof of Theorem 2 we conclude that the one-to-one map  $I \mapsto S_I(U)$  is continuous on the compact space  $\text{comp}(P)$ ; hence, by Theorem XVI.2.4 of [8], our conclusion follows.

## 6. Final remarks

M. Kisielewicz [6] considered the following control problems:

(a) Find a control  $u^*$  in  $U$  that minimizes the cost functional  $I_1$  defined on  $U$  by

$$I_1(u) = \int_0^\infty f^0(t, x_u(t), u(t)) dt.$$

(b) Find a control  $u^*$  in  $U$  that minimizes the functional  $I_2$  defined on  $U$  by

$$I_2(u) = \inf\{t \geq 0: x_u(t) \in \mathcal{F}(t)\}.$$



(c) Find a control  $u^*$  in  $U$  that steers system (+) with initial value  $\varphi$  to the set  $\mathcal{F}(t^*)$  in the minimum time  $t^*$  and also minimizes the cost functional  $I_3$  defined on  $U$  by

$$I_3(u) = \int_0^{t^*} f^0(t, x_u(t), u(t)) dt.$$

(d) Find a control  $u^*$  in  $U$  that maximizes the functional  $I_4$  defined on  $U$  by

$$I_4(u) = \mu(\{t \geq 0: x_u(t) \in \mathcal{F}(t)\}).$$

Here  $f^0$  is a non-negative scalar-valued function on  $J \times \mathbf{R}^n \times \mathbf{R}^m$ ,  $\mathcal{F}(t)$  ( $t \geq 0$ ) is a moving target set in the state space  $\mathbf{R}^n$ ,  $\mu$  is the Lebesgue measure on  $\mathbf{R}$ , and  $x_u$  denotes the solution of (+) with the control  $u \in U$ .

This type of control problems together with the system

$$\begin{cases} x'(t) = f(t, x(t), u(t)), \\ x(0) = x_0 \end{cases}$$

have been investigated by Ahmed in [2].

Let us remark that Ahmed-Kisielewicz optimal control results ([2], [6]) can also be extended to the class of hereditary systems described by (+ +). Note, for example, the following result of this sort:

*Suppose that  $f_i$  ( $i = 1, 2$ ) and  $K$  are functions defined in Section 3. Denote by  $\tilde{x}_u$  the unique solution of (+ +) with initial value  $\varphi$  corresponding to  $u \in U$ . Assume that  $f^0: J \times \mathbf{R}^n \times \mathbf{R}^m \rightarrow J$  is lower-semicontinuous on  $\mathbf{R}^n \times \mathbf{R}^m$  for almost every  $t$  in  $J$ . Then there is a control  $u_0$  in  $U$  such that*

$$\int_0^\infty f^0(t, \tilde{x}_{u_0}(t), u_0(t)) dt = \min_{u \in U} \int_0^\infty f^0(t, \tilde{x}_u(t), u(t)) dt.$$

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