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PROPOSITIONAL DYNAMIC LOGIC WITH RECURSIVE PROGRAMS

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Introduction

In this paper we introduce a generalization of PDL and dynamic algebras in which the operation of iteration procedure α^* is replaced by a more complex recursion operator on programs $\Gamma_\omega(\alpha_1, \ldots, \alpha_n)$. To illustrate the idea of the generalization let us recall the meaning of α^* via the fixed-point semantics.

A natural semantics of PDL is the notion of dynamic set-algebra. In such algebras programs are identified with binary relations in a fixed set U (the domain), the program operations $\circ$ (composition) and $\cup$ (nondeterministic choice) are identified as the composition and union of relations respectively, and the program constants O (nowhere defined program) and I (the program of identity function) are the empty relation O and the identity relation I respectively. The iteration is defined as follows: $\alpha^* = \bigcup_{i=0}^{\infty} \alpha^i$. It is easy to see that α^* is the least solution of the inequality $I \cup \alpha \circ \theta \subseteq \theta$, which also satisfies the equality $I \cup \alpha \circ \alpha^* = \alpha^*$. So α^* is the least fixed point of the program term $\Gamma(\alpha, \theta) = I \cup \alpha \circ \theta$.

Now let $\Gamma(\xi_1, \ldots, \xi_n, \theta)$ be an arbitrary program term built from the variables $\xi_1, \ldots, \xi_n, \theta$ and constants O and I by means of the operations $\circ$ and $\cup$. Then it is easy to see that for any relations $\alpha_1, \ldots, \alpha_n$ the inequality $\Gamma(\alpha_1, \ldots, \alpha_n, \theta) \subseteq \theta$ has a least solution $\Gamma_\omega(\alpha_1, \ldots, \alpha_n)$ which is a fixed point of the map $\theta \mapsto \Gamma(\alpha_1, \ldots, \alpha_n, \theta)$. In agreement with the standard fixed-point semantics of recursive programs, $\Gamma_\omega(\alpha_1, \ldots, \alpha_n)$ can be understood as a nondeterministic recursive procedure, calling itself a number of times in the process of its execution. Obviously, the iteration procedure α^* is a special case of recursion.

An arithmetical axiomatization of a first-order dynamic logic with recursive procedures in the above-described sense is given by Harel [2]

(Harel calls his logic a context-free dynamic logic — CFDL). In this paper we consider a kind of a propositional version of CFDL. Namely, for any program term $\Gamma(\xi_1, \ldots, \xi_n, \theta)$ we define a propositional dynamic logic denoted by Γ_ω-PDL and give for it an infinitary axiomatization in the style of Mirkowska [5] and Goldblatt [1] and a completeness theorem with respect to the fixed-point semantics described above. For that purpose we first introduce the notion of Γ_ω-dynamic algebra and give an analog of the Stone set-representation theorem for some class of such algebras.

1. Γ_ω-dynamic algebras

The notion of Γ_ω-dynamic algebra (Γ_ω-DA) generalizes the notion of dynamic algebra introduced by Kozen [4] and Pratt [6]. Γ_ω-DA is a two-sorted system $(B, P) = ((B, 0, 1, \wedge, \vee, \Rightarrow, \neg), (P, O, I, \circ, \cup, \Gamma, \Gamma_\omega), [\ ], ?)$ combining two algebras: $\boldsymbol{B} = (B, 0, 1, \wedge, \vee, \Rightarrow, \neg)$ which is a Boolean algebra, and $\boldsymbol{P} = (P, O, I, \circ, \cup, \Gamma, \Gamma_\omega)$, called here the *programming part* of $(B. P)$. The elements of P are called *programs*, O and I are elements of P standing here as an abstract of the nowhere defined program and the program of identity function respectively. The binary operations $\circ$ and $\cup$ are called *composition* and *nondeterministic choice*. The operation Γ: $P^{n+1} \to P$ is an $(n+1)$-ary program operation which is a superposition of O, I, $\circ$ and $\cup$. The operation Γ_ω: $P^n \to P$ is an n-ary program operation connected with Γ and standing here as an abstract of the nondeterministic recursion described in the introduction above. The box operation $[\ ]$: $P \times B \to B$ and the test operation $?$: $B \to P$ are inter-sort operations and have the same meaning as in dynamic algebras.

