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EQUIVARIANT VECTOR BUNDLES ON TORIC VARIETIES AND SOME PROBLEMS OF LINEAR ALGEBRA

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This is an exposition of some recent results of the author which join together such subjects as vector bundles, convex polyhedra, geometry and combinatorics of subspace configurations and some types of identities. The proofs, for the most part, are based on standard techniques and so are not presented here.

1. Moduli space of toric vector bundles

1.0. Let X be a complete nonsingular toric variety. This means that there is defined an action of an algebraic torus T on X and this action is free on a dense open orbit $O_0 \subset X$. For example the projective space $\mathbf{P}^n$ is a toric variety relative to an action of any maximal torus $T \subset \mathrm{PGL}(n+1)$.

An *equivariant* or *toric vector bundle* on X is a vector bundle $p\colon \boldsymbol{E} \to X$ equipped with some equivariant T-structure, that is, an action of the torus T on $\boldsymbol{E}$ which makes each diagram below commutative.

$$\begin{array}{ccc} \boldsymbol{E} & \xrightarrow{t} & \boldsymbol{E} \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\ X & \xrightarrow{t} & X, \end{array} \qquad t \in T.$$

The starting point of this paper is a very transparent description of toric bundles in terms of linear algebra.

1.1. To make this description clear we need some basic facts on toric varieties [1, 6]. Let $X = \bigsqcup_\sigma O_\sigma$ be the orbit decomposition (it is always finite). Once for all we identify the unique open orbit $O_0 \subset X$ with the torus T. This allows us to look on characters $\chi \in T$ as rational functions on X. So for each orbit O_σ there is defined a semigroup $\hat{\sigma} \subset \hat{T}$ of all characters which are regular on O_σ. We also need the dual cone

$$(1.1) \qquad \sigma = \{x \in \hat{T}^0_{\mathbf{R}}\colon (\chi, x) \geqslant 0,\ \forall \chi \in \hat{\sigma}\}.$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

The set all cones $\delta \subset \hat{T}^0_{\mathbf{R}}$ is called the *fan* associated with the toric variety X. We denote it by $\Sigma = \Sigma(X)$. There is a one-to-one correspondence between the cones $\delta \in \Sigma$ and the orbits $O_\sigma \subset X$, such that $\sigma \subset \tau \Leftrightarrow \bar{O}_\sigma \supset O_\tau$ and $\dim \sigma = \operatorname{codim} O_\sigma$.

The toric variety $X = X(\Sigma)$ is uniquely determined by its fan $\Sigma = \Sigma(X)$. It may be constructed from the affine pieces $U_\sigma = \operatorname{Spec} k[\hat{\sigma}]$ by identifying $U_{\sigma \cap \tau}$ with $U_\sigma \cap U_\tau$. Any finite set Σ of convex cones $\sigma \subset \hat{T}^0_{\mathbf{R}}$ may be a fan provided it satisfies the following conditions:

(i) every cone $\sigma \in \Sigma$ is generated by a finite number of elements of the lattice $\hat{T}^0$ (dual to the character lattice $\hat{T}$);

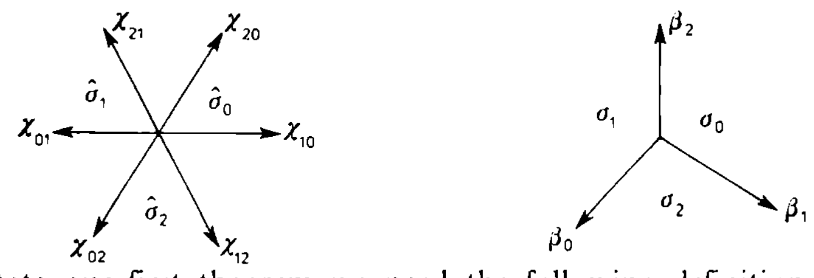
(ii) if $\sigma \in \Sigma$ then all faces of σ also belong to the fan Σ;

(iii) the intersection of two cones of the fan is their common faces.

For simplicity we restrict ourselves to nonsingular varieties X. In terms of the fan $\Sigma = \Sigma(X)$ this means that each cone $\sigma \in \Sigma$ is generated by a part of some basis of the lattice $\hat{T}^0$. We denote this set of generators of σ by $|\sigma|$. The notation $|\Sigma|$ will be used for the set of generators of all one-dimensional cones of the fan Σ.

