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REPRESENTATIONS OF BOUNDED STRATIFIED POSETS, COVERINGS AND SOCLE PROJECTIVE MODULES

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We develop a covering technique for vector space categories and socle projective modules. Following an idea of Nazarova and Roiter we introduce the notion of a bounded stratified poset $I_{\varrho\omega} = (I, \varrho, \omega)$ and we study matrix representations of $I_{\varrho\omega}$ in terms of a vector space category associated to $I_{\varrho\omega}$ and socle projective modules over a right peak ring R associated to $I_{\varrho\omega}$. We do it by representing R as a bound quiver algebra $F(Q, \Omega)$ and by describing its universal cover algebra $F(\tilde{Q}, \tilde{\Omega})$ for a class of bounded stratified posets $I_{\varrho\omega}$. We show how indecomposable representations of bipartite completed posets [12] can be computed in this way.

Introduction

Throughout this paper mod(A) denotes the category of finitely generated right A-modules. Following [9, 13] by a *vector space category* $\boldsymbol{K}_F$ we shall mean an additive Krull–Schmidt category $\boldsymbol{K}$ together with an additive faithful functor $|-|\colon \boldsymbol{K} \to \operatorname{mod}(F)$, where F is a division ring. A *factor space category* $\mathscr{V}(\boldsymbol{K}_F)$ consists of triples (U, X, φ) where U is in mod(F), $X \in \operatorname{ob} \boldsymbol{K}$ and $\varphi\colon |X|_F \to U_F$ is an F-linear map. Morphisms in $\mathscr{V}(\boldsymbol{K}_F)$ are defined in a natural way. If $\boldsymbol{K}$ has up to isomorphism only finitely many indecomposable objects $X_1, \ldots, X_n$ and $A = \operatorname{End}(X_1 \oplus \ldots \oplus X_n)$, ${}_AM_F = |X_1 \oplus \ldots \oplus X_n|_F$ then $R = \boldsymbol{R}_{\boldsymbol{K}} = \begin{bmatrix} A & {}_AM_F \\ 0 & A \end{bmatrix}$ is a right peak ring [16] (i.e. $\operatorname{soc}(R_R) \subseteq R_R$ is essential and $\operatorname{soc}(R_R) \cong P \oplus \ldots \oplus P$, where P is a simple projective right ideal). By [16] there is a full and dense additive functor $H\colon \mathscr{V}(\boldsymbol{K}_F) \to \operatorname{mod}_{\mathrm{sp}}(R)$ such that $\operatorname{Ker} H = [(0, X_1, 0), \ldots$

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..., (0, X_n, 0)], where $\mathrm{mod}_{\mathrm{sp}}(R)$ is the category of socle projective modules in $\mathrm{mod}(R)$.

Let us recall that if $I = \{1, \ldots, n\}$ is a poset (i.e. partially ordered set) then Nazarova and Roiter [8] associate to I a matrix problem $(\mathfrak{M}_I, \mathfrak{G}_I)$, where $\mathfrak{M}_I$ consists of block matrices (with coefficients in a division ring F) of the form

$$A = \boxed{A_1 \mid A_2 \mid \ldots \mid A_n}$$

and $\mathfrak{G}_I$ is a set of elementary transformations on matrices in $\mathfrak{M}_I$ associated to I (see [15, 18]). Let $I^* = I \cup \{*\}$ be the enlargement of I by a unique maximal element $*$ and let FI^* be the incidence ring of I^* ((1.14)). If $K(I)_F$ is the vector space category of the poset I (i.e. the category (2.14) with ϱ and ω trivial) then there is a commutative diagram

$$\begin{tikzcd} \mathfrak{M}_I \arrow[r, hook, "\gamma"] & \mathfrak{M}_I^{\mathrm{ad}} \arrow[r, "H'"] \arrow[d, "\simeq"] & I\text{-sp} \arrow[d, "\simeq"] \\ & \mathscr{V}(K(I)_F) \arrow[r, "H"] & \mathrm{mod}_{\mathrm{sp}}(R), \end{tikzcd}$$

where $R = R_{K(I)} \cong FI^*$, I-sp is the category of I-spaces and $\mathfrak{M}_I^{\mathrm{ad}}$ is the additive category whose objects are triples (V, U, t), where $V = V_1 \oplus \ldots \oplus V_n$ is an I-graded vector F-space and $t\colon V \to U$ is an F-linear map [15, 18]. The functor H' is given by the formula $H'(V, U, t) = (M, M_i)_{i \in I}$, where $M = U$ and

$$M_i = \mathrm{Im}\Big(\bigoplus_{j \preccurlyeq i} V_j \xrightarrow{(t_j)} U\Big).$$

γ is the natural embedding, $H'\gamma$ vanishes only on finitely many trivial indecomposable representations $A^{(1)}, \ldots, A^{(n+1)}$ and establishes a one-to-one correspondence between the indecomposable $\mathfrak{G}_I$-equivalence classes of matrices in $\mathfrak{M}_I$ different from $A^{(1)}, \ldots, A^{(n+1)}$ and the isoclasses of indecomposable I-spaces [15], [16; 7.2].

In the present article we introduce the notion of a bounded stratified poset $I_{\varrho\omega}$ (Definition 1.9), we define a corresponding matrix problem $(\mathfrak{M}_I(I_{\varrho\omega}), \mathfrak{G}(I_{\varrho\omega}))$, a vector space category $K(I_{\varrho\omega})_F$ associated with $I_{\varrho\omega}$ and a right peak F-algebra $F(I^*_{\varrho\omega})$ ((3.2)), and we extend the results above from posets to bounded stratified posets. We show that if F is algebraically closed then every vector space F-category K_F of finite type is of the form $K(I_{\varrho\omega})_F$ and every finite-dimensional basic right peak F-algebra R such that $\mathrm{mod}_{\mathrm{sp}}(R)$ is of finite type is of the form $F(I^*_{\varrho\omega})$. In particular, we get a new interpretation of representations of completed posets in the sense of Nazarova and Roiter [9, 11]. Moreover, our results allow us to study vector space categories $K(I_{\varrho\omega})_F$ by using the narrow overring adjustment functor [21] as well as the covering technique [1–6, 17] applied to the associated right peak algebra $R = F(I^*_{\varrho\omega})$ and to the category $\mathrm{mod}_{\mathrm{sp}}(R)$.

Recall from [17] that if $f\colon \tilde{R} \to R$ is a Galois covering with Galois group G, and $\tilde{R}$, R are locally bounded semiperfect basic algebras (in general without identity) then $\tilde{R}$ is a right multipeak algebra iff R is a right multipeak algebra (i.e. $\mathrm{soc}(eR)$ is a projective essential submodule of eR_R of finite length for any primitive idempotent e in R). Moreover, the pull-up functor $f.$ and the push-down functor f_λ [1, 5] induce functors

$$(0.1) \qquad \mathrm{Mod}_{\mathrm{sp}}(\tilde{R}) \underset{f.}{\overset{f_\lambda}{\rightleftarrows}} \mathrm{Mod}_{\mathrm{sp}}(R).$$

If $\tilde{R}$, R are algebras over an algebraically closed field F then one can prove socle projective analogues of the results of Dowbor and Skowroński [2, 4]. In particular, by the arguments used in [2] we get

THEOREM 0.2. *Let $f\colon \tilde{R} \to R$ be a Galois covering with Galois group G, where $\tilde{R}$, R are locally bounded locally finite-dimensional basic right multipeak algebras over an algebraically closed field. Assume that G acts freely on $\tilde{R}$ and the induced action on* $\mathrm{ind}_{\mathrm{sp}}(\tilde{R})/\cong$ *is also free. Then*:

(a) *If* $\mathrm{mod}_{\mathrm{sp}}(R)$ *is tame then* $\mathrm{mod}_{\mathrm{sp}}(\tilde{R})$ *is tame* [3].

