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EQUIPPED GRAPHS AND MODULAR LATTICES ATTACHED TO THEM

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1. Basic definitions and results

1.1. Let Γ be a graph without loops and without orientation, Γ_0 its set of verticies and Γ_1 its set of edges. For each $b \in \Gamma_1$ we write $\vartheta(b)$ for the 2-point set of ends of the edge b. We define an *equipment* v of Γ as a collection $v = \{v_b\}$ of functions $v_b\colon \vartheta(b) \to \{0, 1\}$. Hence in the equipped graph we have four types of "equipped" edges: $\circ\overset{0\quad 0}{\text{——}}\circ$, $\circ\overset{1\quad 1}{\text{——}}\circ$, $\circ\overset{1\quad 0}{\text{——}}\circ$, $\circ\overset{0\quad 1}{\text{——}}\circ$. Sometimes it is convenient to present these edges as $\circ\!\longleftrightarrow\!\circ$, $\circ\!\rightarrow\!\leftarrow\!\circ$, $\circ\!\rightarrow\!\rightarrow\!\circ$, $\circ\!\leftarrow\!\leftarrow\!\circ$. With every oriented graph Γ_Δ without loops we can associate an equipped graph Γ^v in the following way: an edge $i\circ\!\leftarrow\!\circ j$ in Γ_Δ transforms into the edge $i\circ\overset{0\quad 1}{\text{——}}\circ j$ or $i\circ\!\leftarrow\!\leftarrow\!\circ j$ in Γ^v.

1.2. Let Γ^v be an equipped graph. We associate with Γ^v a *modular lattice* $L(\Gamma^v)$ in the following way. The lattice $L(\Gamma^v)$ is defined by generators $\{v_i, w_b\}$, $i \in \Gamma_0$, $b \in \Gamma_1$, and relations (¹):

L_1: $v_i(\sum_{j\neq i} v_j) = 0$ for each vertex $i \in \Gamma_0$;
L_2: if $\vartheta(b) = \{i, j\}$, then $w_b \subseteq v_i + v_j$ for each edge $b \in \Gamma_1$;
L_{3a}: if $v_b(i) = 0$, then $w_b v_i = 0$;
L_{3b}: if $v_b(i) = 1$, then $w_b + v_j = v_i + v_j$.

In the following example v_i and w_b denote not only the generators of $L(\Gamma^v)$, but also the corresponding vertices and edges of Γ^v:

EXAMPLE 1 (see Fig. 1). The generators are $v_1, \ldots, v_5$; w_{12}, w_{13}, w_{14}, w_{45}. The relations are the following: $v_i(\sum_{j\neq i} v_j) = 0$, $w_{ij} \subseteq v_i + v_j$ for all

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This paper is in final form and no version of it will be submitted for publication elsewhere.

(¹) We denote the operations of intersection and union in a modular lattice L by $\cdot$ and $+$, so that $x \cap y = xy$ and $x \cup y = x + y$ and $\bigvee_i x_i = \sum_i x_i$.

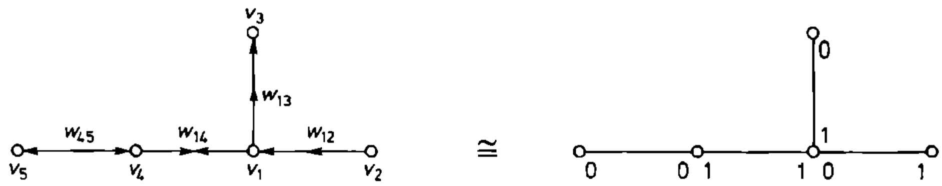

Fig. 1. $L(\Gamma^v) = L(D_5^v)$

$i, j \in \Gamma_0$, and $w_{12}v_1 = 0$, $w_{12}+v_1 = v_1+v_2$; $w_{13}v_3 = 0$, $w_{13}+v_3 = v_1+v_3$; $w_{14}+v_1 = w_{14}+v_4 = v_1+v_4$; $w_{45}v_4 = w_{45}v_5 = 0$.

1.3. Let L be a modular lattice with maximal and minimal elements v and 0. Let V be a finite-dimensional vector space over a commutative field k and let $\mathscr{L}(V)$ be the lattice of all vector subspaces of V. A *representation* ϱ of L in V is a lattice morphism ϱ: $L \to \mathscr{L}(V)$ such that $\varrho(0) = 0$ and $\varrho(v) = V$. Thus ϱ associates with any $x, y \in L$ subspaces $\varrho(x)$ and $\varrho(y)$ of V in such a way that $\varrho(xy) = \varrho(x)\varrho(y)$ and $\varrho(x+y) = \varrho(x)+\varrho(y)$.

If ϱ_1 and ϱ_2 are representations of the same lattice L in spaces V_1 and V_2 (over the same field k) then a *morphism of representations* ψ: $\varrho_1 \to \varrho_2$ is a linear map ψ: $V_1 \to V_2$ such that $\psi\varrho_1(x) \subseteq \varrho_2(x)$ for every $x \in L$. It is easy to check that we have thus defined the *category of representations* $\mathscr{R}(L, k)$ of L in finite-dimensional vector spaces over k. In this category the notions of direct sum of representations and of decomposable and indecomposable representations are defined in the usual way.

Let $\Gamma^v = \{\Gamma_0, \Gamma_1, \vartheta, v\}$ be an equipped graph. In [6] the category of representations $\mathscr{R}(\Gamma^v, k)$ of Γ^v in finite-dimensional vector spaces over k has been defined in the following way. An object $\varrho \in \mathscr{R}(\Gamma^v, k)$, $\varrho = \{V, V_i, W_b;\ i \in \Gamma_0, b \in \Gamma_1\}$, is a k-vector space V with a system of subspaces V_i and W_b satisfying the following relations: 1° $V = \bigoplus_{i \in \Gamma_0} V_i$; 2° if $\vartheta(b) = \{i, j\}$, then $W_b \subseteq V_i + V_j$, and if moreover $v(i) = 0$, then $W_b V_i = 0$, whereas if $v(i) = 1$, then $W_b + V_j = V_i + V_j$. A morphism ψ: $\varrho \to \varrho'$ in $\mathscr{R}(\Gamma^v, k)$ is a linear map $\psi: V \to V'$ such that $\psi V_i \subseteq V_i'$ and $\psi W_b \subseteq W_b'$ for all $i \in \Gamma_0$ and $b \in \Gamma_1$.

PROPOSITION 1.1. *The category $\mathscr{R}(\Gamma^v, k)$ of representations of an equipped graph Γ^v is equivalent to the category $\mathscr{R}(L(\Gamma^v), k)$ of representations of the modular lattice $L(\Gamma^v)$.*

Proof is an easy check of definitions.

It follows from Proposition 1.1 and the results of [6] that the following theorem is true:

THEOREM 1.2. *The category $\mathscr{R}(L(\Gamma^v), k)$ of representations of the modular lattice $L(\Gamma^v)$ of a connected graph Γ has a finite number of mutually nonisomorphic indecomposable representations if and only if Γ is one of the Dynkin diagrams A_n $(n \geqslant 1)$; D_n $(n \geqslant 4)$; E_6, E_7 or E_8.*

Let us define an integral-valued function $d_\varrho: \Gamma_0 \to \mathbf{N}$ by $d_\varrho(i) = \dim \varrho(v_i)$ for every vertex $i \in \Gamma_0$. We shall call the function d_ϱ the *dimension* of the representation ϱ. Similarly to the results of Gabriel [3] the following theorem is true:

THEOREM 1.3. *Let Γ be one of the Dynkin diagrams: $\Gamma \in \{A_n\ (n \geqslant 1),\ D_n\ (n \geqslant 4),\ E_6, E_7, E_8\}$ and let v be an equipment of Γ. Then there exists a one-to-one correspondence between isomorphism classes of indecomposable representations of the lattice $L(\Gamma^v)$ and positive roots of Γ, such that if a representation τ corresponds to a positive root $\alpha = \sum_{i \in \Gamma_0} n_i \alpha_i$, then $d_\tau(i) = n_i$. Here $\{\alpha_i\}_{i \in \Gamma_0}$ is the set of all simple roots attached to the Dynkin diagram Γ.*

Remark. If we have a Dynkin diagram $\Gamma \in \{A_n\ (n \geqslant 1),\ D_n\ (n \geqslant 4),\ E_6, E_7, E_8\}$ and some indecomposable representation τ_α of $L(\Gamma^v)$, then using the function $d_\alpha = d_{\tau_\alpha}$ we can uniquely restore $\dim W_b = \dim \tau_\alpha(w_b)$ for the subspace W_b which corresponds to an equipped edge b of the graph Γ^v.

Theorems 1.2 and 1.3 generalize the theorems of Gabriel [3] in the following way:

PROPOSITION 1.4. *Let Γ_A be an oriented graph without loops and let Γ^v be the corresponding equipped graph. Then the category $\mathscr{L}_k(\Gamma_A)$ of representations of Γ_A is equivalent to the category $\mathscr{R}(L(\Gamma^v), k)$ of representations of the modular lattice $L(\Gamma^v)$.*

Proof. According to the definition (see [1]) a representation (V, f) of Γ_A consists of a family of finite-dimensional vector spaces V_i over the field k labelled by the vertices i of Γ_A ($i \in \Gamma_0$) and of a family of linear maps f_l: $V_{s(l)} \to V_{t(l)}$. Here $s(l),\ t(l) \in \Gamma_0$ are the starting and terminal point of the edge $l \in \Gamma_1$. Let us construct a representation ϱ of $L(\Gamma^v)$ attached to (V, f). The lattice $L(\Gamma^v)$ has generators v_i, $i \in \Gamma_0$, and w_l, $l \in \Gamma_1$, and the following relations: 1) $v_i(\sum_{j \neq i} v_j) = 0$ for all $i \in \Gamma_0$; 2) $w_l \subseteq v_{s(l)} + v_{t(l)}$ for each $l \in \Gamma_1$; 3) $w_l v_{t(l)} = 0$; 4) $w_l + v_{t(l)} = v_{s(l)} + v_{t(l)}$. Notice that the edge $s(l) \to t(l)$ of Γ_A corresponds to the edge $s(l) \overset{1\quad 0}{—} t(l)$ of Γ^v. Let us define ϱ as the representation in the space $V = \bigoplus_{i \in \Gamma_0} V_i$ such that

$$\varrho(v_i) = \{(0, \ldots, 0, x_i, 0, \ldots, 0) \mid x_i \in V_i\},$$

$$\varrho(w_l) = \{(0, \ldots, 0, x_{s(l)}, 0, \ldots, 0, f_l(x_{s(l)}), 0, \ldots, 0) \mid x_{s(l)} \in V_{s(l)},\ f_l(x_{s(l)}) \in V_{t(l)}\}.$$

It is easy to check that $\varrho(w_l)\varrho(v_{t(l)}) = 0$ and $\varrho(w_l) + \varrho(v_{t(l)}) = \varrho(v_{s(l)}) + \varrho(v_{t(l)})$. This means that we have defined a representation ϱ of the modular lattice $L(\Gamma^v)$. This construction admits inversion, i.e. starting from $\varrho \in \mathscr{R}(L(\Gamma^v), k)$ we can construct the object $(V, f) \in \mathscr{L}_k(\Gamma_A)$. It is also easy to check that there is an equivalence between the categories $\mathscr{R}(L(\Gamma^v), k)$ and $\mathscr{L}_k(\Gamma_A)$.