From the above it follows that the fan of a nonsingular toric variety consists of simplicial cones. We also recall that the toric variety is complete if and only if all cones $\sigma \in \Sigma(X)$ cover all the space $\hat{T}^0_{\mathbf{R}}$.

1.2. Example. Let $X = \mathbf{P}^n$ be the projective space with homogeneous coordinates $(x_0 : x_1 : \ldots : x_n)$, and let $T \subset \mathrm{PGL}(n+1)$ be the diagonal torus. Then the quotients $x_i/x_j = \chi_{ij}$ are characters of the torus T (which we identify with the orbit of the point $(1:1:\ldots:1)$). For every fixed j they form a basis of the semigroup $\hat{\sigma}_j \subset \hat{T}$ consisting of all characters regular at the point p_j with coordinates $x_i = \delta_{ij}$. The fan $\Sigma = \Sigma(\mathbf{P}^n)$ consists of all faces of the cones σ_j dual to the $\hat{\sigma}_j$. The cone σ_j is generated by the basis $(\beta_0, \beta_1, \ldots, \hat{\beta}_j, \ldots, \beta_n)$ of the lattice $\hat{T}^0$ where β_k is defined by the relations $(\beta_k, \chi_{ij}) = \delta_{ik}$ for $k \neq j$. From the definition it follows that $\beta_0 + \beta_1 + \ldots + \beta_n = 0$. The figure illustrates the case of $\mathbf{P}^2$.



To state our first theorem we need the following definition.

1.3. Definition. A family of subspaces $E^\beta \subset E$, $\beta \in B$, of a finite-dimensional space E is called *split* if it generates a distributive lattice of subspaces. A family of $\mathbf{Z}$-filtrations $E^\beta(i)$, $\beta \in B$, is *split* if so is the family of subspaces $E^\beta(i)$, $\beta \in B$, for each $i \in \mathbf{Z}$.

All filtrations in this paper will be decreasing and exhaustive: $E^\beta(i) = E$, $i \ll 0$; $E^\beta(i) = 0$, $i \gg 0$.

A split system of subspaces or filtrations may be represented as a direct sum of a systems of rank ($= \dim E$) one.

1.4. THEOREM. *The category of equivariant vector bundles on a toric variety $X = X(\Sigma)$ is naturally equivalent to the category of vector spaces E which are equipped with a family of $\mathbf{Z}$-filtrations $E^\beta(i)$, $\beta \in |\Sigma|$, $i \in \mathbf{Z}$, satisfying the following concordance condition:*

(C) *for any $\sigma \in \Sigma$ the family of filtrations $E^\beta(i)$, $\beta \in |\sigma|$, is split.*

The equivalence of the categories may be established by assigning to any equivariant bundle $\boldsymbol{E}$ its fiber $E = \boldsymbol{E}(x_0)$ at some point x_0 of the open orbit $O_0 \subset X$. The filtrations on E arise as follows. The fiber $E = \boldsymbol{E}(x_0)$ is endowed with the family of subspaces

$$E^\beta(\chi) = \{e \in E: \lim_{tx_0 \to x_\beta, t \in T} \chi^{-1}(t)(te) \text{ exists}\}, \tag{1.2}$$

where $\chi \in \hat{T}$, $\beta \in |\Sigma|$ and $x_\beta \in O_\beta$ is any point of the orbit of codimension one. It is easy to see that the space $E^\beta(\chi)$ depends only on the order of the character χ at the divisor $X_\beta = \bar{O}_\beta$. This order is equal to (χ, β). So (1.2) reduces to the $\mathbf{Z}$-filtration

$$E^\beta(i) := E^\beta(\chi), \quad (\chi, \beta). \tag{1.3}$$

We may interpret the splitting condition for a family of filtrations $E^\beta(i)$ in terms of the parabolic subgroups

$$P^\beta = \{g \in \mathrm{GL}(E): gE^\beta = E^\beta\}$$

as follows:

$$E^\beta,\ \beta \in B, \text{ is split} \Leftrightarrow \bigcap_{\beta \in B} P^\beta \text{ contains a maximal torus.} \tag{1.4}$$

Let $\mathscr{P}(E)$ be an abstract simplicial complex with parabolic subgroups $P \subset \mathrm{GL}(E)$ as vertices and families of subgroups containing a common maximal torus as simplices. Theorem 1.4 implies that the classification of all toric vector bundles with fiber E over a variety $X = X(\Sigma)$ is essentially equivalent to a parametrization of all simplicial maps $f: \Sigma \to \mathscr{P}(E)$. So $\mathscr{P}(E)$ plays the role of a classifying space for toric vector bundles.