The axioms are the following, with $q, r \in B$ and $a, b, c \in P$:

1. $[a](q \wedge r) = [a]q \wedge [a]r$.
2. $[a]1 = 1$.
3. $[O]q = 1$.
4. $[I]q = q$.
5. $[a \circ b]q = [a][b]q$.
6. $[a \cup b]q = [a]q \wedge [b]q$.
7. $[r?]q = r \Rightarrow q$.

To formulate the remaining axiom concerning the operations Γ and Γ_ω we introduce the following sequence for any $a_1, \ldots, a_n \in P$:

$$\Gamma_0(a_1, \ldots, a_n) = O, \quad \Gamma_{k+1}(a_1, \ldots, a_n) = \Gamma(a_1, \ldots, a_n, \Gamma_k(a_1, \ldots, a_n)).$$

Then the last axiom is the following:

8. For any $a_1, \ldots, a_n, c \in P$ and $q \in B$ the infinite meet $\bigcap_{k<\omega} [c \circ \Gamma_k(a_1, \ldots, a_n)]q$ exists in B and is equal to $[c \circ \Gamma_\omega(a_1, \ldots, a_n)]q$.

Following Pratt [6] we call (B, P) *separable* if for $a, b \in P$

$$(\forall q \in B)([a]q = [b]q) \rightarrow a = b.$$

Let (B, P) and (B', P') be Γ_ω-DA's. An *isomorphism* (*homomorphism*) from (B, P) into (B', P') is a pair (h, g), where h and g are isomorphisms (homomorphisms) from B into B' and from P into P', respectively, with $h([a]q) = [g(a)]h(q)$ and $g(q?) = h(q)?$ for any $q \in B$, $a \in P$.

Let $P_0 \subseteq P$, $B_0 \subseteq B$, and define by simultaneous induction:

$$P_{k+1} = P_k \cup \{O, I\} \cup \{q? \mid q \in B_k\} \cup \{a \Box b \mid a, b \in P_k,\ \Box \in \{\circ, \cup\}\} \cup \{\Gamma_\omega(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in P_k\},$$

$$B_{k+1} = B_k \cup \{0, 1\} \cup \{[a]q \mid a \in P_k,\ q \in B_k\} \cup \{q \Box r \mid q, r \in B_k,\ \Box \in \{\wedge, \vee, \Rightarrow\}\} \cup \{\neg q \mid q \in B_k\}.$$

Let Σ be a class of Γ_ω-DA's. We say that (B, P) is *free in* Σ if there exist $B_0 \subseteq B$ and $P_0 \subseteq P$ such that the following two conditions are satisfied:

(1) $B = \bigcup_{k<\omega} B_k$ and $P = \bigcup_{k<\omega} P_k$.

(2) If $(B', P') \in \Sigma$, h_0: $B_0 \rightarrow B'$ and g_0: $P_0 \rightarrow P'$, then (h_0, g_0) can be extended to a homomorphism from (B, P) into (B', P'); we call such a pair (B_0, P_0) a *free generator* of (B, P).

2. Set Γ_ω-dynamic algebras

Let U be a nonempty set. For any $q \subseteq U$ and $a, b \subseteq U \times U$ we define:

$$O = \emptyset \quad \text{(the empty relation)}, \qquad I = \{(x, x) \mid x \in U\},$$

$$a \circ b = \{(x, y) \mid \exists z((x, z) \in a \text{ and } (z, y) \in b)\},$$

$$a \cup b = \{(x, y) \mid (x, y) \in a \text{ or } (x, y) \in b\},$$

$$[a]q = \{x \in U \mid (\forall y \in U)((x, y) \in a \rightarrow y \in q)\},$$

$$q? = \{(x, x) \mid x \in q\}.$$

Let Γ: $(U \times U)^{n+1} \rightarrow U \times U$ be a fixed function which is a superposition of O, I, $\circ$ and $\cup$ and let Γ_k, $k = 0, 1, 2, \ldots$, be defined as in the definition of Γ_ω-DA. Then for any $a_1, \ldots, a_n \subseteq U \times U$ we define:

$$\Gamma_\omega(a_1, \ldots, a_n) = \bigcup_{k<\omega} \Gamma_k(a_1, \ldots, a_n).$$