1.4. Our main aim is the investigation of the lattices $L(\Gamma^v)$ by the methods of representation theory. We will study the structure of the image of $L(\Gamma^v)$

under the representation $\varrho = \bigoplus \tau_\alpha$,where the sum is taken over all indecomposable representations of $L(\Gamma^\nu)$.

A representation τ_α of a lattice L in a vector space $V = \mathbf{Q}^m$ (over the field $\mathbf{Q}$ of rational numbers) is called *irreducible* if the image $\tau_\alpha(L)$ of L coincides with the lattice of all vector subspaces of $\mathbf{Q}^m$.

THEOREM 1.5. *Let Γ be a Dynkin diagram ($\Gamma \in \{A_n$ $(n \geqslant 3)$, D_n $(n \geqslant 4)$, $E_6, E_7, E_8\}$) and let τ_α be an indecomposable representation of the lattice $L(\Gamma^\nu)$ in the vector space $V = \mathbf{Q}^d$ with $\dim V = \sum_{i \in \Gamma_0} d_\alpha(i) = d > 2$. Then τ_α is irreducible.*

A representation ϱ of a modular lattice L is called *preprojective* (resp. *preinjective*) if there exist only a finite number of isomorphism classes of indecomposable representations τ such that $\operatorname{Hom}(\tau, \varrho) \neq 0$ (respectively $\operatorname{Hom}(\varrho, \tau) \neq 0$). For the Dynkin diagrams Γ every indecomposable representation τ is simultaneously preprojective and preinjective. Let us partially order the set $\mathscr{P}_\Gamma^\nu$ of all isomorphism classes of indecomposable representations of the Dynkin diagram Γ in the following way: $\tau_\alpha \geqslant \tau_\beta$ if and only if there exists a sequence $\tau_1, \ldots, \tau_s \in \mathscr{P}_\Gamma^\nu$ such that $\tau_1 \cong \tau_\alpha$, $\tau_s \cong \tau_\beta$ and $\operatorname{Hom}(\tau_i, \tau_{i+1}) \neq 0$ for all i $(1 \leqslant i \leqslant s-1)$. It should be noted that this partial order depends on the choice of the equipment ν of Γ.

An element a of a modular lattice L is called *perfect* if for every indecomposable representation τ in a space V either $\tau(a) = 0$, or $\tau(a) = V$. The set all indecomposable representations $\tau \in \mathscr{P}$ such that $\tau(a) = V$ is called the *characteristic set* of a.

A subset H of a partially ordered set P is called *hereditary* if the conditions $x \in H$, $y \leqslant x$ imply $y \in H$.

Let Γ be a Dynkin diagram. It is easy to prove that the characteristic set of each perfect element h of $L(\Gamma^\nu)$ is hereditary with respect to the partial order on $\mathscr{P}_\Gamma^\nu$. The proof of the following theorem is more involved.

THEOREM 1.6. *Let Γ be a Dynkin diagram ($\Gamma \in \{A_n$ $(n \geqslant 1)$, D_n $(n \geqslant 4)$, $E_6, E_7, E_8\}$). Each hereditary subset H of $\mathscr{P}_\Gamma^\nu$ is characteristic, i.e. there exists a perfect element $h \in L(\Gamma^\nu)$ such that $H = \{\tau \in \mathscr{P}_\Gamma^\nu \mid \tau(h) = V\}$.*

In the next sections we will give explicit proofs of the theorems above in the case of the Dynkin diagram A_n with special equipment.

2. Lattices $L(A_n^0)$

2.1. Let A_n^0 be the Dynkin diagram A_n with the following equipment:

$$\overset{v_1}{\circ} \underset{w_1}{\longleftrightarrow} \overset{v_2}{\circ} \underset{w_2}{\longleftrightarrow} \overset{v_3}{\circ} \quad \ldots \quad \overset{v_{n-1}}{\circ} \underset{w_{n-1}}{\longleftrightarrow} \overset{v_n}{\circ}$$

(i.e. every edge w has the same equipment: $\circ \longleftrightarrow \circ \cong \circ \overset{0\quad 0}{\text{——}} \circ$).

The modular lattice $L(A_n^0)$ corresponding to A_n^0 is by definition the modular lattice with generators $\{v_i, w_j\}$ $(i = 1, \ldots, n;\ j = 1, \ldots, n-1)$ and the following relations: 1° $v_i(\sum_{j \neq i} v_j) = 0$ for all i; 2° $w_j \subseteq v_j + v_{j+1}$ and $w_j v_j = w_j v_{j+1} = 0$ for every j $(j < n)$.

In this section we try to pursue as far as possible the investigation of $L(A_n^0)$ using the methods of lattice theory and representation theory.

2.2. Before we begin to study the case $n \geqslant 3$, which will be our main goal, let us say a few words about $L(A_1^0)$ and $L(A_2^0)$. It is easy to see that $L(A_1^0)$ contains precisely two elements $v = v_1$ and 0. $L(A_2^0)$ is the lattice with three generators v_1, v_2 and $w_1 = w$ and relations $w \subseteq v_1 + v_2$, $v_1 v_2 = w v_1 = w v_2 = 0$. The construction of the free modular lattice with three generators is well known [2]. Using this description it is easy to prove that $L(A_2^0)$ is finite and consists of the following 10 distinct elements: 0, v_1, v_2, w, $v_1 + v_2$, $v_1 + w$, $v_2 + w$, $v_1(w + v_2)$, $v_2(w + v_1)$, $(w + v_1)(w + v_2)$. The diagram of this lattice is shown in Fig. 2, where

$$\begin{aligned} c_{1,2} &= (w+v_1)(w+v_2) = w + v_1(w+v_2) = w + v_2(w+v_1) \\ &= v_1(w+v_2) + v_2(w+v_1). \end{aligned}$$

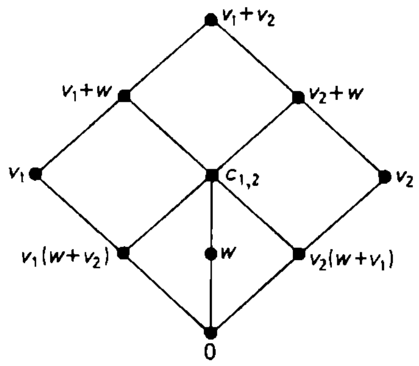

Fig. 2

Put $v_{1,2} = v_1(w+v_2)$ and $v_{2,1} = v_2(w+v_1)$. Notice that the three elements $(w, v_{1,2}, v_{2,1})$ form the *elementary modular triplet*, i.e.

$$(2.1)\qquad \begin{aligned} w v_{1,2} &= w v_{2,1} = v_{1,2} v_{2,1} = 0, \\ w + v_{1,2} &= w + v_{2,1} = v_{1,2} + v_{2,1} = c_{1,2}. \end{aligned}$$

Indeed, $w v_{1,2} = w v_1(w+v_2) = 0(w+v_2) = 0$, and $w + v_{1,2} = w + v_1(w+v_2) = (w+v_1)(w+v_2) = w + v_2(w+v_1) = w + v_{2,1} = c_{1,2}$. Similarly, $w v_{2,1} = v_{1,2} v_{2,1} = 0$ and $v_{1,2} + v_{2,1} = c_{1,2}$.

Let ϱ be a representation of the lattice $L(A_2^0)$ in a vector space V over a field k such that $\varrho(v_i) = V_i$ $(i = 1, 2)$, $\varrho(w) = W$. Then $\varrho(v_1 + v_2) = \varrho(v_1) + \varrho(v_2) = V_1 + V_2 = V$. Further, put $\varrho(v_{1,2}) = V_{1,2} = V_1(W + V_2)$ and

$\varrho(v_{2,1}) = V_{2,1} = V_2(W+V_1)$. It follows from the definition of a representation and equalities (2.1) that the triplet of the subspaces $(W, V_{1,2}, V_{2,1})$ is the elementary modular triplet, i.e.

$$\begin{aligned} &WV_{1,2} = WV_{2,1} = V_{1,2}V_{2,1} = 0, \\ &W+V_{1,2} = W+V_{2,1} = V_{1,2}+V_{2,1} = C_{1,2}. \end{aligned} \tag{2.2}$$

Let ξ_1 be an arbitrary vector in $V_{1,2}$. Since $V_{1,2} \subseteq W+V_{2,1} \cong W\oplus V_{2,1}$, ξ_1 can be uniquely written in the form $\xi_1 = \eta+(-\xi_2)$, where $\eta \in W$ and $\xi_2 \in V_{2,1} = V_2(W+V_1)$. It follows from (2.2) that the obtained correspondence ω: $\xi_1 \mapsto \xi_2$ is an isomorphism between $V_{1,2}$ and $V_{2,1}$. Notice that W is the graph of this isomorphism, i.e. $W = \{\xi_1+\omega(\xi_1) = \xi_1+\xi_2 = \eta | \zeta_1 \in V_{1,2} \text{ and } \xi_2 \in V_{2,1}\}$.

In order to decompose the representation ϱ into a direct sum of indecomposable representations we choose a basis $\{\xi_{i,j} | i = 1, 2; j = 1, \ldots, n_i\}$ in V with the following properties:

(a) the subset $\{\xi_{1,j} | j = 1, \ldots, m\}$ is a basis in the subspace $V_{1,2} = V_1(W+V_2)$;

(b) $\xi_{2,j} = \omega(\xi_{1,j})$ $(j = 1, \ldots, m)$; this implies that $\{\xi_{2,j} | j = 1, \ldots, m\}$ is a basis in $V_{2,1} = V_2(W+V_1)$;

(c) $\{\xi_{1,i} | i = 1, \ldots, n_1\}$ $(m \leqslant n_1)$ is a basis in V_1;

(d) $\{\xi_{2,i} | i = 1, \ldots, n_2\}$ $(m \leqslant n_2)$ is a basis in V_2.

The existence of such a basis follows easily from the properties of the subspaces $V_{1,2} \subseteq V_1$ and $V_{2,1} \subseteq V_2$, i.e. $V_1V_2 = 0$, $V_1+V_2 = V$ and $\omega(V_{1,2}) = V_{2,1}$.