The following theorem exhausts most of the information on $\mathscr{P}(E)$ available to the author.

1.5. THEOREM. *Let E be a vector space of dimension m. Then the following conditions on a family of parabolic subgroups $P^\beta \subset \mathrm{GL}(E)$, $\beta \in B$, are equivalent:*

(i) $\bigcap_{\beta \in B} P^\beta$ *contains a maximal torus,*

(ii) *The intersection of any set of* $m+1$ *subgroups of the family contains a maximal torus.*

If each P^β *is a Borel subgroup then* (i), (ii) *are equivalent to*:

(iii) *The intersection of any triple of subgroups of the family contains a maximal torus.*

(iv) *The relative positions* $\pi_{\alpha|\beta} \in S_m$ *of the subgroups* P^α, P^β *form a cocycle:* $\pi_{\alpha|\beta}\pi_{\beta|\gamma}\pi_{\gamma|\alpha} = 1$, $\forall_{\alpha,\beta,\gamma\in B}$.

A pair of Borel subgroups (P^α, P^β) is in *relative position* $\pi \in S_m$ (the symmetric group) if it is conjugate to a pair (B, B') of upper triangular groups in a basis e and $e' = \pi e$ respectively.

1.6. COROLLARY. *The following conditions on a toric variety* $X = X(\Sigma)$ *are equivalent*:

(i) *All equivariant vector bundles of rank* m *over* X *are split.*

(ii) *Any set of* $m+1$ *vectors of* $|\Sigma|$ *generates a cone of the fan* Σ.

In particular, all toric vector bundles over $\mathbf{P}^n$ of rank less than n are split.

It would be interesting to extend the previous theorem to families of parabolic subgroups of an arbitrary algebraic group G.

1.7. EXAMPLES. (0) The tangent and cotangent bundles are determined by the following filtrations of the spaces $\mathcal{T} = \hat{T}^0 \otimes k$ and $\Omega = \hat{T} \otimes k$:

$$\mathcal{T}^\beta(i) = \begin{cases} \mathcal{T}, & i \leqslant 0, \\ \beta k, & i = 1, \\ 0. & i > 1, \end{cases} \quad \Omega^\beta(i) = \begin{cases} \Phi, & i < 0, \\ \ker \beta, & i = 0, \\ 0, & i > 0. \end{cases}$$

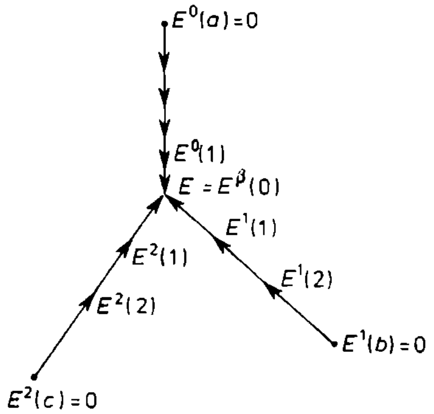
(1) **Line bundles.** A filtration $E^\beta(i)$ of a one-dimensional space E is determined by a number $n_\beta \in \mathbf{Z}$ such that $E^\beta(n_\beta) = E$ and $E^\beta(n_\beta + 1) = 0$. So line toric bundles are parameterized by functions $f\colon |\Sigma| \to \mathbf{Z}$, $\beta \to n_\beta$. The vector bundle corresponding to f will be denoted by $\mathcal{O}(f)$. This description of line bundles is due to Demazure [1].

(2) **Toric bundles of rank two.** Let E be a rank two equivariant vector bundle over a toric variety $X = X(\Sigma)$ given by filtrations $E^\beta(i)$, $\beta \in |\Sigma|$, of a two-dimensional space E. We denote by $\Sigma_E \subset \Sigma$ the subcomplex generated by those $\beta \in |\Sigma|$ for which the filtration $E^\beta(i)$ contains a nontrivial subspace. We will identify this 1-space with a point $x_\beta \in \mathbf{P}(E) = \mathbf{P}^1$. Then the concordance condition of Theorem 1.4 means that the correspondence $f\colon \beta \to x_\beta$ defines a simplicial map of Σ_E on a one-dimensional complex in $\mathbf{P}^1$ (i.e. the image $f(|\sigma|)$, $\sigma \in \Sigma_E$, contains nor more than two points). So irreducible components of the moduli space of rank two toric bundles are parametrized by configurations of points on $\mathbf{P}^1$ up to projective equivalence.