(b) *If G is p-residually finite and $\tilde{R}$ is locally support-finite with respect to indecomposables* $\mathrm{ind}_{\mathrm{sp}}(\tilde{R})$ *in* $\mathrm{mod}_{\mathrm{sp}}(\tilde{R})$ *then* $\mathrm{mod}_{\mathrm{sp}}(\tilde{R})$ *is tame if and only if* $\mathrm{mod}_{\mathrm{sp}}(R)$ *is tame. Moreover, f_λ induces a bijection between the G-orbits of the isoclasses of indecomposables in* $\mathrm{mod}_{\mathrm{sp}}(\tilde{R})$ *and the isoclasses of indecomposables in* $\mathrm{mod}_{\mathrm{sp}}(R)$. *The functor f_λ carries almost split sequences in* $\mathrm{mod}_{\mathrm{sp}}(\tilde{R})$ *to almost split sequences in* $\mathrm{mod}_{\mathrm{sp}}(R)$ *and induces an isomorphism $\Gamma^{\mathrm{sp}}_{\tilde{R}}/G \cong \Gamma^{\mathrm{sp}}_R$, where Γ^{sp}_R is the AR-quiver of* $\mathrm{mod}_{\mathrm{sp}}(R)$.

(c) *A counterpart of* [4; *Theorem* 3.6] *for socle projective modules is valid.*

This result together with a list of minimal sp-representation-infinite simply sp-connected right multipeak algebras given by Weichert [25] is a useful tool for studying sp-representation-finite (i.e. with $\mathrm{mod}_{\mathrm{sp}}(R)$ of finite type) algebras R and vector space categories of finite type and of tame type.

In Sections 4 and 5 we apply this method to get criteria for R to be sp-representation-tame or sp-representation-finite for a class of algebras R of the form $F(I^*_{\varrho\omega})$.

We use the terminology and notation introduced in [16, 17]. For a discussion of the representation type the reader is referred to [3].

1. Bounded stratified posets

Throughout $I = (I, \prec)$ denotes a finite partially ordered set (shortly poset). Without loss of generality we can suppose that $I = \{1, \dots, n\}$ and $i \prec j$ implies $i <_{\mathbf{N}} j$. Consider

$$(1.1) \qquad \blacktriangle I = \{(i, j) \in I \times I;\ i \preccurlyeq j\}$$

and let $\delta\colon I \to \blacktriangle I$ be the diagonal map. Throughout this paper we shall identify I with the diagonal δI of $\blacktriangle I$ via $i \leftrightarrow \delta(i) = (i, i)$. Given $(i, j) \in \blacktriangle I$ we put $[i, j] = \{s \in I;\ i \preccurlyeq s \preccurlyeq j\}$.

Following Nazarova and Roiter [9, 11] we introduce the following definition.

Definition 1.2. A *stratification* of the poset I is an equivalence relation ϱ in $\blacktriangle I$ such that if $(i, j)\varrho(r, t)$ in $\blacktriangle I$ then:

(S1) the relations $i\varrho r$ and $j\varrho t$ hold in $I = \delta I \subseteq \blacktriangle I$,

(S2) there exists a poset isomorphism $\sigma\colon [i, j] \to [r, t]$ such that $(i, k)\varrho(r, \sigma(k))$ and $(k, j)\varrho(\sigma(k), t)$ for all $k \in [i, j]$.

A *stratified poset* is a pair $I_\varrho = (I, \varrho)$, where I is a poset and ϱ is a stratification of I.

Lemma 1.3. *Let (I, ϱ) be a stratified poset. If $(i, j)\varrho(r, t)$ and $i = r$ then j and t are incomparable with respect to the ordering $\prec$ in I. Dually, if $j = t$ then i and r are incomparable.*

Proof. Suppose that $i = r$. If we assume, on the contrary, that j and t are comparable, then obviously $|[i, j]| \neq |[r, t]|$ and we get a contradiction with (S2). The dual statement follows in a similar way.

Note that given a stratified poset (I, ϱ) we have no relation $(i, i)\varrho(r, t)$ with $r \prec t$ because of (S2). Hence, if

$$I = I_1 \dot\cup \ldots \dot\cup I_a, \qquad \Delta I := \blacktriangle I - \delta I = I'_1 \dot\cup \ldots \dot\cup I'_b \tag{1.4}$$

are the decompositions into equivalence classes with respect to ϱ restricted to I and to ΔI respectively, then $I_j \cap I'_r = \varnothing$ for all j and r. We call (1.4) the *ϱ-decomposition* of I and ΔI.

We shall denote by $\tau(i, j)$ the cardinality of the ϱ-equivalence class represented by (i, j).

We shall call ϱ *trivial* (and write $\varrho = \mathrm{id}$) if ϱ is the identity relation. In this case I_j and I'_r are one-element sets. (I, ϱ) is called *simply stratified* if the restriction of ϱ to ΔI is trivial. In this case ϱ is uniquely determined by the disjoint union set decomposition $I = I_1 \dot\cup \ldots \dot\cup I_a$ and we write

$$\varrho = (I_1, \ldots, I_a). \tag{1.5}$$

Definition 1.6. A poset I together with an equivalence relation ϱ in $\blacktriangle I$ is a *completed poset* (in the sense of Nazarova and Roiter [9]) if ϱ satisfies the condition (S1) and the following condition:

(co) If $(i, j)\varrho(i', j')$ and $i \prec r \prec j$ then there exists r' such that $i' \prec r' \prec j'$, $(i, r)\varrho(i', r')$, $(r, j)\varrho(r', j')$ and $\tau(i, j) = \tau(i, r) = \tau(r, j) = 2$.

Lemma 1.7. (i) *If (I, ϱ) is a completed poset then (I, ϱ) is stratified.*

(ii) *A stratified poset is completed if and only if it satisfies the following condition*:

(S3) *If* $\tau(i, j) \geqslant 2$ *and* $i \prec r \prec j$ *then* $\tau(i, j) = \tau(i, r) = \tau(r, j) = 2$.

Proof. (i) If (I, ϱ) is completed and $\tau(i, j) \geqslant 3$ then obviously $[i, j] = \{i, j\}$ and **(S2)** is trivially satisfied. If $\tau(i, j) = 2$, $(i, j)\varrho(i', j')$ and $i \prec r \prec j$ then by **(co)** there is r' satisfying the conditions in **(co)**. If we show that r' is uniquely determined by **(co)** then we can define $\sigma\colon [i, j] \to [i', j']$ by $\sigma(r) = r'$ and we are done. So suppose that $r' \neq r''$ satisfy **(co)**. Since $(i, r)\varrho(i', r')\varrho(i', r'')$ and $\tau(i, r) = 2$ we have $r = r'$ and $i = i'$. Hence $j \neq j'$ and by $(r, j)\varrho(r', j')\varrho(r'', j')$ we get $(r', j') = (r'', j')$ because $\tau(r, j) = 2$. It follows that $r' = r''$; a contradiction.

Given two stratifications ϱ, ϱ' of I we write $\varrho \leqslant \varrho'$ if $x\varrho y$ implies $x\varrho' y$ and we say that ϱ is *weaker* than ϱ'.

We shall use the following drawing convention. We write $i \to j$ if $i \prec j$ and there is no r such that $i \prec r \prec j$. If $i \to j$ and $i\varrho j$ we write $i \overset{\cdot}{\to} j$. For describing ϱ we shall write out only the relations $x\varrho y$ with $x \neq y$.