By applying the Tarski fixed-point theorem ([9], see also [8]) to the mapping $\theta \mapsto \Gamma(a_1, \ldots, a_n, \theta)$, it can easily be seen that $\Gamma_\omega(a_1, \ldots, a_n)$ is the least solution of the inequality $\Gamma(a_1, \ldots, a_n, \theta) \subseteq \theta$ which is a fixed point of Γ, i.e., $\Gamma(a_1, \ldots, a_n, \Gamma_\omega(a_1, \ldots, a_n)) = \Gamma_\omega(a_1, \ldots, a_n)$.

Then the following theorem is true.

THEOREM 1. *The algebra* $(2^U, 2^{U\times U}) = ((2^U, \emptyset, U, \wedge, \vee, \Rightarrow, \neg), (2^{U\times U}, O, I, \circ, \cup, \Gamma, \Gamma_\omega), [\ ], ?)$, *where* $(2^U, \emptyset, U, \wedge, \vee \Rightarrow, \neg)$ *is the Boolean algebra of all subsets of* U *and* $(2^{U\times U}, O, I, \circ, \cup, \Gamma, \Gamma_\omega)$, $[\ ]$, ? *are defined as above, is a separable* Γ_ω*-DA, called the set* Γ_ω*-DA over* U.

The proof is straightforward.

3. Representation theorem

Let (B, P) be a Γ_ω-DA. We call x a *filter* in (B, P) if x is a filter in B; x is a Γ_ω*-filter* in (B, P) if x is a filter in (B, P) and for any $q \in B$, $c, a_1, \ldots, a_n \in P$,

$$\forall k([c \circ \Gamma_k(a_1, \ldots, a_n)]q \in x) \rightarrow [c \circ \Gamma_\omega(a_1, \ldots, a_n)]q \in x.$$

Let $x \subseteq B$, $a \in P$ and define $[a]x = \{q \in B \mid [a]q \in x\}$. It is easy to verify that if x is a Γ_ω-filter then so is $[a]x$.

For $x \subseteq B$, by $th(x)$ we denote the smallest Γ_ω-filter such that $x \subseteq th(x)$.

THE DEDUCTION LEMMA. *If* x *is a* Γ_ω*-filter and* $q \in B$, *then for any* $r \in B$, $q \Rightarrow r \in x$ *iff* $r \in th(x \cup \{q\})$.

Proof. The *if* part is trivial. For the *only if* part we consider the set $y = \{p \in B \mid q \Rightarrow p \in x\}$. Clearly, y is a filter and $x \cup \{q\} \subseteq y$. Since x is a Γ_ω-filter we conclude, from the conditions for ? and $\circ$, that y is a Γ_ω-filter. This justifies the lemma.

A Γ_ω*-ultrafilter* in (B, P) is an ultrafilter in B which is a Γ_ω-filter in (B, P). (*In* [7], Γ_ω-ultrafilters are called *Q-filters.*)

We call (B, P) a *countable* Γ_ω*-DA* if B and P are countable sets.

THE SEPARATION LEMMA. *Let* (B, P) *be a countable* Γ_ω*-DA. If* x *is a* Γ_ω*-filter in* (B, P) *and* $p \in B \setminus x$, *then there exists a* Γ_ω*-ultrafilter* y *such that* $x \subseteq y$ *and* $p \notin y$.

Proof. We shall use a construction similar to that in [1].

Let $p_0, p_1, \ldots$ be an enumeration of all infinite meets of the type $\bigcap_{i<\omega} [c \circ \Gamma_i(a_1, \ldots, a_n)]q$ and let $r_0, r_1, \ldots$ be an enumeration of all elements of B. We set $q_{2m} = r_m$, $q_{2m+1} = p_m$. We define by induction on k an increasing sequence $x_0, x_1, \ldots, x_k, \ldots$ of Γ_ω-filters.

Let $x_0 = th(x \cup \{\neg p\})$. Taking into account the Deduction Lemma, we obtain $0 \notin x_0$. Let x_k be defined.