(3) **Vector bundles on the projective line $\mathbf{P}^1$.** According to Theorem 1.4, a vector bundle over $\mathbf{P}^1$ is determined by a pair of filtrations $E^0(i)$ and $E^1(j)$ of a space E. It is well known that any pair of filtrations is adjoint to some

bigrading $E = \oplus_{i,j} E(i, j)$. This implies that any toric vector bundle over $\mathbf{P}^1$ is split (a toric variant of the Grothendieck theorem).

(4) **Equivariant bundles over toric surfaces.** Just as in the previous example, any pair of filtrations satisfies the concordance condition of Theorem 1.4. So equivariant vector bundles over a toric surface $X = X(\Sigma)$ are parametrized by arbitrary families of filtrations $E^\beta(i)$, $\beta \in |\Sigma|$. These filtrations give rise to representations of a quiver which consists of $N = \# |\Sigma|$ chains joined together at a common point (the figure represents the case of $\mathbf{P}^2$).



A complete classification of all representations of such quivers is a wild problem. Nevertheless much interesting information on representations of quivers is contained in theorems of Gabriel [2], Nazarova [10] and Kac [7]. For example they allow us to describe all indecomposable toric bundles over $\mathbf{P}^2$ defined by a triple of filtrations E^0, E^1, E^2 where the numbers of nonzero factors a, b, c satisfy the condition $1/a + 1/b + 1/c \geqslant 1$. If the inequality is strict then the set of dimensions of all members of the filtrations E^0, E^1, E^2 coincides with the coordinates of a positive root of some system of type A_r, D_r, E_r. If equality holds then the dimensions are equal to the coordinates of an affine root of a system of type E_r.

2. The canonical resolution and characteristic classes

Let $X = X(\Sigma)$ be a complete toric variety of dimension n and let $\boldsymbol{E}$ be an equivariant vector bundle over X defined by a family of filtrations $E^\beta(i)$, $\beta \in |\Sigma|$, of the fiber $E = \boldsymbol{E}(x_0)$. We have an exact sequence induced by the chain complex of the fan Σ:

$$(2.1) \qquad C\colon 0 \to \bigoplus_{\operatorname{codim}\sigma = 0} \sigma \otimes E \xrightarrow{d} \bigoplus_{\operatorname{codim}\sigma = 1} \sigma \otimes E \xrightarrow{d} \dots$$

$$\dots \xrightarrow{d} \bigoplus_{\operatorname{codim}\sigma = n-1} \sigma \otimes E \xrightarrow{d} \emptyset \otimes E \to 0$$

Here the differential $d(\sigma)$ is the sum of all codimension one faces of σ with induced orientation (we suppose all cones $\sigma \in \Sigma$ to be oriented).

Let $f\colon |\Sigma| \to \mathbf{Z}$ be any function bounding the filtrations E^β, $\beta \in |\Sigma|$, that is, $E^\beta(i) = 0$ if $i > f(\beta)$. Then we may define a family of $\mathbf{Z}$-filtrations $C_f^\beta(i)$, $\beta \in |\Sigma|$, of the complex C by the formulae

$$(\sigma \otimes E)_f^\beta(i) = \begin{cases} \sigma \otimes E^\beta(i) & \text{if } \beta \in |\sigma|, \\ \sigma \otimes E & \text{if } \beta \notin |\sigma|,\ i \leqslant f(\beta), \\ 0 & \text{if } \beta \notin |\sigma|,\ i > f(\beta). \end{cases}$$

The filtrations are concordant in the sense of Theorem 1.4 and they are compatible with the differentials of the complex C. So by Theorem 1.4 they give rise to a complex of equivariant vector bundles

$$\mathscr{C}_f\colon\ 0 \to \boldsymbol{E} \to \boldsymbol{F}_0 \to \boldsymbol{F}_1 \to \boldsymbol{F}_2 \to \ldots \to \boldsymbol{F}_{n-1} \to \boldsymbol{F}_n \to 0. \tag{2.2}$$

2.1. THEOREM. *The complex $\mathscr{C}_f$ is a resolution of a toric bundle $\boldsymbol{E}$ and consists of split vector bundles $\boldsymbol{F}_k$.*