EXAMPLE 1.8. Let $I = \{1, 2, 3, 4\}$ and $1 \prec 2 \prec 4$, $1 \prec 3 \prec 4$, $(1, 1)\varrho(2, 2)$, $(3, 3)\varrho(4, 4)$, $(1, 3)\varrho(2, 4)$. Then (I, ϱ) is stratified and we write

$$(I, \varrho)\colon \quad \begin{array}{ccc} 1 & \overset{\cdot}{\to} & 2 \\ \downarrow & & \downarrow \\ 3 & \overset{\cdot}{\to} & 4 \end{array}, \qquad (1, 3)\varrho(2, 4).$$

In this case we have $I_1 = \{1, 2\}$, $I_2 = \{3, 4\}$, $I'_1 = \{(1, 3), (2, 4)\}$. We do not write out one-element equivalence classes.

By a *bound matrix* of (I, ϱ) we shall mean an upper-triangular $n \times n$ matrix $A = [a_{ij}]$ with coefficients in the centre F_0 of F satisfying the following conditions:

(b1) $a_{ij} = 0$ if i, j are unrelated with respect to $\prec$ in I.

(b2) $a_{ij} = 1$ if either $i = j$ or $\tau(i, j) = 1$.

(b3) $a_{ij} \neq 0$ if either $\tau(i, j) = 2$ or $\tau(i, j) \geqslant 3$ and there is r in I such that $i \prec r \prec j$.

(b4) If $(p, q)\varrho(s, t)$ and $[p, q] \neq \{p, q\}$ then there exists a poset isomorphism $\sigma\colon [p, q] \to [s, t]$ such that $a_{pq}(a_{pr}a_{rq})^{-1} = a_{st}(a_{s\sigma(r)}a_{\sigma(r)t})^{-1}$ whenever $p \prec r \prec q$.

DEFINITION 1.9. Let F be a division ring and let (I, ϱ) be a stratified poset with ϱ-coset decompositions

$$I = I_1 \dot\cup \ldots \dot\cup I_a, \qquad \Delta I = I'_1 \dot\cup \ldots \dot\cup I'_b.$$

A *bound set* of (I, ϱ) over the centre F_0 of F is a set

$$\omega = \omega_1 \dot\cup \ldots \dot\cup \omega_a \dot\cup \omega'_1 \dot\cup \ldots \dot\cup \omega'_b \tag{1.10}$$

of F_0-linear forms in $F_0[X_{ij};\ (i,j)\in\blacktriangle I]$ satisfying the following conditions:

(w1) $\omega_i=\omega'_j=\emptyset$ if $|I_i|=|I'_j|=1$.

(w2) If $|I_j|\geqslant 2$ then $\omega_j=\{X_{ss}-X_{tt};\ s,t\in I_j\}$.

(w3) The forms in ω'_j depend only on the variables X_{pq}, $(p,q)\in I'_j$, and if (p,q), $(r,t)\in I'_j$ then X_{pq} and X_{rt} both appear in some form which is an F_0-linear combination of the forms in ω'_j.

(w4) There is a bound matrix $A_\omega=[a_{pq}]$ of (I,ϱ) such that ω'_j contains the set

$$\omega'_j(A_\omega)=\{a_{uv}X_{uv}-a_{ts}X_{ts};\ (u,v),(t,s)\in I'_j\}$$

for $j=1,\ldots,b$. Moreover, $\omega'_j=\omega'_j(A_\omega)$ if either $|I'_j|=2$ or $|I'_j|\geqslant 3$ and there is a relation $p\prec r\prec q$ with $(p,q)\in I'_j$.

The triple $I_{\varrho\omega}=(I,\varrho,\omega)$, where ϱ is a stratification of I and ω is a bound set of (I,ϱ), will be called a *bounded stratified poset*.

If A is a bound matrix of (I,ϱ) then it induces a bound set $\varrho(A)$ (called a *principal bound*) defined by the formula

$$\varrho(A)=\{a_{pq}X_{pq}-a_{st}X_{st};\ (p,q)\varrho(s,t)\}. \tag{1.12}$$

In this case we say that $(I,\varrho,\varrho(A))$ is *principally bounded* with respect to A and we often denote it by $I_{\varrho A}=(I,\varrho,A)$.

Every stratified poset (I,ϱ) admits a bound matrix $E(I,\varrho)=[b_{ij}]$ with $b_{ij}=1$ for $i\prec j$. The corresponding principal bound set $\varrho(E(I,\varrho))$ will be called *primitive*. We usually write (I,ϱ,E) instead of $(I,\varrho,E(I,\varrho))$.

A bounded stratified poset $(I,\varrho,\bar{\varrho})$ is called *NR-bounded* if (I,ϱ) satisfies the condition **(S3)** in Lemma 1.7 and $\bar{\varrho}$ is defined by the formulas

$$\begin{aligned}\bar{\varrho}_j&=\{X_{ss}-X_{tt};\ s,t\in I_j\},\\ \bar{\varrho}'_j&=\{X_{pq}-X_{rt};\ (p,q),(r,t)\in I'_j\} &&\text{if } |I'_j|=2,\\ &=\{\textstyle\sum_{(p,q)\in I'_j}X_{pq}\} &&\text{if } |I'_j|\geqslant 3.\end{aligned} \tag{1.13}$$

The bound set $\bar{\varrho}$ is called an NR-bound because it was used by Nazarova and Roiter [9, 11] in their definition of completed poset representations.

We recall that the *incidence ring* of the poset $I=\{1,\ldots,n\}$ over F is the upper-triangular matrix subring

$$FI=[{}_iF_j]=e_1FI\oplus\ldots\oplus e_nFI \tag{1.14}$$

of $M_n(F)$, where ${}_iF_j=F$ for $i\preccurlyeq j$ and ${}_iF_j=0$ otherwise. Here e_i is the idempotent matrix with 1 in the (i,i) place and zeros elsewhere.

Given a bounded stratified poset (I,ϱ,ω) we consider

$$F(I,\varrho,\omega)=\{\lambda=[\lambda_{pq}]\in FI;\ f(\lambda)=0 \text{ for all } f\in\omega\}. \tag{1.15}$$

By Lemma 1.16 below $F(I, \varrho, \omega)$ is an F-subalgebra of FI. We call it the *incidence ring* of (I, ϱ, ω) with coefficients in F.

LEMMA 1.16. *Let F be a division ring and let (I, ϱ, ω) be a bounded stratified poset with ϱ-decomposition* (1.4). *Then*:

(i) *$F(I, \varrho, \omega)$ is a finite-dimensional F-subalgebra of FI.*

(ii) *If $e_1, \ldots, e_n$ is the standard set of primitive orthogonal idempotents in FI then the elements*

$$\hat{e}_j = \sum_{i \in I_j} e_i, \quad j = 1, \ldots, a, \tag{1.17}$$

form a complete set of primitive orthogonal idempotents in $F(I, \varrho, \omega)$.