Put $T_{1,2;i} = k\xi_{1,i}\oplus k\xi_{2,i}$ for $i = 1, \ldots, m$; $T_{1,1;j} = k\xi_{1,j}$ $(j = m+1, \ldots, n_1)$; and $T_{2,2;j} = k\xi_{2,j}$ $(j = m+1, \ldots, n_2)$. It is easy to prove that the representation ϱ decomposes into the direct sum of indecomposable representations $\tau_{1,2;i}$, $\tau_{1,1;j}$, $\tau_{2,2;k}$ on the subspaces $T_{1,2;i}$, $T_{1,1;j}$, $T_{2,2;k}$ respectively such that

$$\tau_{1,2;i} \cong \tau_{1,2;j} \cong \tau_{1,2}; \quad \tau_{1,1;i} \cong \tau_{1,1;j} \cong \tau_{1,1}; \quad \tau_{2,2;i} \cong \tau_{2,2;j} \cong \tau_{2,2}.$$

Here the representations $\tau_{1,1}$ and $\tau_{2,2}$ are one-dimensional and $\tau_{1,2}$ is two-dimensional, namely:

$$\begin{aligned} &\tau_{1,1}(v_1) = T_{1,1} \cong k, && \tau_{1,1}(v_2) = \tau_{1,1}(w) = 0, \\ &\tau_{2,2}(v_2) = T_{2,2} \cong k, && \tau_{2,2}(v_1) = \tau_{2,2}(w) = 0, \\ &\tau_{1,2}(v_i) = k\xi_i \quad (i = 1, 2), && \tau_{1,2}(w) = k(\xi_1+\xi_2), \end{aligned}$$

$$\tau_{1,2}(v_1+v_2) = T_{1,2} = k\xi_1\oplus k\xi_2.$$

Thus, the lattice $L(A_2^0)$ has exactly three mutually nonisomorphic indecomposable representations: $\tau_{1,1}$, $\tau_{2,2}$, $\tau_{1,2}$. All (nonzero) morphisms between these indecomposable representations are shown in the following diagram:

$$\begin{array}{ccc} \tau_{1,1} & & \tau_{2,2} \\ & \searrow\ \swarrow & \\ & \tau_{1,2} & \end{array}$$

This diagram can be interpreted as the diagram of the partially ordered set $\mathscr{P}(A_2^0)$ (where $\tau_{1,1} \geqslant \tau_{1,2}$ and $\tau_{2,2} \geqslant \tau_{1,2}$). If we take a hereditary subset H in $\mathscr{P}(A_2^0)$, then we can find a perfect element h such that H is characteristic for h. The nontrivial perfect elements in $L(A_2^0)$ are: $c_{1,1} = v_1 + w$, $c_{2,2} = v_2 + w$ and $c_{1,2} = (v_1 + w)(v_2 + w)$. The partially ordered set of these elements is represented by the diagram

$$\begin{array}{ccc} c_{1,1} & & c_{2,2} \\ & \diagdown\ \diagup & \\ & c_{1,2} & \end{array}$$

2.3. *New generators in* $L(A_n^0)$. Let $w = \sum_{i=1}^{n-1} w_i$.

PROPOSITION 2.1. *The lattice* $L(A_n^0)$ $(n \geqslant 2)$ *is isomorphic to the lattice* L *with generators* $(v_1, \ldots, v_n, w)$ *and relations*: 1° $v_i(\sum_{j \neq i} v_j) = 0$ *for all* i; 2° $wv_i = 0$ *for all* i; 3° $w = \sum_{i=1}^{n-1} w(v_i + v_{i+1})$.

Our proof of this proposition is based on the following lemma.

LEMMA 2.2. *The subset* $\{x_t\} = \{w_1, \ldots, w_{i-1}, v_i, \ldots, v_j, w_j, \ldots, w_{n-1}\}$ *of* $L(A_n^0)$, *where* $1 \leqslant i \leqslant j < n$, *is independent, i.e.* $x_t(\sum_{s \neq t} x_s) = 0$ *for every* t.

We omit the simple proof of this lemma. It follows from the lemma that the subset $\{w_1, \ldots, w_{i-1}, v_i, v_{i+1}, w_{i+1}, \ldots, w_{n-1}\}$ is independent. Hence $(v_i + v_{i+1})(\sum_{j \neq i} w_j) = 0$. Therefore,

$$w(v_i + v_{i+1}) = (\sum_{j \neq i} w_j + w_i)(v_i + v_{i+1}) = w_i + (\sum_{j \neq i} w_j)(v_i + v_{i+1}) = w_i.$$

We know that $w_i v_i = 0$. If we put here $w_i = w(v_i + v_{i+1})$, we obtain $w_i v_i = w(v_i + v_{i+1})v_i = wv_i = 0$. Analogously we obtain $wv_{i+1} = 0$. So $wv_i = 0$ for all i.

We have shown that if $w = \sum_{i=1}^{n-1} w_i$, then $wv_i = 0$ for all i and $w_i = w(v_i + v_{i+1})$.

Now let L be the lattice with generators $\{v_1, \ldots, v_n, w\}$ and relations 1°–3° (see Proposition 2.1). Let $w_i' = w(v_i + v_{i+1})$; then 3° can be written as $w = \sum_{i=1}^{n-1} w_i'$. Let us find $w_i' v_i$ and $w_i' v_{i+1}$. If we substitute $w_i' = w(v_i + v_{i+1})$, we obtain $w_i' v_i = w(v_i + v_{i+1})v_i = wv_i = 0$. Analogously, we get $w_i' v_{i+1} = 0$. We have obtained the lattice with generators $\{v_1, \ldots, v_n, w_1', \ldots, w_{n-1}'\}$ and relations 1°–2° (see 2.1). This ends the proof of Proposition 2.1.

2.4. *The representations* $\tau_{i,j}$ $(i \leqslant j)$ *and their properties*. Let us define a representation $\tau_{i,j}$ of the lattice $L(A_n^0)$ in the vector space $V_{i,j} = k^{j-i+1}$ for every pair (i, j) of vertices of the graph A_n^0 (such that $1 \leqslant i \leqslant j \leqslant n$) as follows. Let $\{\varepsilon_s\} = \{\varepsilon_{i,j}\}$ (where $i \leqslant s \leqslant j$) be a basis of the space $V_{i,j}$. We set

$$\tau_{i,j}(v_s) = \begin{cases} k\varepsilon_s & \text{if } i \leqslant s \leqslant j, \\ 0 & \text{otherwise,} \end{cases} \qquad \tau_{i,j}(w_s) = \begin{cases} k(\varepsilon_s + \varepsilon_{s+1}) & \text{if } i \leqslant s < j, \\ 0 & \text{otherwise.} \end{cases}$$

(in case $i = j$ we set $\tau_{i,i}(w_s) = 0$ for all s).

THEOREM 2.3. (a) *All representations* $\tau_{i,j}$ $(1 \leqslant i \leqslant j \leqslant n)$ *are indecomposable.*

(b) *An arbitrary indecomposable representation* τ *of the lattice* $L(A_n^0)$ *is isomorphic to one of* $\tau_{i,j}$.

(c) *The representations* $\tau_{i,j}$ *are ordered in the following way*: $(\tau_{i_1,j_1} \leqslant \tau_{i_2,j_2}) \Leftrightarrow (i_1 \leqslant i_2 \leqslant j_2 \leqslant j_1)$; *so* $\mathrm{Hom}(\tau_\beta, \tau_\alpha) \cong k$ *if* $\tau_\beta \geqslant \tau_\alpha$, *and* $\mathrm{Hom}(\tau_\beta, \tau_\alpha) = 0$ *otherwise (i.e. if either* $\tau_\beta \leqslant \tau_\alpha$ $(\tau_\alpha \not\leqq \tau_\beta)$ *or* τ_α *and* τ_β *are incomparable).*

We will prove part (a) of Theorem 2.3 in the case of the representation $\tau_{1,n}$. By the definition, it is the representation in $V_{1,n} \cong k^n$ with the basis $\varepsilon_1, \ldots, \varepsilon_n$ such that $\tau_{1,n}(v_i) = V_i = k\varepsilon_i$ and $\tau_{1,n}(w) = W$ is the subspace spanned by the vectors $\varepsilon_1+\varepsilon_2, \varepsilon_2+\varepsilon_3, \ldots, \varepsilon_{n-1}+\varepsilon_n$. Clearly $\dim W = n-1$. If follows from Proposition 2.1 that $W(V_j+V_{j+1}) = k(\varepsilon_j+\varepsilon_{j+1})$ for every $j < n$.

Notice that the vector $\varepsilon_j-\varepsilon_{j+2} = (\varepsilon_j+\varepsilon_{j+1})-(\varepsilon_{j+1}+\varepsilon_{j+2})$ belongs to $W(V_j+V_{j+2})$. Hence $W(V_j+V_{j+2}) \cong k(\varepsilon_j-\varepsilon_{j+2})$. We can prove similarly that $W(V_j+V_m) \cong k(\varepsilon_j+(-1)^\lambda\varepsilon_m)$, where $\lambda = m-j+1$, for every j and m $(j \neq m)$.

Assume that $\tau_{1,n}$ decomposes on two subspaces G_1 and G_2 $(G_1G_2 = 0, G_1+G_2 = V)$. We deduce by the definition of decomposition that

$$\tau_{1,n}(x) = X = XG_1+XG_2 \tag{2.3}$$

for every $x \in L(A_n^0)$. In particular, (2.3) has to be true for all generators of $L(A_n^0)$, i.e.

$$\forall j \quad V_j = V_jG_1+V_jG_2, \qquad W = WG_1+WG_2. \tag{2.3a, b}$$

Since by the definition $\dim V_j = 1$, equality (2.3a) implies that either $V_jG_1 = V_j$ (i.e. $V_j \subseteq G_1$) and $V_jG_2 = 0$, or $V_jG_1 = 0$ and $V_jG_2 = V_j$ (i.e. $V_j \subseteq G_2$). Hence there exists a decomposition of the set $J = \{1, \ldots, n\}$ into two disjoint subsets J_1 and J_2 such that $G_1 = \sum_{j\in J_1} k\varepsilon_j$ and $G_2 = \sum_{j\in J_2} k\varepsilon_j$.