Case 1. $0 \notin th(x_k \cup \{q_k\})$. Then $x_{k+1} = th(x_k \cup \{q_k\})$.

Case 2. $0 \in th(x_k \cup \{q_k\})$. Then $\neg q_k \in x_k$.

Subcase 2.1. $k = 2m$. Then $x_{k+1} = x_k$.

Subcase 2.2. $k = 2m+1$. Then $q_k = \bigcap_{i<\omega} [c \circ \Gamma_i(a_1, \ldots, a_n)]q$ for some $c, a_1, \ldots, a_n \in P$ and $q \in B$. If $0 \in th(x_k \cup \{\neg[c \circ \Gamma_i(a_1, \ldots, a_n)]q\})$ for all $i < \omega$, then $[c \circ \Gamma_i(a_1, \ldots, a_n)]q \in x_k$ for all $i < \omega$, by the Deduction Lemma, and so $[c \circ \Gamma_\omega(a_1, \ldots, a_n)]q \in x_k$. But, from $\neg \bigcap_{i<\omega} [c \circ \Gamma_i(a_1, \ldots, a_n)]q \in x_k$ and Axiom 8, this would imply that $0 \in x_k$. Hence $0 \notin th(x_k \cup \{\neg[c \circ \Gamma_{i_k}(a_1, \ldots, a_n)]q\})$ for some $i_k < \omega$. We put $x_{k+1} = th(x_k \cup \{\neg[c \circ \Gamma_i(a_1, \ldots, a_n)]q\})$.

Let $y = \bigcup_{k<\omega} x_k$. Clearly, $x \subseteq y$, $\neg p \in y$, $0 \notin y$ and y is a filter. Any $q \in B$ is q_m for some m. But either $q_m \in x_{m+1}$ or $\neg q_m \in x_{m+1}$. So at least one of q, $\neg q$ is in y. Hence y is an ultrafilter. Suppose $[c \circ \Gamma_i(a_1, \ldots, a_n)]q \in y$ for all $i < \omega$ and let $\bigcap_{i<\omega} [c \circ \Gamma_i(a_1, \ldots, a_n)]q = p_m$. Then if $p_m \notin y$, we have $\neg q_{2m+1} \in x_{2m+2}$. But then $\neg[c \circ \Gamma_{i_{2m+1}}(a_1, \ldots, a_n)]q \in x_{2m+2}$. Hence $0 \in y$, which contradicts $0 \notin y$. We conclude that $[c \circ \Gamma_\omega(a_1, \ldots, a_n)]q \in y$. Therefore y is a Γ_ω-ultrafilter.

This completes the proof.

THEOREM 2. *Let (B, P) be a countable separable Γ_ω-DA free in the class of all separable Γ_ω-DA's. Then there exist a set U and an isomorphism (h, g) from (B, P) into the set Γ_ω-DA over U, $(2^U, 2^{U \times U})$.*

Proof. Let U be the set of all Γ_ω-ultrafilters in (B, P). Let (B_0, P_0) be a free generator of (B, P). For $a \in P_0$ and $p \in B_0$ put

$$g_0(a) = \{(x, y) \mid x, y \in U,\ [a]x \subseteq y\}, \qquad h_0(p) = \{x \mid x \in U,\ p \in x\}.$$

There exists a homomorphism (h, g) which extends (h_0, g_0). We will prove that (h, g) is an isomorphism from (B, P) into $(2^U, 2^{U \times U})$.

LEMMA 1. *For any $q \in B$ and any $a \in P$*

(i) $h(q) = \{x \mid q \in x\}$.

(ii) $(\forall r \in B)([a]r \in x \leftrightarrow (\forall y)((x, y) \in g(a) \to r \in y))$.

Proof. $B = \bigcup_{k<\omega} B_k$ and $P = \bigcup_{k<\omega} P_k$; hence there exists k such that $q \in B_k$ and $a \in P_k$. We prove (i) and (ii) simultaneously by induction on k.

Base of induction. $k = 0$, i.e., $q \in B_0$ and $a \in P_0$. (i) is trivial. (ii) follows from the Separation Lemma.