Proof. Suppose that $b = [b_{iu}]$, $c = [c_{ij}] \in F(I, \varrho, \omega)$ and let $bc = [d_{ij}] \in FI$. It is clear that $d_{ii} = d_{jj}$ if $i\varrho j$. In order to prove that $f(bc) = 0$ for any form f in $\omega'_j(A_{\omega})$ (see **(w4)**) suppose that $(p, q), (s, t) \in I'_j$. By **(b4)** there exists a poset isomorphism σ: $[p, q] \to [s, t]$ such that

$$u_{prq} := a_{pq}(a_{pr}a_{rq})^{-1} = a_{st}(a_{s\sigma(r)}a_{\sigma(r)t})^{-1} \quad \text{whenever } p \prec r \prec q.$$

Hence we have

$$\begin{aligned} a_{pq}d_{pq} &= \sum_{p \preccurlyeq r \preccurlyeq q} a_{pq}b_{pr}c_{rq} = \sum_{p \preccurlyeq r \preccurlyeq q} (a_{pr}b_{pr}a_{rq}c_{rq})u_{prq} \\ &= \sum_{p \preccurlyeq r \preccurlyeq q} (a_{s\sigma(r)}b_{s\sigma(r)}a_{\sigma(r)t}c_{\sigma(r)t})u_{s\sigma(r)t} \\ &= a_{st} \sum_{s \preccurlyeq u \preccurlyeq t} b_{su}c_{ut} = a_{st}d_{st}. \end{aligned}$$

Now suppose that $f \in \omega'_j$, $|I'_j| \geqslant 3$ and $[p, q] = \{p, q\}$ for all $(p, q) \in I'_j$. Then given (p,q) in I'_j we have $d_{pq} = b_{pp}c_{pq} + b_{pq}c_{qq}$. Since f is linear and depends only on X_{pq}, $(p, q) \in I'_j$, we have

$$f([d_{pq}]) = b_{pp}f([c_{pq}]) + f([b_{pq}])c_{qq} = 0.$$

Consequently, $g(bc) = 0$ for all $g \in \omega$ and (i) follows. Since (ii) is an easy observation, the proof is complete.

EXAMPLES 1.18. (a) Let (I, ϱ): $1 \dashrightarrow 2 \dashrightarrow 3 \dashrightarrow 4$ and let ϱ restricted to ΔI be defined by taking $I'_1 = \{(1, 2), (2, 3), (3, 4)\}$, $I'_2 = \{(1, 4)\}$ and $I'_3 = \{(1, 3), (2, 4)\}$. Consider the primitive bound set $\varrho(A)$ with

$$A = \begin{bmatrix} 1 & a_{12} & a & 1 \\ 0 & 1 & a_{23} & b \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where a_{12}, a_{23}, a_{34} are fixed nonzero elements in the centre of F and

$a = a_{12}a_{23}$, $b = a_{23}a_{34}$. Then the incidence algebra $F(I, \varrho, \varrho(A))$ consists of all matrices in FI of the form

$$\begin{bmatrix} h & x & u & v \\ 0 & h & y & w \\ 0 & 0 & h & z \\ 0 & 0 & 0 & h \end{bmatrix}$$

and satisfying $a_{12}x = a_{23}y = a_{34}z$, $au - bv = 0$. It follows from Lemma 1.7 that the bounded stratified poset $(I, \varrho, \varrho(A))$ is not a completed poset.

(b) Let $I = \{1, 2, 3, 4, 5\}$, $1 \prec 2 \prec 3 \prec 5$, $2 \prec 4 \prec 5$, and define ϱ by taking $I_1 = \{1, 2\}$, $I_2 = \{3, 4\}$, $I_3 = \{5\}$, $I'_1 = \{(2, 3), (2, 4)\}$. Then (I, ϱ) is a stratified poset. If **pr** is the primitive bound set of I then

$$F(I, \varrho, \mathbf{pr}) = \begin{bmatrix} F & F & F & F & F \\ & F & F = & F & F \\ & & F & 0 & F \\ \text{zeros} & & & F & F \\ & & & & F \end{bmatrix}.$$

2. Representations

DEFINITION 2.1. Let F be a division ring and let (I, ϱ, ω) be a bounded stratified poset with $I = \{1, \ldots, n\}$, ϱ-decomposition (1.4) and a bound set decomposition (1.10). A *matrix F-representation* of (I, ϱ, ω) over F is a block matrix

$$(2.2) \qquad A = \left.\begin{array}{|c|c|c|c|} \hline \underbrace{A_1}_{s_1} & \underbrace{A_2}_{s_2} & \cdots & \underbrace{A_n}_{s_n} \\ \hline \end{array}\right\} s_* = s_{n+1}$$

with coefficients in F and such that $s_i = s_j$ for $i\varrho j$. We call

$$(2.3) \qquad \mathbf{cdn}(A) = (s_1, s_2, \ldots, s_n, s_{n+1})$$

the *coordinate vector* of A (see [15, 16]). In the set $\mathfrak{M}(I, \varrho, \omega)$ of all matrix F-representations of (I, ϱ, ω) a direct sum is defined by the formula

$$A \oplus A' = \begin{array}{|c|c|c|c|c|} \hline A_1 & 0 & \cdots & A_n & 0 \\ \hline 0 & A'_1 & \cdots & 0 & A'_n \\ \hline \end{array}.$$

Let $\mathfrak{G}(I, \varrho, \omega)$ be the set of all (ϱ, ω)*-transformations* on matrices in $\mathfrak{M}(I, \varrho, \omega)$ of the following types:

(e_1) All simultaneous elementary transformations on rows.

($\mathbf{e}_2$) For every I_j all *ϱ-simultaneous elementary transformations* inside blocks A_i for $i \in I_j$, i.e.

($\mathbf{e}_2'$) Given a nonzero $\lambda \in F$ and given a number $s \leqslant s_i$, $i \in I_j$, one multiplies the sth column by λ simultaneously in all blocks A_i, $i \in I_j$.

($\mathbf{e}_2''$) Given any two numbers $s \neq r \leqslant s_i$, $i \in I_j$, and $\lambda \in F$ one adds the sth column multiplied by λ to the rth column simultaneously in all blocks A_t, $t \in I_j$.

($\mathbf{e}_2'''$) Given any two numbers $s \neq r \leqslant s_i$ one interchanges the sth and rth columns simultaneously in all blocks A_t, $t \in I_j$.

($\mathbf{e}_3$) For every I_j' all *ϱ-simultaneous elementary ω-transformations* with a source in I_j', i.e. given numbers $s \leqslant s_p$, $r \leqslant s_q$ where $(p, q) \in I_j'$, and given a vector $(b_{pq})_{(p,q)\in I_j'}$, $b_{pq} \in F$, which is a common zero of all forms in ω_j' one adds the sth column in A_p multiplied by b_{pq} to the rth column in A_q simultaneously for $(p, q) \in I_j'$, $p \prec q$.

We have defined a *matrix problem* $(\mathfrak{M}(I, \varrho, \omega), \mathfrak{G}(I, \varrho, \omega))$ of classifying indecomposable $\mathfrak{G}(I, \varrho, \omega)$-equivalence classes $A/\sim$ of matrices A in $\mathfrak{M}(I, \varrho, \omega)$, where $\sim \; = \; \sim_{\mathfrak{G}(I,\varrho,\omega)}$ is the relation defined by the compositions of transformations in $\mathfrak{G}(I, \varrho, \omega)$ (see [15, 18]). We say that (I, ϱ, ω) is of *finite type* if $\mathfrak{M}(I, \varrho, \omega)$ has a finite number of indecomposable $\mathfrak{G}(I, \varrho, \omega)$-equivalence classes.

Here we follow an idea of Nazarova and Roiter [9, 11]. In the case where $(I, \varrho, \bar{\varrho})$ is an NR-bounded poset the definition above and the definition of representations of a completed poset coincide.

Given $s = (s_1, \ldots, s_{n+1})$ we put

$$\mathfrak{M}_s(I, \varrho, \omega) = \{A \in \mathfrak{M}(I, \varrho, \omega);\ \mathbf{cdn}(A) = s\} \tag{2.4}$$

and we denote by $\mathfrak{G}_s(I, \varrho, \omega)$ the set of all transformations in $\mathfrak{G}(I, \varrho, \omega)$ which operate on matrices A with $\mathbf{cdn}(A) = s$.