Let $J_1 = \{j_1, \ldots, j_{m_1}\}$ and $J_2 = \{t_1, \ldots, t_{m_2}\}$, where $m_1+m_2 = n$. It is easy to see that the vectors $\eta_\alpha = \varepsilon_{j_\alpha}+(-1)^\lambda\varepsilon_{j_{\alpha+1}}$ (where $\lambda = \lambda_\alpha = j_{\alpha+1}-j_\alpha+1$) belong to WG_1 and the vectors $\vartheta_\beta = \varepsilon_{t_\beta}+(-1)^\mu\varepsilon_{t_{\beta+1}}$ (where $\mu = \mu_\beta = t_{\beta+1}-t_\beta+1$) belong to WG_2. Set $\zeta = \varepsilon_{j_1}+(-1)^\varkappa\varepsilon_{t_1}$, where $\varkappa = j_1-t_1+1$. It is easy to check that all vectors $\eta_1, \ldots, \eta_{m_1-1}, \zeta, \vartheta_1, \ldots, \vartheta_{m_2-1} \in W$ are linearly independent and hence form a basis of W. Consequently, $\dim WG_j \geqslant m_j-1$. If $W = WG_1+WG_2$, then $\dim W = n-1 = m_1+m_2-1 = \dim WG_1+\dim WG_2$. Since $\dim WG_j \geqslant m_j-1$, the last equality means that either $\dim WG_1 = \dim G_1$ (i.e. $WG_1 = G_1$) or $\dim WG_2 = \dim G_2$ (i.e. $WG_2 = G_2$). The equality $WG_1 = G_1$ means that $W \supseteq G_1 = \sum_{j\in J_1} k\varepsilon_j = \sum_{j\in J_1} V_j$. But as we know $WV_j = 0$ for all j, a contradiction. Analogous arguments show that the equality $WG_2 = G_2$ is also impossible. Hence the assumption that $\tau_{1,n}$ is decomposable implies a contradiction. The indecomposability of all other representations $\tau_{i,k}$ $(i < k)$ can be shown in a similar way. All representations $\tau_{i,i}$ are obviously indecomposable.

The *dimension* of an indecomposable representation $\tau_\alpha = \tau_{i,j}$ is by definition the integral-valued function d_α: $\Gamma_0 \to \mathbf{N}$ such that

$$d_\alpha(s) = \dim \tau_\alpha(v_s) = \dim \tau_{i,j}(v_s) = \begin{cases} 1 & \text{if } i \leqslant s \leqslant j, \\ 0 & \text{otherwise.} \end{cases}$$

An arbitrary positive root α of the Dynkin diagram A_n can be written in the form $\alpha = \sum_{i \leqslant s \leqslant j} \alpha_s$, where α_s $(s = 1, \ldots, n)$ are the simple roots. Hence we associate with such a positive root an indecomposable representation $\tau_\alpha = \tau_{i,j}$.

The proof of part (c) of Theorem 2.3 is not complicated and we omit it. The proof of (b) can be easily obtained from the main theorem 5.4.

It follows from Theorem 2.3 that we can partially order $\mathscr{P}_n$ (the set of different indecomposable representations $\tau_\alpha = \tau_{i,j}$) in the following way: if $\alpha = (i_1, j_1)$ and $\beta = (i_2, j_2)$, then

$$\tau_\alpha \geqslant \tau_\beta \Leftrightarrow i_2 \leqslant i_1 \leqslant j_1 \leqslant j_2 \Leftrightarrow \alpha \geqslant \beta.$$

In the case $n = 5$ the partially ordered set $\mathscr{P}_5$ is shown in Fig. 3. Here the points (i, j) $(i \leqslant j)$ correspond to the indecomposable representations $\tau_{i,j}$ or to the positive roots $\alpha = \alpha_{i,j} = \sum_{i \leqslant s \leqslant j} \alpha_s$.

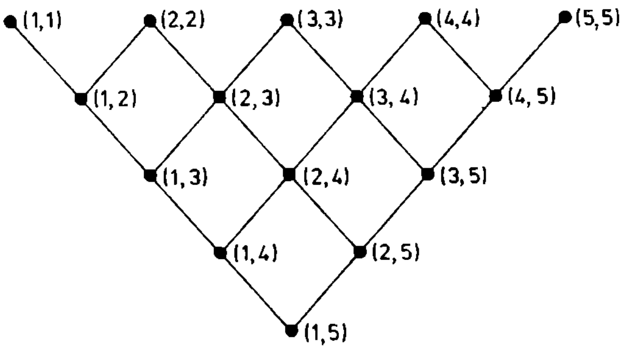

Fig. 3

3. Construction of the lattice of perfect elements

3.1. Before we start to construct perfect elements in the lattice $L(A_n^0)$ we prove

PROPOSITION 3.1. *Let h be a perfect element and let H be the characteristic subset of indecomposable representations corresponding to h ($\tau_\alpha \in H \Leftrightarrow \tau_\alpha$ is indecomposable and $\tau_\alpha(h) = V_\alpha$, where V_α is the space of τ_α). Then H is a hereditary subset of the partially ordered set $\mathscr{P}_n$, i.e. if $\tau_\alpha \in H$ and $\tau_\alpha \geqslant \tau_\beta$, then $\tau_\beta \in H$.*

Proof. Let $\tau_\alpha \in H$ and let τ_β be an indecomposable representation such that $\mathrm{Hom}(\tau_\alpha, \tau_\beta) \neq 0$. Let $\psi \in \mathrm{Hom}(\tau_\alpha, \tau_\beta)$ and $\psi \neq 0$. Since the element h is perfect and τ_γ belongs to the characteristic subset of h, we have $\tau_\alpha(h) = V_\alpha$. According to the definition of a morphism $\psi\tau_\alpha(h) \subseteq \tau_\beta(h)$. Since $\psi \neq 0$, the subspace $\psi\tau_\alpha(h)$ is not zero, and hence $\tau_\beta(h) \neq 0$. By assumption h is perfect, so either $\tau_\beta(h) = 0$ or $\tau_\beta(h) = V_\beta$. Consequently, $\tau_\beta(h) = V_\beta$, i.e. $\tau_\beta \in H$. Thus the subset H is hereditary. We have proved the proposition.

The rest of the paper will be devoted to the proof that each hereditary subset T of $\mathscr{P}_n$ is characteristic, i.e. there exists a perfect element $t \in \mathscr{P}_n$ such that $T = \{\tau \in \mathscr{P}_n | \tau(t) = V_\tau\}$.

3.2. *The elements $v_{i,j}$ and the mappings φ_i^- and φ_i^+.* For each $x_j \subseteq v_j \in L(A_n^0)$ we let

$$\varphi_j^-(x_j) = v_{j-1}(w+x_j) \quad \text{if } j > 1; \tag{3.1a}$$

$$\varphi_j^+(x_j) = v_{j+1}(w+x_j) \quad \text{if } j < n. \tag{3.1b}$$

Proposition 3.2. *If $x_j \subseteq v_j \in L(A_n^0)$, then*

$$v_{j-1}(w+x_j) = v_{j-1}(w_{j-1}+x_j), \qquad v_{j+1}(w+x_j) = v_{j+1}(w_j+x_j).$$

We prove the first formula. Since $w(v_{j-1}+v_j) = w_{j-1}$ and $(v_{j-1}+v_j) \supseteq (v_{j-1}+x_j)$, we have

$$\begin{aligned} v_{j-1}(w+x_j) &= v_{j-1}(v_{j-1}+v_j)(w+x_j) \\ &= v_{j-1}\big(w(v_{j-1}+v_j)+x_j\big) = v_{j-1}(w_{j-1}+x_j). \end{aligned}$$

The second formula can be proved analogously.

Let ϱ be a representation of $L(A_n^0)$ in a space V. We write $\varrho(v_j) = V_j$, $\varrho(w) = W$ and $\varrho(x_j) = X_j$. If we transfer the mappings φ_j^- and φ_j^+ on V, we obtain

$$\varphi_j^-(X_j) = V_{j-1}(W_{j-1}+X_j), \qquad \varphi_j^+(X_j) = V_{j+1}(W_j+X_j).$$

It is easy to check that $V_{j-1}(W_{j-1}+X_j)$ and $V_{j+1}(W_j+X_j)$ are the images of the subspace $X_j \subseteq V_j$ under the action of the relations W_{j-1} and W_j respectively.

Let us define elements $v_{i,j}$ of $L(A_n^0)$ by induction on the integer $n = |i-j|$. Let $v_{j,j} = v_j$, $v_{j-1,j} = \varphi_j^-(v_{j,j}) = v_{j-1}(w+v_j)$, and

$$v_{i-1,j} = \varphi_i^-(v_{i,j}) = v_{i-1}(w+v_{i,j}) \quad \text{for } i < j. \tag{3.2}$$

We define analogously $v_{j+1,j} = \varphi_j^+(v_j) = v_{j+1}(w+v_j)$ and

$$v_{i+1,j} = \varphi_i^+(v_{i,j}) = v_{i+1}(w+v_{i,j}) \quad \text{for } i > j. \tag{3.3}$$

We shall prove in Proposition 4.1 that the elements $v_{i,j}$ may be expressed through the generators v_i, v_j and w for all $i, j \in \{1, \ldots, n\}$ in the following way:

$$v_{i,j} = v_i(w+v_j). \tag{3.4}$$

It is easy to check that the mappings φ_j^- and φ_j^+ are monotone, i.e. if $x_j \subseteq y_j \subseteq v_j$ then $\varphi_j^-(x_j) \subseteq \varphi_j^-(y_j) \subseteq \varphi_j^-(v_j)$ and analogously $\varphi_j^+(x_j) \subseteq \varphi_j^+(y_j) \subseteq \varphi_j^+(v_j)$. Hence we have the chains

$$(3.5) \qquad \begin{gathered} v_1 = v_{1,1} \supseteq v_{1,2} \supseteq \ldots \supseteq v_{1,n-1} \supseteq v_{1,n}, \\ v_{n,1} \subseteq v_{n,2} \subseteq \ldots \subseteq v_{n,n-1} \subseteq v_{n,n} = v_n. \end{gathered}$$

For any other element v_j ($j \neq 1$ and $j \neq n$) we get two chains:

$$(3.6) \qquad \begin{gathered} v_j = v_{j,j} \supseteq v_{j,j+1} \supseteq \ldots \supseteq v_{j,n}, \\ v_{j,1} \subseteq v_{j,2} \subseteq \ldots \subseteq v_{j,j-1} \subseteq v_{j,j} = v_j. \end{gathered}$$

We denote by E_j the lattice generated by that pair of chains $\{v_{j,k}\}$ ($k \geqslant j$) and $\{v_{j,i}\}$ ($i \leqslant j$). We shall return to the investigation of these lattices later on.