Here is an explicit decomposition of the bundles $\boldsymbol{F}_k$. Let T_σ be the stabilizer of any point x_σ in the orbit O_σ, $\sigma \in \Sigma$. For any character $\chi \in \hat{T}$ we define a function $f_\chi^\sigma\colon |\Sigma| \to \mathbf{Z}$ by

$$f_{\chi(\beta)}^\sigma = \begin{cases} f(\beta) & \text{if } \beta \notin |\sigma|, \\ (\chi, \beta) & \text{if } \beta \in |\sigma|. \end{cases}$$

Let $\mathcal{O}(f_\chi^\sigma)$ be the corresponding line bundle (see ex. 1.7.1). Then

$$\boldsymbol{F}_k = \bigoplus_{\chi \in \hat{T},\, \operatorname{codim} \sigma = k} \mathcal{O}(f_\chi^\sigma)\, m(\chi; \boldsymbol{E}(x_\sigma)), \tag{2.3}$$

where $m(\chi, \boldsymbol{E}(x_\sigma))$ is the multiplicity of the character χ in the fiber $\boldsymbol{E}(x_\sigma)$.

The canonical resolution (2.2) together with the formula (2.3) allows us to find characteristic classes of the equivariant vector bundle $\boldsymbol{E}$.

2.2. THEOREM. *Let $X_\beta = \bar{O}_\beta$, $\beta \in |\Sigma|$, be the class in the Chow or cohomology ring of the closure of codimension one orbit O_β. Then the full Chern class of a toric vector bundle $\boldsymbol{E}$ is equal to*

$$c(\boldsymbol{E}) = \prod_{\sigma \in \Sigma} \det\Big(1 + \sum_{\beta \in |\sigma|} \beta X_\beta \,|\, \boldsymbol{E}(x_\sigma)\Big)^{(-1)^{\operatorname{codim} \sigma}} \tag{2.4}$$

where we identify a vector $\beta \in |\sigma| \subset \hat{T}_\sigma^0 \otimes \mathbf{R}$ with an element of the Lie algebra $\operatorname{Lie} T_\sigma = \hat{T}_\sigma^0 \otimes \mathbf{C}$ *naturally acting in the fiber $\boldsymbol{E}(x_\sigma)$.*

It is worth while to note that if $\tau \subset \sigma$ then $T_\sigma \subset T_\tau$ and $\boldsymbol{E}(x_\tau) = \boldsymbol{E}(x_\sigma)|_{T_\tau}$. So all representations $\boldsymbol{E}(x_\sigma)$, $\sigma \in \Sigma$, and hence the characteristic class $c(\boldsymbol{E})$, are uniquely determined by the action of the torus T in the fibers of fixed points $\boldsymbol{E}(x)$, $x \in X^T$ (in the preceding notation they correspond to points x_Δ where $\Delta \in \Sigma$ is a cone of maximal dimension).

2.3. COROLLARY. *Characteristic classes of an equivariant vector bundle $\boldsymbol{E}$ on a toric variety X depend on the action of the torus in fibers of fixed points $\boldsymbol{E}(x)$, $x \in X^T$, only.*

3. Cohomology and the trace formula

In this section we will find the cohomology groups $H^p(X, \mathbf{E})$ of an equivariant vector bundle $\mathbf{E}$ on a toric variety $X = X(\Sigma)$, or rather their isotypic components $H^p(X, \mathbf{E})_\chi$, $\chi \in \hat{T}$.

We begin with some notation. Let $\mathbf{E}$ be defined by a family of filtrations $E^\beta(i)$, $\beta \in |\Sigma|$, of the space $E = \mathbf{E}(x_0)$. For a character $\chi \in \hat{T}$ and a cone $\sigma \in \Sigma$ we put

(3.1) $$E^\sigma(\chi) = \bigcap_{\beta \in |\sigma|} E^\beta(\chi); \qquad E_\sigma(\chi) = E / \sum_{\beta \in |\sigma|} E^\beta(\chi),$$

where $E^\beta(\chi)$ is defined in (1.2) or (1.3).

We use the spaces $E_\sigma(\chi)$ to form a complex

(3.2) $$C^*(\mathbf{E}, \chi)\colon\ 0 \to E \to \bigoplus_{\dim \sigma = 1} E_\sigma(\chi) \to \ldots \to \bigoplus_{\dim \sigma = n} E_\sigma(\chi) \to 0$$

with the differential

$$d(e_\sigma) = \sum_{\tau \supset \sigma} e_\sigma|_\tau, \qquad e_\sigma \in E_\sigma(\chi),$$

where the sum is taken over all cones $\tau \in \Sigma$ which contain σ as a face of codimension one; $|_\tau\colon E_\sigma(\chi) \to E_\tau(\chi)$ is the natural projection taking into account the orientations of σ and τ.