Induction hypothesis. For any $q \in B_k$ and for any $a \in P_k$, (i) and (ii) are true.

SUBLEMMA. *Let T be an m-ary program operation which is a superposition of O, I, $\circ$ and $\cup$. For any $a_1, \ldots, a_m \in P_k$ and for any $r \in B$*

$$[T(a_1, \ldots, a_m)]r \in x \quad \textit{iff} \quad (\forall y)((x, y) \in g(T(a_1, \ldots, a_m)) \to r \in y).$$

Proof is by induction on the construction of T. We omit it.

Induction step. Let $q \in B_{k+1}$ and $a \in P_{k+1}$. The verification of claims (i) and (ii) is straightforward.

Now we can easily complete the proof of the theorem by proving that (h, g) is an isomorphism from (B, P) into $(2^U, 2^{U \times U})$. Let $q \in B$ and $q \neq 0$. By the Separation Lemma there exists $x \in U$ such that $q \in x$, hence $x \in h(q)$ and $h(q) \neq \emptyset$. Hence h is an isomorphism. Let $a, b \in P$ and $a \neq b$. Since (B, P) is a separable Γ_ω-DA, there exists $q \in B$ such that $[a]q \neq [b]q$. But h is an isomorphism, $[g(a)]h(q) = h([a]q)$ and $[g(b)]h(q) = h([b]q)$, whence $[g(a)]h(q) \neq [g(b)]h(q)$. Theorem 1 yields $g(a) \neq g(b)$. This completes the proof of Theorem 2.

4. Propositional dynamic logic with the operator Γ_ω (standing for the least fixed point)

Language. Φ_0 is a countable set of propositional variables, Π_0 is a finite or countable nonempty set of atomic programs, 0, 1, $\wedge$, $\vee$, $\Rightarrow$, $\neg$, [ ] – logical connectives, O, I, $\circ$, $\cup$, Γ_ω, ?-program operators, ( ) – parentheses.

The set of programs Π and the set of formulas Φ are defined by simultaneous induction as follows:

(1) $\Pi_0 \subseteq \Pi$, $\{O, I\} \subseteq \Pi$, $\Phi_0 \subseteq \Phi$, $\{0, 1\} \subseteq \Phi$,

(2) If $a, b, a_1, \ldots, a_n \in \Pi$ and $q, r \in \Phi$, then $(a \circ b)$, $(a \cup b)$, $q?$, $\Gamma_\omega(a_1, \ldots, a_n) \in \Pi$ and $(q \wedge r)$, $(q \vee r)$, $(q \Rightarrow r)$, $\neg q$, $[a]q \in \Phi$.

We abbreviate: $q \Leftrightarrow r = (q \Rightarrow r) \wedge (r \Rightarrow q)$.

Axioms. All (or enough) propositional tautologies,

$$[a](q \Rightarrow r) \Rightarrow ([a]q \Rightarrow [a]r), \quad [O]q \Leftrightarrow 1, \quad [I]q \Leftrightarrow q, \quad [a \circ b]q \Leftrightarrow [a][b]q,$$

$$[a \cup b]q \Leftrightarrow ([a]q \wedge [b]q), \quad [q?]r \Leftrightarrow (q \Rightarrow r),$$

$$[\Gamma_\omega(a_1, \ldots, a_n)]q \Rightarrow [\Gamma(a_1, \ldots, a_n, \Gamma_\omega(a_1, \ldots, a_n))]q.$$

Rules.

$$\text{(MP)}\ \frac{q \Rightarrow r,\ q}{r}, \quad \text{(Nor)}\ \frac{q}{[a]q},$$

$$(\Gamma_\omega R)\ \frac{[c \circ \Gamma_k(a_1, \ldots, a_n)]q, \text{ for } k = 0, 1, 2, \ldots}{[c \circ \Gamma_\omega(a_1, \ldots, a_n)]q}.$$

A *logic* is a set $L \subseteq \Phi$ containing all axioms, closed under substitution and the rules (MP), (Nor) and $(\Gamma_\omega R)$. The intersection of any collection of logics is itself a logic. Hence there exists a smallest logic which we call Γ_ω-PDL.