3.3. *The elements a_j, b_j ($j = 1, \ldots, n$) and the lattice generated by them.* Let us define elements a_j and b_j of $L(A_n^0)$ in the following way:

$$\begin{aligned} a_1 &= v_1 + \sum_{i=2}^{n} v_{i,1}, & b_n &= \sum_{i=1}^{n-1} v_{i,n} + v_n, \\ a_2 &= v_1 + v_2 + \sum_{i=3}^{n} v_{i,2}, & b_{n-1} &= \sum_{i=1}^{n-2} v_{i,n-1} + v_{n-1} + v_n, \\ &\ldots\ldots\ldots\ldots & &\ldots\ldots\ldots\ldots\ldots\ldots \\ a_s &= \sum_{i=1}^{s} v_i + \sum_{i=s+1}^{n} v_{i,s}, & b_s &= \sum_{i=1}^{s-1} v_{i,s} + \sum_{i=s}^{n} v_i, \\ & & &\ldots\ldots\ldots\ldots\ldots\ldots \\ a_n &= \sum_{i=1}^{n} v_i = v, & b_1 &= \sum_{i=1}^{n} v_i = v. \end{aligned}$$

Using the monotonicity properties (3.5) and (3.6) of the $v_{i,j}$ we can see that the elements a_j form an increasing chain:

$$(3.8) \qquad a_1 \subseteq a_2 \subseteq \ldots \subseteq a_j \subseteq \ldots \subseteq a_{n-1} \subseteq a_n = v.$$

Similarly the b_j form a decreasing chain:

$$(3.9) \qquad v = b_1 \supseteq b_2 \supseteq \ldots \supseteq b_j \supseteq \ldots \supseteq b_{n-1} \supseteq b_n.$$

We denote by $D_n = D(A_n^0)$ the lattice generated by the two chains $\{a_j\}$ and $\{b_j\}$. It is known [2] that if a modular lattice D is generated by two chains of elements, then D is distributive. We shall show in Sec. 5 that all elements of D_n are perfect. First, we investigate D_n as an abstract lattice with generators and relations. Let

$$(3.10) \qquad c_j = a_j b_j.$$

It is not difficult to prove that $c_j = \sum_{i=1}^{n} v_{i,j}$. By (3.4), c_j can be expressed through the generators of $L(A_n^0)$ as

$$c_j = \sum_{i=1}^{n} v_i(w+v_j). \tag{3.11}$$

PROPOSITION 3.3. *Let D_n be the distributive lattice with maximum element v and with generators $a_1 \subseteq a_2 \subseteq \ldots \subseteq a_{n-1} \subseteq a_n = v$ and $v = b_1 \supseteq b_2 \supseteq \ldots$ $\ldots \supseteq b_{n-1} \supseteq b_n$ and relations $a_{j-1}+b_j = v$ for all j $(1 < j \leqslant n)$. Then D_n is isomorphic to the distributive lattice D'_n with generators c_j and relations $c_i c_j c_k = c_i c_k$ for $i \leqslant j \leqslant k$.*

Proof. Define lattice morphisms $\varphi\colon D \to D'$ and $\psi\colon D' \to D$ by the formulas $\varphi(a_j) = c_1+\ldots+c_j$, $\varphi(b_j) = c_j+\ldots+c_n$ and $\psi(c_j) = a_j b_j$. It is easy to see that φ is order-preserving, i.e. $\varphi(a_j) \subseteq \varphi(a_{j+1})$ and $\varphi(b_j) \supseteq \varphi(b_{j+1})$ for all $j < n$. It is also obvious that $\varphi(a_{j-1}+b_j) = \varphi(a_{j-1})+\varphi(b_j) = c_1+\ldots+c_n = \varphi(a_n) = \varphi(b_1)$ $= \varphi(v)$ for all j $(1 < j)$. Let us show that if $i \leqslant j \leqslant k$, then $\psi(c_i c_j c_k) = \psi(c_i c_k)$. Indeed,

$$\begin{aligned}\psi(c_i c_j c_k) &= \psi(c_i)\psi(c_j)\psi(c_k) = a_i b_i a_j b_j a_k b_k = a_i a_j a_k b_i b_j b_k\\ &= a_i b_k = a_i a_k b_i b_k = a_i b_i a_k b_k = \psi(c_i)\psi(c_k) = \psi(c_i c_k).\end{aligned}$$

It remains to check that the compositions $\varphi \circ \psi$ and $\psi \circ \varphi$ are the identical morphisms. Indeed,

$$\varphi\circ\psi(c_j) = \varphi(a_j b_j) = \varphi(a_j)\varphi(b_j) = (c_1+\ldots+c_j)\cap(c_j+\ldots+c_n) = c_j.$$

We now verify that $\psi\circ\varphi$ is identical on a_j, by induction on j. According to the definitions $\psi\circ\varphi(a_1) = \psi(c_1) = a_1 b_1 = a_1 v = a_1$. Supppose that we have proved $\psi\circ\varphi(a_j) = a_j$. Then

$$\begin{aligned}\psi\circ\varphi(a_{j+1}) &= \psi(c_1+\ldots+c_j+c_{j+1}) = \psi(c_1+\ldots+c_j)+\psi(c_{j+1})\\ &= \psi\circ\varphi(a_j)+a_{j+1}b_{j+1} = a_j+a_{j+1}b_{j+1} = a_{j+1}(a_j+b_{j+1})\\ &= a_{j+1}v = a_{j+1}.\end{aligned}$$

We can check similarly that $\psi\circ\varphi(b_j) = b_j$, by descending induction starting from $j = n$.

PROPOSITION 3.4. *The distributive lattice D_n with generators c_i $(i = 1, \ldots$ $\ldots, n)$ and relations $c_i c_j c_k = c_i c_k$ $(i \leqslant j \leqslant k)$ is isomorphic to the distributive lattice $D(P_n)$ of hereditary subsets of the partially ordered set $P_n = \{(i, k) \mid$ $i, k \in \mathbf{N},\ 1 \leqslant i \leqslant k \leqslant n\}$ with $(i_1, k_1) \leqslant (i_2, k_2)$ if and only if $i_1 \leqslant i_2 \leqslant k_2 \leqslant k_1$.*

Proof. It is known [7] that an arbitrary finite distributive lattice D is isomorphic to the lattice $D(P)$, where P is the set of indecomposable (into a union) elements of D, and $D(P)$ is the set of hereditary subsets of the partially ordered set P.

In particular, the indecomposable (into a union) elements of D_n (Proposition 3.4) are $c_i c_k$ $(1 \leqslant i \leqslant k \leqslant n)$. The set P_n of those elements is partially ordered: $c_{i_1} c_{k_1} \subseteq c_{i_2} c_{k_2} \Leftrightarrow i_1 \leqslant i_2 \leqslant k_2 \leqslant k_1$. If we associate with $c_i c_k$ its index $\alpha = (i, k)$ (where $i \leqslant k$) and if we transfer the partial order from P_n to the set of those indices we obtain the conclusion of Proposition 3.4.

Notice that the elements $c_i c_k$ $(i \leqslant k)$ of D_n are indecomposable (into a union) as long as we consider D_n as an abstract distributive lattice. But if we have a representation of D_n in the modular lattice $L(A_n^0)$, then the image of $c_j = \sum_{i=1}^{n} v_{i,j}$ is decomposable (into a union) in $L(A_n^0)$. The same holds for $c_i c_k$.

4. The lattices $\bar{D}_n$

4.1. Before we start to prove that all elements of the lattice $D_n = D(A_n^0)$ are perfect, we study a certain distributive sublattice $\bar{D}_n$ in the modular lattice $L(A_n^0)$ such that $D_n \subseteq \bar{D}_n \subseteq L(A_n^0)$. We call $\bar{D}_n$ the *covering* of D_n.

Denote by E_j $(j = 1, \ldots, n)$ the sublattice of $L(A_n^0)$ generated by the following chains of elements:

$$\text{(4.1)} \qquad \begin{aligned} v_j &= v_{j,j} \supseteq v_{j,j+1} \supseteq \ldots \supseteq v_{j,n-1} \supseteq v_{j,n}, \\ v_j &= v_{j,j} \supseteq v_{j,j-1} \supseteq \ldots \supseteq v_{j,2} \supseteq v_{j,1}. \end{aligned}$$

By definition the sublattices E_1 and E_n are the chains $v_1 = v_{1,1} \supseteq v_{1,2} \supseteq \ldots \supseteq v_{1,n-1} \supseteq v_{1,n}$ and $v_{n,1} \subseteq v_{n,2} \subseteq \ldots \subseteq v_{n,n-1} \subseteq v_{n,n} = v_n$ respectively.

It is known [2] that an arbitrary modular lattice generated by two chains of elements is distributive. Hence, one can describe each lattice E_j $(j \neq 1, n)$ through the partially ordered subset M_j of indecomposable (into a union) elements. For E_j, these are the intersections $v_{j,i} v_{j,k}$ $(i \leqslant j \leqslant k)$, which will be denoted by $v_{i_j k}$. It follows from the definition that the elements $v_{i_j k}$ with j fixed form a partially ordered set M_j in the following way:

$$\text{(4.3)} \qquad \text{if } i_1 \leqslant i_2 \leqslant j \leqslant k_2 \leqslant k_1, \quad \text{then } v_{i_{1j} k_1} \subseteq v_{i_{2j} k_2}.$$

This implies that E_j is isomorphic to the lattice of hereditary subsets of the partially ordered set M_j. The diagram of M_j for two cases $j = 2$, $j = 3$ and $n = 5$ is shown in Fig. 4. Note that according to the definition $v_{j_j j} = v_j$, $v_{j_j k} = v_{j,k}$ and $v_{i_j j} = v_{j,i}$.

We denote by $\bar{D}_n$ the sublattice of $L(A_n^0)$ equal to the direct product of the lattices E_j: $\bar{D}_n = E_1 \times E_2 \times \ldots \times E_{n-1} \times E_n$. It is easy to prove that $\bar{D}_n$ is distributive and that the set $\bar{M}_n$ of its elements which are indecomposable into a union is $\bar{M}_n = \bigcup_{j=1}^{n} M_j$. This implies that any $x \in \bar{D}_n$ can be written as a sum $x = \sum v_{i_j k}$. The diagram of the partially ordered set $\bar{M}_5 \subset L(A_5^0)$ is illustrated in Fig. 5.

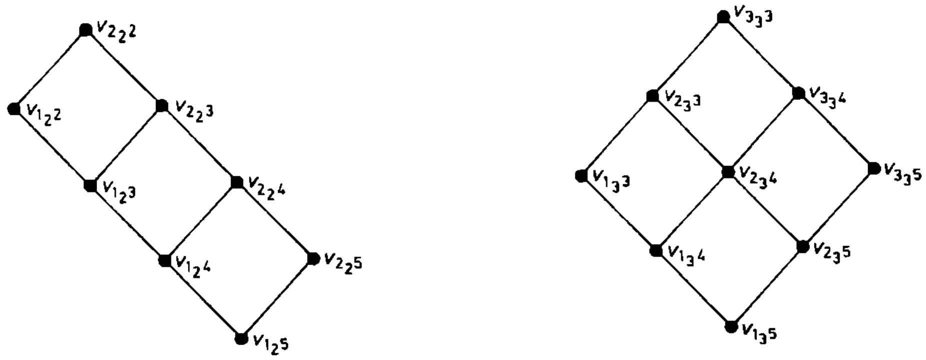

Fig. 4

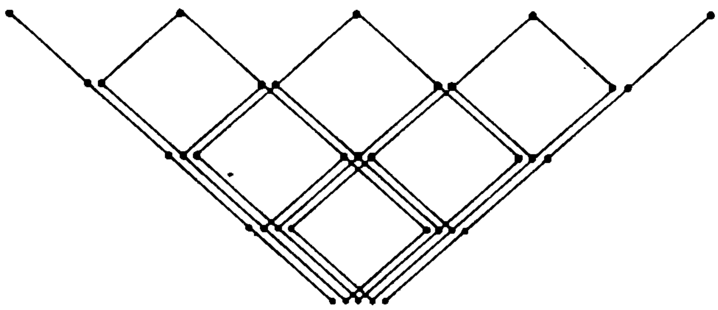

Fig. 5

4.2. *Main properties of the elements* v_{i_jk}.