Note that transformations in $\mathfrak{G}_s(I, \varrho, \omega)$ of type ($\mathbf{e}_1$) act on matrices in $\mathfrak{M}_s(I, \varrho, \omega)$ as left multiplication by all elementary matrices in $\mathrm{Gl}(s_{n+1}, F)$, whereas transformations of types ($\mathbf{e}_2$) and ($\mathbf{e}_3$) act as right multiplication by all (ϱ, ω)-elementary matrices in the group

$$\mathfrak{H}_s = \{[g_{ij}] \in M_{|s|}(F);\ g_{ij} = 0 \text{ for } i > j,\ g_{ii} \in \mathrm{Gl}(s_i, F),\ g_{ij} \in M_{s_i \times s_j}(F)\}, \quad |s| = s_1 + \ldots + s_n, \tag{2.5}$$

of the following forms:

($\mathbf{e}_2'$) $\quad e^r_{I_j}(\lambda) = \mathrm{diag}(E_1, \ldots, E_n), \quad 0 \leqslant j \leqslant a,\ 0 \leqslant r \leqslant s_t,\ t \in I_j,$

where $\lambda \in F$, $E_i \in \mathrm{Gl}(s_i, F)$ and $E_i = \mathrm{diag}(1, \ldots, 1, \lambda, 1, \ldots, 1)$ with λ in the rth place for $i \in I_j$, whereas $E_i = E$ is the identity matrix for $i \notin I_j$;

($\mathbf{e}_2''$) $\quad e^{sr}_{I_j}(\lambda) = \mathrm{diag}(H_1, \ldots, H_n), \quad 0 \leqslant j \leqslant a,\ r, s \leqslant s_u,\ r \neq s,\ u \in I_j,$

where $\lambda \in F$, $H_i = E$ for $i \notin I_j$ and $H_i = E + \lambda e_{sr} \in \mathrm{Gl}(s_i, F)$ for $i \in I_j$ (here e_{sr} is the matrix in $M_i(F)$ with 1 in the (r, s) place and zeros elsewhere);

$(\mathbf{e}_2''')$ $\quad e_{I_j}^{sr} = \operatorname{diag}(H_1', \ldots, H_n'), \quad 0 \leqslant j \leqslant a,\ r, s \leqslant s_u,\ r \neq s,\ u \in I_j,$

where $H_i' = E$ for $i \notin I_j$ and $H_i' = \sum_{t \leqslant s_i, t \neq r,s} e_{tt} + e_{rs} + e_{sr}$ for $i \in I_j$;

$(\mathbf{e}_3)$ $\quad e_{I_j'}^{sr}(\bar{b}) = [g_{it}], \quad 0 \leqslant j \leqslant b,\ 0 \leqslant s \leqslant s_p,\ 0 \leqslant r \leqslant s_q,\ (p, q) \in I_j',$

where $\bar{b} = (b_{pq})_{(p,q) \in I_j'}$ is such that $f(\bar{b}) = 0$ for all f in ω_j' and

$$\begin{aligned} g_{it} &= E && \text{for } t = i, \\ &= b_{pq} e_{pq} \in M_{s_p \times s_q}(F) && \text{for } (i, t) = (p, q) \in I_j', \\ &= 0 && \text{for } (i, t) \notin I_j'. \end{aligned}$$

Denote by $\tilde{\mathfrak{G}}_s(I, \varrho, \omega)$ the subgroup of $\mathfrak{H}_s$ generated by all matrices of types $(\mathbf{e}_2')$–$(\mathbf{e}_2''')$ and $(\mathbf{e}_3)$ above.

Proposition 2.6. *For a bounded stratified poset (I, ϱ, ω) and a fixed coordinate vector $s = (s_1, \ldots, s_{n+1})$ we have*

$$\tilde{\mathfrak{G}}_s(I, \varrho, \omega) = \{g = [g_{ij}] \in \mathfrak{H}_s;\ f(g) = 0 \ \textit{for all}\ f \in \omega\}.$$

Proof. Denote the right side set above by $\mathfrak{G}_s'$. It follows from the definition of the bound set that $\mathfrak{G}_s'$ is closed under multiplication and under taking inverses (see the proof of Lemma 1.16). It is clear that $\tilde{\mathfrak{G}}_s(I, \varrho, \omega) \subseteq \mathfrak{G}_s'$. Now, if $g \in \mathfrak{G}_s'$ then a standard analysis shows that multiplication of g on the left and right by suitably chosen elements in $\tilde{\mathfrak{G}}_s(I, \varrho, \omega)$ reduces it to the identity matrix. This proves the converse inclusion and finishes the proof. The details are left to the reader.

Now given $s = (s_1, \ldots, s_{n+1})$ we consider the group

$$\mathfrak{G}_s = \mathrm{Gl}(s_{n+1}, F) \times \tilde{\mathfrak{G}}_s(I, \varrho, \omega) \tag{2.7}$$

and the group action $*$: $\mathfrak{G}_s \times \mathfrak{M}_s \to \mathfrak{M}_s$ given by $(g, h) * A = g^{-1} A h$, where $\mathfrak{M}_s = \mathfrak{M}_s(I, \varrho, \omega)$. Then Proposition 2.6 and the discussion preceding it yield

Corollary 2.8. *Matrices A, B in $\mathfrak{M}_s(I, \varrho, \omega)$ are $\mathfrak{G}(I, \varrho, \omega)$-equivalent if and only if $\mathfrak{G}_s * A = \mathfrak{G}_s * B$.*

Note also that if F is a commutative field then $\mathfrak{M}_s$ is an algebraic F-variety, $\mathfrak{G}_s$ is an algebraic group and $*$ is an algebraic group action.

Let A be a matrix representation of $I_{\varrho\omega} = (I, \varrho, \omega)$. We call $A/\!\sim$ *indecomposable* if there is an indecomposable matrix B such that $B \sim A$. The class $A/\!\sim$ is called *faithful* if $A/\!\sim$ is indecomposable and all s_j in $\mathbf{cdn}(A)$ ((2.3)) are nonzero. We call $I_{\varrho\omega}$ *faithful* if $\mathfrak{M}(I_{\varrho\omega})/\!\sim$ has a faithful class $A/\!\sim$.

Now we are going to connect $\mathfrak{M}(I_{\varrho\omega})$ with the factor space category $\mathscr{V}(\boldsymbol{K}_F)$ [16] of some vector space category $\boldsymbol{K}_F$ and with socle projective modules over a right peak ring [16].

DEFINITION 2.9. The *additive category of a bounded stratified poset* $I_{\varrho\omega} = (I, \varrho, \omega)$ over F is the category $\boldsymbol{K} = \boldsymbol{K}(I_{\varrho\omega})_F$ whose objects are I-graded finite-dimensional vector F-spaces

(2.10) $$\mathscr{V} = V_1 \oplus \ldots \oplus V_n \quad \text{with } V_i = V_j \text{ for } i\varrho j.$$

A map $g\colon \mathscr{V} \to \mathscr{W}$ in $\boldsymbol{K}$ is an F-linear map given by an upper-triangular $n \times n$ matrix

(2.11) $$g = [g_{ij}]$$

with $g_{ji} \in \mathrm{Hom}_F(V_i, W_j)$ such that $g_{ji} = 0$ for i, j unrelated with respect to $\prec$ and $h(g) = 0$ for all h in ω. The composition in $\boldsymbol{K}$ is given by the matrix multiplication.