3.1. THEOREM. *There is a natural isomorphism of the cohomology spaces*

$$H^*(X, \mathbf{E})_\chi = H(C^*(E, \chi)).$$

3.2. COROLLARY. (i) $H^0(X, \mathbf{E})_\chi = \bigcap_{\beta \in |\Sigma|} E^\beta(\chi)$.

(ii) *For a complete variety X of dimension n*

$$H^n(X, \mathbf{E})_\chi = E / \sum_{\beta \in |\Sigma|} E^\beta(\chi);$$

(iii) *We have the Euler characteristic relation*:

$$\sum_{i=1}^{n} (-1)^i \dim H^i(X, \mathbf{E})_\chi = \sum_{\sigma \in \Sigma} (-1)^{\dim \sigma} \dim E_\sigma(\chi)$$
$$= \sum_{\sigma \in \Sigma} (-1)^{\operatorname{codim} \sigma} \dim E^\sigma(\chi).$$

(*for X complete*)

For projective spaces all the cohomology may be found explicitly.

3.2′. THEOREM. *Let $\mathbf{E}$ be a toric vector bundle over the projective space $\mathbf{P}^n$ and let E^β, $\beta = 0, 1, \ldots, n$ be the corresponding system of filtrations. Then for $0 < i < n$*

$$H^i(\mathbf{P}^n, \mathbf{E})_\chi = \bigcap_{j \geq i} (E^0(\chi) + \ldots + E^{i-1}(\chi) + E^j(\chi)) / (E^0(\chi) + \ldots + E^{i-1}(\chi) + \bigcap_{j \geq i} E^j(\chi)).$$

From this formula it follows that the cohomology $H^*(\mathbf{P}^n, E)_\chi$ is an obstruction to the distributivity of the lattice generated by the spaces $E^\beta(\chi)$, $\beta = 0, 1, \ldots, n$.

3.3. EXAMPLE. *Cohomology of general vector bundles over* $\mathbf{P}^2$.

Let E be a toric vector bundle of rank r over $\mathbf{P}^2$ corresponding to a triple of filtrations E^0, E^1, E^2 which are in general position. Put.

$$d(E, \chi) = \dim E^0(\chi) + \dim E^1(\chi) + \dim E^2(\chi).$$

Then by Theorem 3.2

$$\dim H^0(\mathbf{P}^2, E)_\chi = \dim E^0(\chi) \cap E^1(\chi) \cap E^2(\chi) = d(E, \chi) - 2r$$

if $d(E, \chi) \geqslant 2r$;

$$\dim H^1(\mathbf{P}^2, E)_\chi = \dim \frac{E^0(\chi) \cap (E^1(\chi) + E^2(\chi))}{E^0(\chi) \cap E^1(\chi) + E^0(\chi) \cap E^2(\chi)} = d(E, \chi) - r$$

if $2r > d(E, \chi) \geqslant r$;

$$\dim H^2(\mathbf{P}^2, E)_\chi = \dim E/E^0(\chi) + E^1(\chi) + E^2(\chi) = r - d(E, \chi)$$

if $r \geqslant d(E, \chi)$.

In all the other cases the cohomology is zero.

We see that every character with $d(E, \chi) \neq r, 2r$ appears in the cohomology $H^*(\mathbf{P}^2, E)$ exactly once.

Let us return to the Euler characteristic relation of Corollary 3.2 (iii). It has a useful interpretation as a trace formula.

3.4. THEOREM. *Let E be an equivariant vector bundle over a complete toric variety $X = X(\Sigma)$. Then for $t \in T$*

$$\sum_i (-1)^i \operatorname{Tr}(t | H^i(X, E)) = \sum_\Delta \operatorname{Tr}(t | E(x_\Delta)) / \prod_{\chi \in \Delta^*} (1 - \chi^{-1}(t)) \tag{3.3}$$

where the sum on the right-hand side is over all cones $\Delta \in \Sigma$ of maximal dimension $n = \dim X$; $x_\Delta \in X^T$ is a fixed point corresponding to Δ; Δ^* is the basis of the character lattice $\hat{T}$ dual to the basis $|\Delta|$ of $\hat{T}^0$.