PROPOSITION 4.1. $v_{i_jk} = (v_i+w)v_j(v_k+w)$ *for all* $i \leqslant j \leqslant k$ $(1 \leqslant i \leqslant k \leqslant n)$.

Proof. It suffices to show that

$$v_{i,j} = v_i(v_j+w) \tag{4.4}$$

for any i and j $(1 \leqslant i, j \leqslant n)$. Our proof of (4.4) is based on the following lemma.

LEMMA 4.2.

(4.5) *If* $i<j$, *then* $v_{i,j} = v_i(w_i+w_{i+1}+\ldots+w_{j-1}+v_j)$.

(4.6) *If* $j<i$, *then* $v_{i,j} = v_i(v_j+w_j+w_{j+1}+\ldots+w_{i-1})$.

Let us prove e.g. formula (4.5). If $i=j-1$, then by definition $v_{j-1,j} = \varphi_j^- v_j = v_{j-1}(w+v_j) = v_{j-1}(w_{j-1}+v_j)$. Suppose that we have proved $v_{i+1,j} = v_{i+1}(w_{i+1}+\ldots+w_{j-1}+v_j)$ for $i+1<j$. Then using the definition of $v_{i,j}$ we have

$$v_{i,j} = \varphi_{i+1}^-(v_{i+1,j}) = v_i(w_i+v_{i+1,j}) = v_i\big(w_i+v_{i+1}(w_{i+1}+\ldots+w_{j-1}+v_j)\big).$$

Set temporarily $\sigma = w_{i+1}+\ldots+w_{j-1}+v_j$. Notice that the triplet v_i, v_{i+1}, σ is distributive, since

$$\begin{aligned} v_i(v_{i+1}+\sigma) &= v_i(v_{i+1}+w_{i+1}+\ldots+w_{j-1}+v_j) \\ &\subseteq v_i(v_{i+1}+v_{i+2}+\ldots+v_{j-1}+v_j) = 0. \end{aligned}$$

Hence $(v_i+v_{i+1})\sigma = v_i\sigma+v_{i+1}\sigma = 0+v_{i+1}\sigma = v_{i+1}\sigma$. Using this equality, we obtain

$$\begin{aligned} v_{i,j} &= v_i(w_i+v_{i+1}\sigma) = v_i(w_i+(v_i+v_{i+1})\sigma) \\ &= v_i(v_i+v_{i+1})(w_i+\sigma) \qquad (\text{since } w_i \subseteq v_i+v_{i+1}) \\ &= v_i(w_i+\sigma) = v_i(w_i+w_{i+1}+\ldots+w_{j-1}+v_j). \end{aligned}$$

Formula (4.5) is proved. (4.6) can be proved analogously.

Proof of Proposition 4.1. Let $i<j$. It is easy to see that

$$v_i(w+v_j) = v_i(v_i+v_{i+1}+\ldots+v_j)(w+v_j) = v_i(w(v_i+v_{i+1}+\ldots+v_j)+v_j).$$

According to Lemma 2.2 the subset $\{w_1, w_2, \ldots, w_{i-1}, v_i, \ldots, v_j, w_j, w_{j+1}, \ldots \ldots, w_{n-1}\}$ is independent for any i and j such that $1 \leqslant i \leqslant j \leqslant n$. Hence, if we write

$$\sum_{s=i}^{j-1} w_s = \zeta, \qquad \sum_{s=1}^{i-1} w_s + \sum_{s=j}^{n-1} w_s = \zeta',$$

then: (a) $w = \zeta+\zeta'$ and this sum is direct; (b) $\zeta \subseteq v_i+v_{i+1}+\ldots+v_j$; (c) the set $\{\zeta', v_i, \ldots, v_j\}$ is independent. Therefore

$$\begin{aligned} w(v_i+\ldots+v_j) &= (\zeta'+\zeta)(v_i+\ldots+v_j) = \zeta+\zeta'(v_i+\ldots+v_j) \\ &= \zeta+0 = \zeta = \sum_{s=i}^{j-1} w_s. \end{aligned}$$

If we substitute this formula in the equality $v_i(w+v_j) = v_i(w(v_i+\ldots+v_j)+v_j)$, which we have just proved, we obtain $v_i(w+v_j) = v_i(w_i+\ldots+w_{j-1}+v_j)$. Using (4.5), we obtain

$$v_{i,j} = v_i(w_i+w_{i+1}+\ldots+w_{j-1}+v_j) = v_i(w+v_j)$$

in case $i<j$. If $j<i$, the proof is similar.

PROPOSITION 4.3. (a) *If* $i<k<j$, *then* $v_{j,i}v_{j,k} = v_{j,i}$, *i.e.* $(v_i+w)v_j(v_k+w) = (v_i+w)v_j$.

(b) *If* $j<i<k$, *then* $v_{j,i}v_{j,k} = v_{j,k}$, *i.e.* $(v_i+w)v_j(v_k+w) = v_j(v_k+w)$.

PROPOSITION 4.4. (a) $\varphi_j^- v_{jjk} = v_{j-1_{j-1}k}$.

(b) *If* $i<j\leqslant k$, *then* $\varphi_j^- v_{i_jk} = v_{i_{j-1}k}$.

(c) $\varphi_j^+ v_{i_jj} = v_{i_{j+1}j+1}$.

(d) *If* $i\leqslant j<k$, *then* $\varphi_j^+ v_{i_jk} = v_{i_{j+1}k}$.

PROPOSITION 4.5. *The lattice* $\bar{D}_n$ *is closed with respect to the mappings* φ_j^+ *and* φ_j^-, *namely if* $x_j \in E_j$, *then its images* $\varphi_j^- x_j$ *and* $\varphi_j^+ x_j$ *belong to* E_{j-1} *and* E_{j+1} *respectively.*

The proofs are left to the reader.

5. The elements of the lattice D_n are perfect

5.1. *Representations of distributive lattices.* Let ϱ be a representation of a finite distributive lattice D in a finite-dimensional vector space V (over a commutative field k), such that $\varrho(v) = V$ for the maximum element $v \in D$. Hence the image $\varrho(D)$ is a distributive sublattice of the modular lattice $L(V)$.

PROPOSITION 5.1. (a) *All indecomposable representations of a distributive lattice D over a field k are one-dimensional (i.e. they are representations in the space $V \cong k^1$).*

(b) *The partially ordered set $\mathscr{P}_D$ of all isomorphism classes of indecomposable representations τ_α of D is isomorphic to the set P of indecomposable (into a union) elements $\alpha \in D$.*

The main idea of the proof is the following. Set $\alpha' = \sum_{\beta \subset \alpha} \beta$, where the sum is taken over all indecomposable (into a union) elements β (i.e. $\beta \in P$), such that $\beta \subset \alpha$ ($\beta \neq \alpha$). We can easily prove that such an element α' is unique, and for any $x \in D$ such that $\alpha' \subseteq x \subseteq \alpha$, either $x = \alpha'$ or $x = \alpha$. In each subspace $\varrho(\alpha)$, where $\alpha \in P$, choose a subspace R_α such that $R_\alpha \varrho(\alpha') = 0$ and $R_\alpha + \varrho(\alpha') = \varrho(\alpha)$. It is not difficult to prove that any subspace $\varrho(a)$ (where $a \in D$) can be written in the form $\varrho(a) = \sum R_\beta \cong \bigoplus R_\beta$, where the sum is taken over all indecomposable $\beta \in P$ such that $\beta \subset a$. In particular, this is true for $\varrho(v) = V$. Hence $\varrho \cong \bigoplus \varrho_\alpha$, where $\varrho_\alpha = \varrho|_{R_\alpha}$. We can check easily that $\varrho_\alpha \cong \tau_\alpha \oplus \ldots \oplus \tau_\alpha$ where all τ_α are the same indecomposable representation, and their number is equal to $\dim R_\alpha$. We omit the easy proof of part (b) of the proposition.

5.2. *A concordant choice of the subspaces R_{i_jk}.* We have shown that the set of indecomposable (into a union) elements of the distributive sublattice $\bar{D}_n = \prod_i E_i$ consists of the elements $v_{i_jk} = (v_i + w)v_j(v_k + w)$ $(1 \leqslant i \leqslant j \leqslant k \leqslant n)$. This set is partially ordered: $v_{i_jk} \subseteq v_{i'_{j'}k'} \Leftrightarrow (i \leqslant i' \leqslant k' \leqslant k$ and $j = j')$ (if $j \neq j'$, then v_{i_jk} and $v_{i'_{j'}k'}$ are incomparable). Let ϱ be a representation of the lattice $L(A_n^0)$ in a vector space V over a field k. Set $\varrho(v_i) = V_i$, $\varrho(w_i) = W_i$ and $\varrho(v_{i_jk}) = V_{i_jk}$. Using Proposition 5.1 it is possible to choose a family of subspaces R_{i_jk} $(i \leqslant j \leqslant k)$ of V_j such that each $V_{i'_jk'}$ (in particular $V_{j_jj} = V_j$) can be written in the form $V_{i'_jk'} = \sum_{(i,k)} R_{i_jk} \cong \bigoplus_{(i,k)} R_{i_jk}$, where the sum is taken over (i, k) such that $i \leqslant i' \leqslant j \leqslant k' \leqslant k$.

In order to find such a family it is sufficient to choose R_{1_jn}, R_{1_jk} and R_{i_jn} in the following way:

$$R_{1_jn} = V_{1_jn}, \tag{5.1}$$

$$V_{1_jk+1} R_{1_jk} = 0 \qquad \text{if } k < n, \tag{5.2}$$

$$V_{1_jk+1} + R_{1_jk} = V_{1_jk} \qquad \text{if } k < n, \tag{5.3}$$

$$V_{i-1_jn} R_{i_jn} = 0 \qquad \text{if } i > 1, \tag{5.4}$$

$$V_{i-1_jn} + R_{i_jn} = R_{i_jn} \qquad \text{if } i > 1. \tag{5.5}$$

If $i \neq 1$ and $k \neq n$, the subspaces R_{i_jk} can be chosen so that

(5.6) $$R_{i_jk}(V_{i-1_jk}+V_{i_jk+1}) = 0,$$

(5.7) $$R_{i_jk}+(V_{i-1_jk}+V_{i_jk+1}) = V_{i_jk}.$$

PROPOSITION 5.2. *The subspaces R_{i_jk} $(1 \leqslant i \leqslant j \leqslant k \leqslant n)$ can be chosen concordant with the mappings φ_j^- and φ_j^+, i.e. such that*

$$i < j \Rightarrow \varphi_j^- R_{i_jk} = R_{i_{j-1}k},$$

$$j < k \Rightarrow \varphi_j^+ R_{i_jk} = R_{i_{j+1}k}.$$

Our proof of this proposition is based on the following lemma.