Using the same type of arguments as in the proof of Lemma 1.16 one can prove that gf is a map in $\boldsymbol{K}$ provided f and g are. It is easy to see that $\boldsymbol{K}$ is a Krull–Schmidt category, the object

(2.12) $$\mathscr{F}_i = F_1 \oplus \ldots \oplus F_n$$

with $F_j = F$ for $j \in I_i$ and $F_j = 0$ otherwise is indecomposable for $i = 1, \ldots, a$,

(2.13) $$\mathrm{End}\,\mathscr{F}_i \cong \{\lambda = [\lambda_{pq}] \in FI_i;\ h(\lambda) = 0 \text{ for all } h \in \omega \text{ depending on } X_{pq} \text{ with } (p, q) \in \blacktriangle I_i\}$$

is a local finite-dimensional F-algebra and

$$\mathscr{V} \cong \mathscr{F}_1^{t_1} \oplus \ldots \oplus \mathscr{F}_a^{t_a}$$

where $t_j = \dim V_j$.

Denote by

(2.14) $$\boldsymbol{K}_F = \boldsymbol{K}(I_{\varrho\omega})_F$$

the vector space category $(\boldsymbol{K}, |-|\colon \boldsymbol{K} \to \mathrm{mod}(F))$, where $|-|$ is the forgetful functor. If $\mathscr{V}(\boldsymbol{K}_F)$ is the factor space category of $\boldsymbol{K}_F$ (see [16]) then we have a map

(2.15) $$\xi\colon \mathfrak{M}(I_{\varrho\omega}) \to \mathscr{V}(\boldsymbol{K}_F)$$

given by $\xi(A) = (F^{s_{n+1}}, F^{s_1} \oplus \ldots \oplus F^{s_n}, \varphi)$, where A is a block matrix of the form (2.2), $\mathbf{cdn}(A) = (s_1, \ldots, s_{n+1})$ and $\varphi\colon |F^{s_1} \oplus \ldots \oplus F^{s_n}| \to F^{s_{n+1}}$ is the linear map defined by A in the standard bases.

THEOREM 2.16. *Let A, B be matrices in $\mathfrak{M}(I_{\varrho\omega})$ and let $\sim\ =\ \sim_{\mathfrak{G}(I,\varrho,\omega)}$. Then*:

(a) *$A \sim B$ if and only if $\xi(A) \cong \xi(B)$.*

(b) *The equivalence class $A/\!\sim$ is indecomposable if and only if $\xi(A)$ is an indecomposable object in $\mathscr{V}(\boldsymbol{K}(I_{\varrho\omega})_F)$.*

(c) ξ *establishes a one-to-one correspondence between the indecomposable classes* $A/\sim$ *in* $\mathfrak{M}(I_{\varrho\omega})$ *and the isoclasses of indecomposable objects in* $\mathscr{V}(\boldsymbol{K}(I_{\varrho\omega})_F)$.

Proof. Let $\boldsymbol{K}_F = \boldsymbol{K}(I_{\varrho\omega})_F$. In order to prove (a) recall that if $\mathbf{cdn}(B) = (t_1, \ldots, t_{n+1})$ then a map $\xi(A) \to \xi(B)$ in $\mathscr{V}(\boldsymbol{K}_F)$ is a pair (h, g), where h: $F^{s_{n+1}} \to F^{t_{n+1}}$ is an F-linear map and g is a map in $\boldsymbol{K}$ of the form (2.11) such that the diagram

$$\begin{array}{ccc} F^{s_1} \oplus \ldots \oplus F^{s_n} & \xrightarrow{(A_1, \ldots, A_n)} & F^{s_{n+1}} \\ \downarrow g & & \downarrow h \\ F^{t_1} \oplus \ldots \oplus F^{t_n} & \xrightarrow{(B_1, \ldots, B_n)} & F^{t_{n+1}} \end{array} \tag{2.17}$$

is commutative. It follows that (h, g) is an isomorphism if and only if $t_1 = s_1, \ldots, t_{n+1} = s_{n+1}$ and h, g are bijective. Now we conclude from Proposition 2.6 that this is the case if and only if $h \in \mathrm{Gl}(s_{n+1}, F)$ and $g \in \tilde{\mathfrak{G}}_s(I_{\varrho\omega})$. Hence if (h, g) is an isomorphism then $A = h^{-1}Bg = (h, g) * B$ and according to Corollary 2.8 we get $A \sim B$. The converse follows in a similar way and (a) is proved.

(b) is an immediate consequence of the definitions, whereas (c) follows from (a) and (b) because any object $(U_F, \mathscr{V}, \varphi\colon |\mathscr{V}|_F \to U_F)$ in $\mathscr{V}(\boldsymbol{K}_F)$ is isomorphic to $\xi(A)$, where A is defined as follows. Fix linear isomorphisms $U_F \cong F^{\dim U}$, $V_j \cong F^{\dim V_j}$, take $s_j = \dim V_j$, $s_{n+1} = \dim U_F$ and take for A the matrix of φ in the bases corresponding to the standard ones under the isomorphisms fixed above.

Remark 2.18. (a) It follows from Theorem 2.16 that the matrix problem $(\mathfrak{M}(I_{\varrho\omega}), \mathfrak{G}(I_{\varrho\omega}))$ is equivalent to the classification of indecomposable objects in $\mathscr{V}(\boldsymbol{K}(I_{\varrho\omega})_F)$ and therefore the results in [16, 17] apply to it. In particular, we have a Kleisli category interpretation of $\mathscr{V}(\boldsymbol{K}(I_{\varrho\omega})_F)$ presented in [17; Proposition 1.9].

(b) Let $\tilde{S} = (I, \varrho, \bar{\varrho})$ be an NR-bounded poset. A simple analysis of the commutative diagram (2.17) with a map g in $\boldsymbol{K} = \boldsymbol{K}(\tilde{S})$ shows that $\mathscr{V}(\boldsymbol{K}_F)$ is dual to the category $\mathscr{R}(\tilde{S}, \mathrm{mod}(F))$ of representations of the completed poset $\tilde{S}$ in the sense of Nazarova and Roiter [11; p. 21].

3. A right peak ring of a bounded stratified poset and socle projective modules

We are going to give an interpretation of representations of a bounded stratified poset (I, ϱ, ω) over a division ring F in terms of socle projective modules over a finite-dimensional right peak F-algebra [16] associated to (I, ϱ, ω). In the case where ϱ and ω are both trivial our interpretation coincides with the well-known interpretation of matrix representations of a poset I in terms of I-spaces (see [15, 18]). Our results answer a question of Nazarova and Roiter in [10; p. 9].

Recall some notation from [16, 17, 20]. Suppose that

$$R = e_1 R \oplus \ldots \oplus e_a R \oplus e_{+1} R \oplus \ldots \oplus e_{+r} R$$

is a semiperfect right multipeak ring with a fixed set $e_1, \ldots, e_a, e_{+1}, \ldots, e_{+r}$ of primitive orthogonal idempotents. Suppose that $e_{+1}R, \ldots, e_{+r}R$ are *right peaks of* R, i.e. they "generate" $\mathrm{soc}(R_R)$ [17; p. 23]. Let X be an indecomposable module in $\mathrm{mod}_{\mathrm{sp}}(R)$ and denote by $P(X)$ the projective cover of X. If

$$\mathrm{soc}(X) \cong (e_{+1}R)^{t_1^+} \oplus \ldots \oplus (e_{+r}R)^{t_r^+}, \qquad P(X) \cong (e_1 R)^{t_1} \oplus \ldots \oplus (e_a R)^{t_a}$$

then we put

$$\mathbf{cdn}(X) = (t_1, \ldots, t_a, t_1^+, \ldots, t_r^+) \tag{3.0}$$

and call it the *coordinate vector* of X. The module X is called *sp-sincere* if $t_i > 0$ and $t_j^+ > 0$ for all i and j. The ring R is called *sp-sincere* if there is an sp-sincere indecomposable module in $\mathrm{mod}_{\mathrm{sp}}(R)$.