Semantics. Let (B, P) be a Γ_ω-DA. A *valuation in* (B, P) is a pair (h_0, g_0) where $h_0\colon \Phi_0 \to B$ and $g_0\colon \Pi_0 \to P$. We extend inductively (h_0, g_0) to the mappings $h\colon \Phi \to B$ and $g\colon \Pi \to P$ in a usual way. We say that a formula p is *true in* (B, P) if for every valuation (h_0, g_0) in (B, P) we have $h(p) = 1$.

Let L be a logic and (B, P) a Γ_ω-DA. We say that (B, P) is a Γ_ω-*dynamic L-algebra* if for any $p \in L$, p is true in (B, P).

5. The Lindenbaum algebra for Γ_ω-logics

Let L be a consistent logic, i.e., $L \neq \Phi$. For any $q, r \in \Phi$ and $a, b \in \Pi$ define $q \overset{L}{\vdash\dashv} r$ iff $q \Leftrightarrow r \in L$ and $a \overset{L}{\sim} b$ iff $[a]p \Leftrightarrow [b]p \in L$ for some $p \in \Phi_0$ not occurring in a, b.

LEMMA 2 (i) $\overset{L}{\vdash\dashv}$ *is a congruence relation in* Φ *with respect to* $\wedge, \Rightarrow, \vee, \neg$.

(ii) $\overset{L}{\sim}$ *is a congruence relation in* Π *with respect to* $\circ, \cup, \Gamma_\omega$,

(iii) *If* $q, r \in \Phi$ *and* $q \overset{L}{\vdash\dashv} r$, *then* $q? \overset{L}{\sim} r?$.

(iv) *If* $q, r \in \Phi$, $a, b \in \Pi$, $q \overset{L}{\vdash\dashv} r$ *and* $a \overset{L}{\sim} b$, *then* $[a]q \overset{L}{\vdash\dashv} [b]r$.

Proof. Omitted.

Lemma 2 enables us to form quotients $\Phi/\overset{L}{\vdash\dashv}$ and $\Pi/\overset{L}{\sim}$ and to define in them the boolean operations and the program operators in the usual way [10]. The algebra $(\Phi/\overset{L}{\vdash\dashv}, \Pi/\overset{L}{\sim})$ is called the *Lindenbaum algebra of L.*

THEOREM 3. *The Lindenbaum algebra of L is a countable separable* Γ_ω-*DA free in the class of all separable* Γ_ω-*dynamic L-algebras.*

Proof. It easy to see that $(\Phi/\overset{L}{\vdash\dashv}, \Pi/\overset{L}{\sim})$ is a countable separable Γ_ω-DA. To complete the proof, see [10].

6. Completeness theorem for Γ_ω-PDL

Let p be a formula. The following conditions are equivalent:

(i) *p is a theorem of* Γ_ω-*PDL, i.e.,* $p \in \Gamma_\omega$-*PDL.*

(ii) *p is true in any* Γ_ω-*DA.*

(iii) *p is true in any countable separable* Γ_ω-*DA free in the class of all* Γ_ω-*DA's.*

(iv) *p is true in the Lindenbaum algebra of* Γ_ω-*PDL.*

(v) *p is true in any set* Γ_ω-*DA.*

Proof. (i) $\to$ (ii) and (iv) $\to$ (i) – in a standard way.

(ii) $\to$ (iii) $\to$ (iv) and (iii) $\to$ (v) are obvious.

$\neg$(iii) $\to$ $\neg$(v) by the Representation Theorem.

Note that (i) $\leftrightarrow$ (v) states the completeness of Γ_ω-PDL with respect to the standard Kripke semantics.

7. Remarks

1 Let $\Gamma = \xi_1 \circ (\xi_2 \circ \theta \circ \xi_3)$. Then Γ_ω-PDL is Π_1^1-complete [3].

2. In an analogous way we also consider logics and algebras with many Γ_ω-operators. In particular, the context-free PDL (propositional analog of CFDL from [2]) obtained by adding all possible Γ_ω-operators is complete with respect to the standard Kripke semantics.

3. We can extend Γ_ω-PDL adding special programs and boolean constants interpreted as counters and simple pushdown stores. The above results remain true.

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