LEMMA 5.3. *Let (a, b, w) be a modular triplet such that $w \subseteq a+b$ and $wa = wb = ab = 0$, and let $\{a_i\}$ be a family of elements such that $a_i \subseteq a(w+b)$ for all i. Set $b_i = b(w+a_i)$. Then: (1) $b_i b_j = b(w+a_i a_j)$ and $b_i+b_j = b(w+a_i+a_j)$ for all i and j; (2) $a(w+b_i) = a_i$.*

We omit the easy proof.

Denote by L_a and L_b the sublattices generated by the elements a_i and b_i (Lemma 5.3). It follows from this lemma that L_a and L_b are isomorphic.

We can begin a concordant choice of subspaces R_{i_jk} from the subspace V_1. As we know the sublattice E_1 is the chain $V_{1_1n} \subseteq V_{1_1n-1} \subseteq \ldots \subseteq V_{1_12} \subseteq V_{1_11} = V_1$, where $V_{1_1k} = V_{1,k} = V_1(W+V_k)$. R_{1_1k} can be chosen so that $R_{1_1n} = V_{1_1n}$, and $R_{1_1k}V_{1_1k+1} = 0$ and $V_{1_1k+1}+R_{1_1k} = V_{1_1k}$ for all $k < n$. Set $\varphi_1^+ R_{1_1k} = R'_{1_2k}$ if $1 < k$. Using Lemma 5.3 we can easily prove that the R'_{1_2k} have the following properties: $R'_{1_2n} = V_{1_2n}$, $R'_{1_2k}V_{1_2k+1} = 0$ and $R'_{1_2k}+V_{1_2k+1} = V_{1_2k}$ for all k such that $2 \leqslant k < n$, and besides $\varphi_2^- R'_{1_2k} = R_{1_1k}$. Choose $R_{2_2k} \subseteq V_2$ so that $R_{2_2n}V_{1_2n} = 0$, $R_{2_2n}+V_{1_2n} = V_{2_2n}$, and $R_{2_2k}(V_{1_2k}+V_{2_2k+1}) = 0$, $R_{2_2k}+V_{1_2k}+V_{2_2k+1} = V_{2_2k}$ for $2 \leqslant k < n$. If we choose now $R_{1_2k} = R'_{1_2k}$, then it is not difficult to check that each subspace $V_{i'_2k'}$ (in particular $V_{2_22} = V_2$) can be written as

$$V_{i'_2k'} = \sum_{(i,k)} R_{i_2k} \cong \bigoplus_{(i,k)} R_{i_2k},$$

where the sum is taken over all (i, k) for which $1 \leqslant i \leqslant i' \leqslant 2 \leqslant k' \leqslant k \leqslant n$.

Suppose that all subspaces $R_{i_{j'}k}$ have been chosen in this way for $2 \leqslant j' \leqslant j-1$. Transfer the subspaces $R_{i_{j-1}k}$ from $V_{i_{j-1}k} \subseteq V_{j-1}$ to $V_{i_jk} \subseteq V_j$ by means of the mapping φ_{j-1}^+, setting $R_{i_jk} = \varphi_{j-1}^+ R_{i_{j-1}k}$. Choose R_{j_jn} in such a way that $R_{j_jn}V_{j-1_jn} = 0$ and $R_{j_jn}+V_{j-1_jn} = V_{j_jn}$, and if $j \leqslant k < n$, then $R_{j_jk}(V_{j-1_jk}+V_{j_jk+1}) = 0$ and $R_{j_jk}+V_{j-1_jk}+V_{j_jk+1} = V_{j_jk}$. It is not difficult to prove that each $V_{i'_jk'}$ (in particular $V_{j_jj} = V_j$) can be represented as $V_{i'_jk'} = \sum_{(i,k)} R_{i_jk} \cong \bigoplus_{i,k} R_{i_jk}$, where the sum is taken over all (i, k) for which $1 \leqslant i \leqslant i' \leqslant j \leqslant k' \leqslant k \leqslant n$. It is also easy to check that $\varphi_j^- R_{i_jk} = R_{i_{j-1}k}$. If we continue this process, we get a family of subspaces R_{i_jk} in $V = \bigoplus_{i=1}^n V_i$, concordant with the mappings φ_j^- and φ_j^+. The proposition is proved.

5.3. *Main theorems.* Set $R_{i,k} = \bigoplus_j R_{i_jk}$. We will denote by $\varrho_{i,k}$ the restriction of the representation ϱ to the subspace $R_{i,k}$.

THEOREM 5.4. *The representation ϱ decomposes into the direct sum $\varrho \cong \bigoplus_{(i,k)} \varrho_{i,k}$, where $\varrho_{i,k} \cong \tau_{i,k} \oplus \ldots \oplus \tau_{i,k}$ ($p_{i,k}$ summands), $\tau_{i,k}$ is an indecomposable representation and $p_{i,k} = \dim R_{i_jk} \geqslant 1$.*

It is sufficient to prove that if x is a generator in $L(A_n^0)$ then the subspace $\varrho(x)$ can be represented in the form $\varrho(x) = \sum_{(i,k)} \varrho(x) R_{i,k}$. We have proved in Proposition 5.2 that

$$\varrho(v_j) = V_j = \sum_{(i,k)} R_{i_jk} \cong \bigoplus_{(i,k)} R_{i_jk}$$

for each fixed j. Since all sums $\sum_{i,j,k} R_{i_jk}$ are direct,

$$V_j R_{r,t} = \Big(\sum_{i,k} R_{i_jk}\Big)\Big(\sum_{s=r}^{t} R_{r_st}\Big) = \begin{cases} R_{r_jt} & \text{if } r \leqslant j \leqslant t, \\ 0 & \text{otherwise.} \end{cases} \tag{5.8}$$

Hence $\varrho(v_j) = V_j = \sum_{(i,k)} R_{i_jk} = \sum_{(i,k)} V_j R_{i,k}$, where only (i, k) with $i \leqslant j \leqslant k$ enter in the sum. We can extend the last sum to all pairs (i, k) $(i \leqslant k)$, since according to (5.8) we have $\varrho(v_j) R_{i,k} = 0$ for $j < i$ or $k < j$.

The proof of the formula $\varrho(w_j) = W_j = \sum_{(i,k)} W_j R_{i,k}$ can easily be obtained from the following lemma (we omit the easy proof).

LEMMA 5.5. *Let ϱ be a representation of the lattice $L(A_2^0)$ in a vector space V $(V = V_1 + V_2 = \varrho(v_1 + v_2))$. Let $R_{1,i}$ $(-q \leqslant i \leqslant k)$ and $R_{2,j}$ $(1 \leqslant j \leqslant m)$, $k \leqslant m$, be direct families of subspaces in V_1 and V_2 respectively such that*:

(a) *if* $1 \leqslant i \leqslant k$ *then* $R_{2,i} = V_2(W + R_{1,i})$;
(b) $\sum_{i=1}^{k} R_{1,i} = V_1(W + V_2)$;
(c) $V_1 = \sum_{i=-q}^{k} R_{1,i}$;
(d) $V_2 = \sum_{j=1}^{m} R_{2,j}$.

Then $\varrho = \sum_{i=-q}^{m} \varrho_i$, *where* $\varrho_i = \varrho|_{R_{1,i}}$ *for* $-1 \leqslant i \leqslant 0$, $\varrho_i = \varrho|_{R_{1,i}+R_{2,i}}$ *for* $1 \leqslant i \leqslant k$, *and* $\varrho_i = \varrho|_{R_{2,i}}$ *for* $k < i \leqslant m$.

Hence we obtain $\varrho \cong \bigoplus \varrho_{i,k}$, where $\varrho_{i,k} = \varrho|_{R_{i,k}}$ and $R_{i,k} = \bigoplus_j R_{i_jk}$. Notice that the subspaces R_{i_jk} have been chosen concordant, i.e. $\varphi_j^+ R_{i_jk} = R_{i_{j+1}k}$ and $\varphi_{j+1}^- R_{i_{j+1}k} = R_{i_jk}$ for $i \leqslant j < k$. This means that all R_{i_jk} with fixed i and k are isomorphic. In addition, $W_{i_jk} = W_j(R_{i_jk} + R_{i_{j+1}k})$ is the graph of the isomorphism between R_{i_jk} and $R_{i_{j+1}k}$.

Now we prove that $\varrho_{i,k} \cong \tau_{i,k} \oplus \ldots \oplus \tau_{i,k}$ ($p_{i,k}$ summands), where $\tau_{i,k}$ is an indecomposable representation and $p_{i,k} = \dim R_{i_ik} \geqslant 1$. Let $p_{i,k} > 1$ (in the case $p_{i,k} = 1$ it is easy to show that $\varrho_{i,k} = \varrho|_{R_{i,k}} \cong \tau_{i,k}$). Choose a basis $\{\xi_{i,t}\}$ $(t = 1, \ldots, p_{i,k})$ in R_{i_ik}. Each $\xi_{i,t}$ can be uniquely written as $\xi_{i,t} = \eta_{i,t} + (-\xi_{i+1,t})$, where $\eta_{i,t} \in W_{i_ik}$, $\xi_{i+1,t} \in V_{i_{i+1}k}$. We obtain the correspondence $\xi_{i,t} \mapsto \xi_{i+1,t}$, and this is the isomorphism φ_i^+ between R_{i_ik} and $R_{i_{i+1}k}$, with graph W_{i_ik}. Hence the vectors $\{\xi_{i+1,t}\}$ $(t = 1, \ldots, p_{i,k}\}$ are a basis in $R_{i_{i+1}k}$. If we continue this process (with $\varphi_{j-1}^{+}(\xi_{j-1,t}) = \xi_{j,t}$ and so on) we obtain a basis $\{\xi_{j,t}\}$ in each R_{i_jk} $(i \leqslant j \leqslant k)$. We denote by $T_{i,k,t}$ the span of $\xi_{i,t}, \xi_{i+1,t}, \ldots, \xi_{k,t}$ and by $\tau_{i,k,t}$ the restriction $\varrho|_{T_{i,k,t}}$. It is not difficult to prove that $\tau_{i,k,t} \cong \tau_{i,k}$, where $\tau_{i,k}$ is an indecomposable representation, i.e. $\varrho_{i,k} = \tau_{i,k,1} \oplus \ldots \oplus \tau_{i,k,p} \cong \tau_{i,k} \oplus \ldots \oplus \tau_{i,k}$, with $p = p_{i,k}$ summands.