3.5. COROLLARY. *The Poincaré polynomial of X may be written as follows:*

$$P_X(s) = \sum_i s^i \dim H^{2i}(X, \mathbf{C}) = \sum_\Delta \prod_{\chi \in \Delta^*} \frac{1 - s\chi}{1 - \chi}. \tag{3.4}$$

To prove the corollary it is sufficient to apply the theorem to the bundle of differential forms $\Omega^p(X)$. It may be interesting to compare (3.4) with another formula (see [8, 9]):

$$P_X(s) = \sum_{k=0}^n \# \Sigma^{(k)} (s-1)^{n-k}$$

where $\# \Sigma^{(k)}$ is the number of cones of dimension k in the fan Σ.

3.6. *Remark.* The right-hand sides of formulae (3.3) and (3.4) are rational functions on the torus T which are in fact constants. So these formulae are a source of various exotic identities including many classical ones.

For example, (3.4) for $X = \mathbf{P}^n$ implies

$$P_{\mathbf{P}^n}(s) = 1+s+\ldots+s^n = \sum_{i=0}^{n} \prod_{\substack{j \\ j\neq i}} \frac{x_j - sx_i}{x_j - x_i}.$$

The specialization $x_i = q^{-i}$ and letting $n \to \infty$ gives rise to the Cauchy identity

$$\prod_{i=1}^{\infty} \frac{1-q^i}{1-sq^{i-1}} = \sum_{i=0}^{\infty} \frac{(s-q)(s-q^2)\ldots(s-q^i)}{(1-q)(1-q^2)\ldots(1-q^i)}$$

which is widely known for $s = 0$ as the Euler identity.

Another example: the Euler characteristic of the structure sheaf $\mathcal{O}_X$ is equal to one; so by Theorem 3.4 we get an identity

$$\sum_{\Delta} \prod_{\chi \in \Delta^*} (1-\chi)^{-1} = 1.$$

As a consequence of the trace formula we also get the following intersection index theorem.

3.7. THEOREM. *Let $X_\beta = \bar{O}_\beta$, $\beta \in |\Sigma|$ be the closure of an orbit of codimension one in a complete toric variety $X = X(\Sigma)$ of dimension n. Then the intersection index of the cycles X_{β_i}, $i = 1, \ldots, n$, is equal to*

$$(3.5) \qquad (X_{\beta_1} X_{\beta_2} \ldots X_{\beta_n}) = \sum_{\Delta \ni \beta_1, \beta_2, \ldots, \beta_n} \frac{\beta_1^\Delta \beta_2^\Delta \ldots \beta_n^\Delta}{\prod_{\beta \in |\Delta|} \beta^\Delta}$$

where the sum is taken over all cones $\Delta \in \Sigma$ of maximal dimension and $\beta^\Delta \in \hat{T}$, $\beta \in |\Delta|$, is the element of the basis of the character lattice $\hat{T}$ dual to the basis $|\Delta|$ of $\hat{T}^0$.

This theorem may be used to calculate Chern numbers.

3.8. COROLLARY. *Let $P(c_1, c_2, \ldots, c_n)$ be a weighted homogeneous polynomial of degree n in the characteristic classes $c_i = c_i(E)$, $\deg c_i = i$, of an equivariant vector bundle E over a complete toric variety $X = X(\Sigma)$. Then in the notation of the theorem*

$$(3.6) \qquad P(c_1, c_2, \ldots, c_n) = \sum_{\Delta} P(\lambda_1^\Delta, \lambda_2^\Delta, \ldots, \lambda_n^\Delta) / \prod_{\beta \in |\Delta|} \beta^\Delta$$

where λ_i^Δ is the character of the representation $\Lambda^i E(x_\Delta)$ and x_Δ is a fixed point corresponding to a maximal cone $\Delta \in \Sigma$.

The corollary may be obtained by combining the theorem with the characteristic class formula (2.4). It may also be deduced from the Bott residue formula [5].

3.9. EXAMPLE. Let $P(c_1, c_2, \dots, c_n)$ be the Todd polynomial in the characteristic classes of a variety X. Then the left-hand side of (3.6) is the Euler characteristic of the structure sheaf, $\chi(\mathcal{O}_X) = 1$. If $\dim X = 2$ it follows that $c_1^2 + c_2 = 12$ and by (3.6) we get an identity

$$\sum_{i=1}^{N} \frac{(x_i+y_i)^2 + x_i y_i}{x_i y_i} = 12 \Leftrightarrow \sum_{i=1}^{N} \left(\frac{x_i}{y_i} + \frac{y_i}{x_i}\right) = 12 - 3N$$

where (x_i, y_i) are the coordinates of any point $p \in \hat{T}^0_{\mathbf{R}}$ in the ith basis of the fan $\Sigma = \Sigma(X)$; N is the cardinality of $|\Sigma|$.