It follows from [17; Proposition 1.15 (d) and Corollary 1.16] that the study of indecomposable socle projective R-modules can be reduced to the case where R is sp-sincere.

Let $I_{\varrho\omega} = (I, \varrho, \omega)$ be a bounded stratified poset with $I = \{1, \ldots, n\}$, with ϱ-decomposition (1.4) and bound set decomposition (1.10). Let $I^* = I \cup \{*\}$ be the enlargement of I by a unique maximal element $*$ (which will be equivalently denoted by $n+1$). If we extend ϱ and ω trivially to I^* we get the bounded stratified poset $I^*_{\varrho\omega} = (I^*, \varrho, \omega)$ The incidence ring

$$FI^* = \begin{bmatrix} FI & {}_{FI}K_F \\ 0 & F \end{bmatrix} \tag{3.1}$$

of I^* with coefficients in a division ring F is a finite-dimensional right peak F-algebra (see [16]), where $K = \begin{bmatrix} F \\ \vdots \\ F \end{bmatrix}$ (n rows) is a left FI-module via the matrix multiplication. We have an F-algebra embedding

$$R := F(I^*_{\varrho\omega}) = \begin{bmatrix} S & {}_SK_F \\ 0 & F \end{bmatrix} \subseteq FI^*, \tag{3.2}$$

where $S = F(I_{\varrho\omega})$. Since K viewed as a left S-module is obviously faithful, R is a finite-dimensional right peak F-algebra. We call it the *right peak algebra of* $I_{\varrho\omega}$.

Denote by $\mathrm{mod}^{\mathrm{pr}}(R)$ the full subcategory of $\mathrm{mod}(R)$ consisting of modules of the form $X_R = (X'_S, X''_F, \varphi\colon X' \otimes_S K_F \to X''_F)$ with X' in $\mathrm{pr}(S)$. Let

$$(3.3) \qquad \Theta\colon \mathrm{mod}^{\mathrm{pr}}(R) \to \mathrm{mod}_{\mathrm{sp}}(R)$$

be the functor defined by $\Theta(X_R) = (Y_S, X''_F, \psi)$, where Y_S is the image of the map $\bar{\varphi}$ adjoint to φ and ψ is the map adjoint to the inclusion $Y_S \to \mathrm{Hom}_F({}_SK_F, X''_F)$ [16]. Note that $\mathrm{mod}^{\mathrm{pr}}(R)$ is an additive subcategory of $\mathrm{mod}(R)$ containing $\mathrm{pr}(R)$. It is closed under extension and under taking kernels of epimorphisms. Although in general $\mathrm{mod}^{\mathrm{pr}}(R)$ is not abelian it is a nice hereditary subcategory of $\mathrm{mod}(R)$ because of the following observation (see [20], [14; 2.5]).

LEMMA 3.4. *If* $R = \begin{bmatrix} S & {}_SK_F \\ 0 & F \end{bmatrix}$ *is an arbitrary right peak semiperfect ring with a unique simple projective right ideal* P_* *then for any module* $X_R = (X'_S, X''_F, \varphi\colon X' \otimes_S K_F \to X''_F)$ *in* $\mathrm{mod}^{\mathrm{pr}}(R)$ *there exists an exact sequence in* $\mathrm{mod}(R)$

$$0 \to P_*^t \to P(X_R) \xrightarrow{h} X_R \to 0.$$

If X *and* Y *are in* $\mathrm{mod}^{\mathrm{pr}}(R)$ *then* $\mathrm{Ext}^2_R(X, Y) = 0$.

Proof. Without loss of generality we can suppose that X_R has no summand isomorphic to P_*. Since X'_S is projective, $P(X_R) = (X'_S, X'_S \otimes {}_SK_F, \mathrm{id})$ and h has the form (id, h''), where $h''\colon X' \otimes {}_SK_F \to X''_F$ is an F-linear epimorphism. Hence $\mathrm{Ker}\, h = (0, \mathrm{Ker}\, h'', 0) \cong P_*^t$, where $t = \dim \mathrm{Ker} h''$, and the first statement follows. The second part follows from the first one.

A basic role in our considerations is played by the following result.

THEOREM 3.5. *Let* $\boldsymbol{K}_F = \boldsymbol{K}(I_{\varrho\omega})_F$ *be the vector space* F*-category of a bounded stratified poset* $I_{\varrho\omega} = (I, \varrho, \omega)$. *Then*:

(a) *If* $\mathscr{F} = \mathscr{F}_1 \oplus \ldots \oplus \mathscr{F}_a$ (see (2.12)) *then there is an* F*-algebra isomorphism* $\boldsymbol{K}(\mathscr{F}, \mathscr{F}) \cong F(I_{\varrho\omega}) = S$ *and the Yoneda functor* $\boldsymbol{w}\colon \boldsymbol{K} \to \mathrm{pr}(S)$, $\boldsymbol{w}(-) = \boldsymbol{K}(-, \mathscr{F})$, *is an equivalence of categories such that the diagram*

$$\begin{tikzcd} \boldsymbol{K} \arrow[r, "|-|"] \arrow[d, "w"'] & \mathrm{mod}(F) \\ \mathrm{pr}(S) \arrow[ur, "(-)\otimes_S K_F"'] & \end{tikzcd}$$

is commutative up to the natural equivalence of functors $|-|_F \to \boldsymbol{w}(-) \otimes {}_SK_F$.

(b) *The functor* $\boldsymbol{w}^+\colon \mathscr{V}(\boldsymbol{K}_F) \to \mathrm{mod}(R)$ *given by*

$$(3.6) \qquad \boldsymbol{w}^+(U, \mathscr{V}, \varphi) = \big(\boldsymbol{w}(\mathscr{V}), U, \boldsymbol{w}(\mathscr{V}) \otimes {}_SK_F \cong |\mathscr{V}|_F \xrightarrow{\varphi} U_F\big)$$

is full, faithful and establishes an equivalence of categories

$$\boldsymbol{w}^+\colon \mathscr{V}(\boldsymbol{K}_F) \to \mathrm{mod}^{\mathrm{pr}}(R).$$

Moreover, $\mathrm{mod}^{\mathrm{pr}}(R)$ *is a hereditary subcategory of* $\mathrm{mod}(R)$.

(c) *The composed functor*

$$H = \Theta w^{+}\colon \mathscr{V}(\boldsymbol{K}_F) \to \mathrm{mod}_{\mathrm{sp}}(R) \tag{3.7}$$

is full dense and $\mathrm{Ker}\, H = [(0, \mathscr{F}_a, 0), \ldots, (0, \mathscr{F}_a, 0)]$. *In particular, H establishes a representation equivalence* $H_0\colon \mathscr{V}_0(\boldsymbol{K}_F) \to \mathrm{mod}_{\mathrm{sp}}(R)$, *where* $\mathscr{V}_0(\boldsymbol{K}_F)$ *is the full subcategory of* $\mathscr{V}(\boldsymbol{K}_F)$ *consisting of objects having no summands of the form* $(0, \mathscr{F}_j, 0)$.