Recall that D_n is the distributive lattice with generators c_i $(i = 1, \ldots, n)$ and relations $c_i c_j c_k = c_i c_k$ for $i \leqslant j \leqslant k$. We denote by $D(A_n^0)$ the image of D_n in the modular lattice $L(A_n^0)$ such that $c_j = \sum_{i=1}^{n} v_i(w + v_j)$.

THEOREM 5.6. *Each element of the sublattice $D(A_n^0)$ is perfect.*

Proof. Let us show that the element $c_i c_k$ $(i \leqslant k)$ of $D(A_n^0)$ is perfect. We have shown that

$$c_i c_k = \Big(\sum_{s=1}^{n} v_{s,i}\Big)\Big(\sum_{t=1}^{n} v_{t,k}\Big).$$

All elements $v_{i,j} = v_i(w + v_j)$ belong to the distributive sublattice $\bar{D}_n$. By the properties of $v_{i,j}$ (see Propositions 4.1 and 4.3) we have:

1) If $s \neq t$, then $v_{s,i} v_{t,k} = 0$.
2) If $i \leqslant j \leqslant k$, then $v_{j,i} v_{j,k} = v_{i_jk} = (v_i + w) v_j (v_k + w)$.
3) If $j < i \leqslant k$, then $v_{j,i} v_{j,k} = v_{j,k} = v_{j_jk} = v_j(v_k + w)$.
4) If $i \leqslant k < j$, then $v_{j,i} v_{j,k} = v_{j,i} = v_{i_jj} = v_j(v_i + w)$.

Hence

$$c_i c_k = \sum_{j=1}^{n} v_{j,i} v_{j,k} = \sum_{j=1}^{i-1} v_{j_jk} + \sum_{j=i}^{k} v_{i_jk} + \sum_{j=k+1}^{n} v_{i_jj}.$$

Let ϱ be a representation of the lattice $L(A_n^0)$ in a vector space V. Set $\varrho(v_{i_jk}) = V_{i_jk}$. The R_{i_jk} have been chosen in such a way that every V_{i_jk} can be written in the form $V_{i_jk} = \sum_{(i',k')} R_{i'_jk'} = \bigoplus_{(i',k')} R_{i'_jk'}$, where the sum is taken over all (i', k') such that $(i', k') \leqslant (i, k)$, i.e. $i' \leqslant i \leqslant j \leqslant k \leqslant k'$.

Using this, we obtain

$$\varrho(c_i c_k) = \sum_{j=1}^{i-1} \sum_{\substack{(i',k') \\ (i',k') \leqslant (j,k)}} R_{i'_jk'} + \sum_{j=i}^{k} \sum_{\substack{(i',k') \\ (i',k') \leqslant (i,k)}} R_{i'_jk'} + \sum_{j=k+1}^{n} \sum_{\substack{(i',k') \\ (i',k') \leqslant (i,j)}} R_{i'_jk'}.$$

If we collect the terms $R_{i'_jk'}$ with the same (i', k') (for j such that $i' \leqslant j \leqslant k'$), we get

$$\varrho(c_i c_k) = \sum_{(i',k') \leqslant (i,k)} \sum_{j} R_{i'_jk'},$$

where the first sum is taken over $(i', k') \leqslant (i, k)$, i.e. $i' \leqslant i \leqslant k \leqslant k'$, and the second over j with $i' \leqslant j \leqslant k'$. We have chosen the subspaces R_{i_jk} in a concordant way. Hence if we set $R_{i',k'} = \sum_{j=i'}^{k'} R_{i'jk'}$ and $\sigma_{i,k} = \varrho|_{\varrho(c_i c_k)}$, then

$$\varrho(c_i c_k) = \sum_{(i',k')\leqslant(i,k)} R_{i',k'}, \qquad \sigma_{i,k} = \sum_{(i',k')\leqslant(i,k)} \varrho_{i',k'} \cong \bigoplus_{(i',k')\leqslant(i,k)} \varrho_{i',k'}.$$

It also follows from Theorem 5.4 that $\varrho_{i',k'} = \varrho|_{R_{i',k'}}$ is a representation isomorphic to a direct sum of indecomposables $\tau_{i,k}$. It is easy to prove that the representation $\sigma_{i,k}$ is a direct summand of ϱ, i.e. $\varrho = \sigma_{i,k}\oplus\bar{\varrho}$, where $\bar{\varrho}$ is such that if we decompose it in a direct sum of indecomposables $\tau_{s,t}$, then no $\tau_{s,t}$ with $(s, t) \leqslant (i, k)$ appears. This shows that $c_i c_k$ is perfect. Notice that the characteristic subset $H_{i,k}$ corresponding to $c_i c_k$ is $H_{i,k} = \{\tau_{i',k'} | (i', k') \leqslant (i, k) \Leftrightarrow i' \leqslant i \leqslant k \leqslant k'\}$. Since the $c_i c_k$ are indecomposable in the abstract distributive lattice D_n, any $h \in D(A_n^0)$ can be written as $h = \sum_{(i,k)\in H_R} c_i c_k$, where H_R is a hereditary subset of the partially ordered set R of positive roots (i.e. if $\alpha = (i, k) \in H_R \subseteq R$ and $(i', k') = \alpha' \leqslant \alpha$, then $\alpha' \in H_R$). The characteristic subset of this perfect element h is $H \cong H_R$ (i.e. $\tau_\alpha = \tau_{(i,k)} \in H \Leftrightarrow \alpha = (i, k) \in H_R$). This ends the proof.

These arguments show that the following theorem is true.

THEOREM 5.7. *Each hereditary subset H of the partially ordered set $\mathscr{P}_n$ is characteristic, i.e. there exists a perfect element $h \in D(A_n^0) \subseteq L(A_n^0)$ such that $H = \{\tau \in \mathscr{P}_n | \tau(h) = V_\tau\}$.*

Hence we have three isomorphic partially ordered sets: 1) the partially ordered set $\mathscr{P}_n$ of indecomposable representations $\tau_\alpha = \tau_{i,k}$; 2) the set R of positive roots $\alpha = (i, k)$, ordered in the following way: $\alpha' = (i', k') \leqslant (i, k) = \alpha \Leftrightarrow i' \leqslant i \leqslant k \leqslant k'$; 3) the partially ordered set P_n of perfect elements $c_i c_k$, ordered in a similar way: $c_{i'} c_{k'} \subseteq c_i c_k \Leftrightarrow i' \leqslant i \leqslant k \leqslant k'$. Hence the lattice $D(A_n^0)$ of perfect elements with generators $c_i = \sum_{j=1}^{n} v_j(w + v_i)$ is isomorphic to the lattice $D(\mathscr{P}_n)$ of hereditary subsets of the partially ordered set $\mathscr{P}_n$.

We can sum up all these facts in

COROLLARY 5.8. *The distributive sublattice $D(A_n^0) \subseteq L(A_n^0)$ of perfect elements is isomorphic to the lattice $D(\mathscr{P}_n)$ of all hereditary subsets H of the partially ordered set $\mathscr{P}_n$ (the set of isomorphism classes of indecomposable representations). If $H \subseteq \mathscr{P}_n$ is a hereditary subset, then the corresponding perfect element $h \in D(A_n^0)$ is $h = \sum_{(i,k)} c_i c_k$, where the sum is taken over all (i, k) such that $\tau_{i,k} \in H$.*

5.4. We will discuss here some remaining problems. The whole structure of the lattices $L(A_n^0)$ $(n \geqslant 3)$ is not known yet. We say that the elements $x, y \in L(A_n^0)$ are *linearly equivalent* and write $x \sim y$ if for any representation ϱ (over an arbitrary field k) $\varrho(x) = \varrho(y)$. We will denote by $\overline{L(A_n^0)}$ the lattice $L(A_n^0)$ factored by the linear equivalence.

Consider the representation $\tau_{\mathscr{P}} = \bigoplus \tau_{i,k}$, where (i, k) runs through the whole set $\mathscr{P}_n$. Let $V_{\mathscr{P}}$ be the space of this representation over the field $\mathbf{Q}$. If we use the definition of the representations $\tau_{i,k}$, we can provide $V_{\mathscr{P}}$ with a basis $\{\xi_{ijk}\}$, where $1 \leqslant i \leqslant j \leqslant k \leqslant n$. We have $\tau_{\mathscr{P}}(v_j) = V_j = \bigoplus_{(i,k)} \mathbf{Q}\xi_{ijk}$ for every j, where the sum is taken over all (i, k) such that $i \leqslant j \leqslant k$. Also, $\tau_{\mathscr{P}}(w) = W = \bigoplus_{(i,j,k)} \mathbf{Q}(\xi_{ijk} + \xi_{ij+1k})$, where $i \leqslant j < k$. It is easy to show that $\dim V_j = j(n+1-j)$ for every j, $\dim V = \frac{1}{6}n(n+1)(n+2)$ and $\dim W = \frac{1}{6}(n-1)n(n+1)$. If we set $T_{i,k} = \sum_j \mathbf{Q}\xi_{ijk}$, where the sum is taken over all j such that $i \leqslant j \leqslant k$, then $\tau_{\mathscr{P}}|_{T_{i,k}} \cong \tau_{i,k}$. Define linear maps ψ^-: $T_{i,k} \to T_{i-1,k}$ $(i > 1)$ and ψ^+: $T_{i,k} \to T_{i,k+1}$ $(k < n)$ by $\psi^-(\xi_{ijk}) = \xi_{i-1jk}$ and $\psi^+(\xi_{ijk}) = \xi_{ijk+1}$. It is easy to prove that ψ^- and ψ^+ are morphisms of representations: ψ^-: $\tau_{i,k} \to \tau_{i-1,k}$ and $\psi^+: \tau_{i,k} \to \tau_{i,k+1}$. We can also prove that the representations $\tau_{i,k}$ are irreducible (i.e. $\tau_{i,k}(L(A_n^0)) \cong \mathscr{L}(\mathbf{Q}^{k-i})$) when $k - i \geqslant 3$.

We call a family of subspaces $U_{i,k} \subseteq T_{i,k}$ *admissible* if 1) $U_{i,i}$ is equal to 0 or $T_{i,i} \cong \mathbf{Q}\xi_{iii}$; 2) $U_{i,i+1}$ is one of the five subspaces: 0, $\mathbf{Q}\xi_{iii+1}$, $\mathbf{Q}\xi_{ii+1i+1}$, $\mathbf{Q}(\xi_{iii+1} + \xi_{ii+1i+1})$, $T_{i,i+1}$; 3) if $1 < i$, then $\psi^- U_{i,k} \subseteq U_{i-1,k}$ for all i, k, and if $k < n$, then $\psi^+ U_{i,k} \subseteq U_{i,k+1}$ for all i, k.

CONJECTURE. The lattice $\overline{L(A_n^0)}$ is isomorphic to the lattice of all admissible subspaces in $V_{\mathscr{P}} = \bigoplus_{(i,k)} T_{i,k}$.

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