For a three-dimensional variety X we have $c_1 c_2 = 24$ and

$$\sum_i \left(\frac{x_i}{y_i} + \frac{y_i}{x_i} + \frac{y_i}{z_i} + \frac{z_i}{y_i} + \frac{z_i}{x_i} + \frac{x_i}{z_i}\right) = 24 - 3N.$$

4. A structure theorem for some families of subspaces and toric vector bundles over $\mathbf{P}^n$

Let E^β, $\beta \in B$, be a family subspaces of a finite-dimensional vector space E. The family is called indecomposable if it cannot be represented in the form $E = E_1 + E_2$, $E^\beta = E_1^\beta + E_2^\beta$; $\dim E_i > 0$. Any system of subspaces is a direct sum of indecomposable systems which are uniquely determined up to isomorphism.

Indecomposable systems of four subspaces were described by Gel'fand and Ponomarev [3]. But in general such a description leads to wild problems of linear algebra. In this section we will describe the indecomposable families for which all proper subfamilies are split (in the sense of 1.3). The simplest example of such a family is a configuration of $n+1$ subspaces of codimension one in general position in a space F of dimension n. We denote this configuration by $\Omega_n^1 = (F; F^\beta, \beta = 0, 1, \dots, n)$ and put $\Omega_n^k = (\Lambda^k F; \Lambda^k F^\beta, \beta = 0, 1, \dots, n)$. The configuration Ω_n^k is closely related to the bundle of k-forms on $\mathbf{P}^n$ (see ex. 1.7.0).

4.1. THEOREM. *An indecomposable family of subspaces $E^\beta \subset E$, $\beta = 0, 1, \dots, n$, whose all proper subfamilies are split is either of rank ($= \dim E$) one or is isomorphic to one of the systems Ω_n^k, $k = 0, 1, \dots, n$.*

The proof makes use of some ideas of Gel'fand and Ponomarev [4].

Here is a typical situation in which configurations of the theorem appear. Let E be a toric vector bundle over $\mathbf{P}^n$ and let $E^\beta(i)$, $\beta = 0, 1, \dots, n$, be the corresponding family of filtrations. Then by the concordance condition of Theorem 1.4 for any $\chi \in \hat{T}$ all proper subsystems of the family $E(\chi) = (E^\beta(\chi), \beta = 0, 1, \dots, n)$ are split. So by Theorem 4.1 we get a decomposition

$$E(\chi) = \Big(\bigoplus_k m_k \Omega_n^k\Big) \oplus S \tag{4.1}$$

where S is a split system.

The multiplicities m_k are of cohomological nature. Indeed, by Theorem 3.2 the cohomology $H^*(\mathbf{P}^n, E)_\chi$ is completely determined by the configuration $E(\chi)$. The split part S does not affect the cohomology. Then from the classical equality $\dim H^p(\mathbf{P}^n, \Omega^q) = \delta_{pq}$ it follows that

$$m_k = \dim H(\mathbf{P}^n, E)_\chi. \tag{4.2}$$

The relations (4.1) and (4.2) imply

4.2. Corollary. *For a toric vector bundle* $\mathbf{E}$ *over* $\mathbf{P}^n$ *and any character* $\chi \in \hat{T}$ *the following inequality holds*:

$$\sum_k \binom{n}{k} \dim H^k(\mathbf{P}^n, E)_\chi \leqslant \operatorname{rk} E.$$

In particular, $H^k(\mathbf{P}^n, E) = 0$ *if* $\operatorname{rk} E < \binom{n}{k}$.

There is another way of describing the components of the decomposition (4.1):

$$\begin{aligned} m_k \Omega_n^k &= \Omega_n^k \otimes H^k(\mathbf{P}^n, E)_\chi \\ &= \bigcap_{|I|=k+1} \sum_{i \in I} E^i(\chi) \Big/ \sum_{|J|=n-k+1} \bigcap_{j \in J} E^j(\chi), \qquad I, J \subset \{0, 1, \dots, n\}. \end{aligned} \tag{4.3}$$

It is sufficient to verify this equality for split bundles and for Ω_n^k. It may be used to calculate the cohomology of toric vector bundles over $\mathbf{P}^n$ (cf. Theorem 3.2).

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