Proof. The isomorphism $\boldsymbol{K}(\mathscr{F}, \mathscr{F}) \cong S$ follows immediately from the definitions of $\boldsymbol{K}$ and S. Since we know from our discussion following Definition 2.9 that $\mathscr{F}$ is an additive generator of $\boldsymbol{K}$, by standard arguments w is an equivalence of categories. Next we note that there is a bimodule isomorphism ${}_{K(\mathscr{F},\mathscr{F})}|\mathscr{F}|_F \cong {}_S K_F$ and therefore the ring $R = F(I^*_{\varrho\omega})$ is isomorphic to the right peak ring $\boldsymbol{R}_{\boldsymbol{K}}$ of the category $\boldsymbol{K}_F$ in the sense of [16; Section 3]. Therefore the theorem is a consequence of [16; Lemma 3.2, Theorem 3.3] and Lemma 3.4.

COROLLARY 3.8. *Let* $I_{\varrho\omega} = (I, \varrho, \omega)$ *be a bounded stratified poset over a division ring F and let* $\boldsymbol{K}_F = \boldsymbol{K}(I_{\varrho\omega})$, $R = FI^*_{\varrho\omega} = \hat{e}_1 R \oplus \ldots \oplus \hat{e}_a R \oplus e_* R$ (see (1.17)). *The composed map* $\mathfrak{M}(I_{\varrho\omega}) \xrightarrow{\xi} \mathscr{V}(\boldsymbol{K}_F) \xrightarrow{H} \mathrm{mod}_{\mathrm{sp}}(R)$ *establishes a one-to-one correspondence between the indecomposable classes* $A/\sim$ *in* $\mathfrak{M}(I_{\varrho\omega})$ *with* $A/\sim \notin \{\xi^{-1}(\mathscr{F}_1)/\sim, \ldots, \xi^{-1}(\mathscr{F}_a)/\sim\}$ *and the isoclasses of indecomposable modules in* $\mathrm{mod}_{\mathrm{sp}}(R)$. *If* $\mathbf{cdn}\, A = (s_1, \ldots, s_n, s_*)$, $H\xi(A) \neq 0$ *and* $\mathbf{cdn}\, H\xi(A) = (t_1, \ldots, t_a, t_*)$ *then* $t_* = s_*$ *and* $t_i = s_j$ *for* $j \in I_i$.

Remark 3.9. (a) One of the main advantages of our socle projective module interpretation of representations of bounded stratified posets is the fact that one can study $\mathrm{mod}_{\mathrm{sp}}(R)$ by applying almost split sequences [13, 16], triangular reductions [16, 17], differentiation procedures [7, 19] as well as the covering technique ([17; Theorem 1.10] and Theorem 0.2). On the other hand, the description of indecomposables in $\mathrm{mod}_{\mathrm{sp}}(R)$ can sometimes be done easier by calculating the canonical forms of the corresponding matrices in $\mathfrak{M}(I_{\varrho\omega})$.

(b) The map $w^{+}\xi\colon \mathfrak{M}(I_{\varrho\omega}) \to \mathrm{mod}^{\mathrm{pr}}(R)$ reduces the study of $\mathfrak{M}(I_{\varrho\omega})$ to the study of a hereditary subcategory of $\mathrm{mod}(R)$ (see (3.4)).

(c) The construction of $F(I_{\varrho\omega}) \subseteq FI$ is analogous to Gabriel's construction [5] of the Galois quotient Λ/G of an algebra Λ with a free action of a group G on Λ. Here (ϱ, ω) plays the role of the G-action on Λ.

(d) The correspondence $(I, \varrho, \omega) \mapsto F(I^*, \varrho, \omega)$ yields a wide class of finite-dimensional right peak algebras. Although this class does not exhaust all right peak algebras our observation below shows that every sp-representation-finite algebra over an algebraically closed field F is of the form $F(I^*, \varrho, \omega)$.

THEOREM 3.10. *If R is a finite-dimensional basic right peak algebra over an algebraically closed field F and R is sp-representation finite then there exists* *a bounded stratified poset $I_{\varrho\omega}$ such that $R \cong F(I^*_{\varrho\omega})$. Moreover, $|I_i| \leqslant 3$ for $i \leqslant a$, $|I'_j| \leqslant 3$ for $j \leqslant b$, I_i is linearly ordered and the condition* **(S3)** *in Lemma* 1.7 *is satisfied. If $|I_i| = |I_j| = 3$ for some $i \neq j$ then the poset $I_i \cup I_j$ is linearly orderd.*

Outline of proof. By [16], R has the form $R = \begin{bmatrix} A & {}_AN_F \\ 0 & F \end{bmatrix}$, where $\dim N_F = n$ is finite and ${}_AN$ is faithful. Fix a complete set $\hat{e}_1, \ldots, \hat{e}_a$ of primitive orthogonal idempotents in A. Without loss of generality we can suppose that $A \subseteq \mathrm{End}_F(N_F)$ and $A_j := \hat{e}_j A \hat{e}_j \subseteq \mathrm{End}_F(N^j_F)$, where $N^j_F = \hat{e}_j N_F$.

Since R is sp-representation-finite, applying the arguments in [9; Sec. 2] (see also [17; p. 54]) one can show that the radical $J_i = J(A_i)$ of A_i is a principal one-sided ideal, $\dim J^t_i N^i_F / J^{t+1}_i N^i_F \leqslant 1$ and $\dim N^i_F \leqslant 3$ for all $t \geqslant 0$ and $i = 1, \ldots, a$. It follows that for every i one can choose a basis $I_i = \{v_{i1}, \ldots, v_{im_i}\}$ of N^i_F in such a way that:

(i) $v_{i1} \notin J_i N^i_F$ and $v_{it+1} = f^t_i(v_{i1})$, where f_i: $N^i_F \to N^i_F$ is a linear map such that $J_i = A_i f_i$, and

(ii) in the basis I_i A_i has a matrix form of one of the types

$$F, \quad \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}, \quad \begin{bmatrix} F & F & F \\ & F & F \\ & & F \end{bmatrix}, \quad \begin{bmatrix} F & F & F \\ & F & F \\ & & F \end{bmatrix}$$

(see [9; Lemma 4.2] and [13; p. 131]).

Define a partial order relation $\prec$ on the basis

$$I = I_1 \cup \ldots \cup I_a$$

of N_F by $v \prec v' \Leftrightarrow av' = v$ for some $a \in A$. From $A \subseteq \mathrm{End}_F(N_F)$ we easily derive an algebra embedding $A \subseteq FI$ such that $\hat{e}_t = \sum_{j \in I_t} e_j$, $e_j \in FI$, for $t = 1, \ldots, a$ and the space $\hat{e}_t A \hat{e}_p$ has a basis of one of the types (a), (b) or (c) in [9; Lemma 4.6] depending on matrices e_{js} in FI with $j \prec s$, j, $s \in I_t \cup I_p$, and on some constants in the field F. We put $j\varrho j'$ if j and j' belong to some I_t, and $(j, r)\varrho(s, v)$ if e_{jr} and e_{sv} appear in a basic element of some $\hat{e}_t A \hat{e}_p$ with a basis chosen as above. It is easy to check that the constants yield a bound ω of the stratified poset (I, ϱ) such that the image of the natural embedding

$$R = \begin{bmatrix} A & {}_AN_F \\ 0 & 0 \end{bmatrix} \subseteq FI^* = \begin{bmatrix} FI & {}_{FI}N_F \\ 0 & 0 \end{bmatrix}$$

is the algebra $F(I^*_{\varrho\omega})$. The remaining part of the theorem can be easily derived from [9; Section 4].

Remark 3.11. Applying the same type of arguments as in [9; Section 2] one can show that if $\mathrm{mod}_{\mathrm{sp}}(FI^*_{\varrho\omega})$ is not wild then $|I_j| \leqslant 4$ and either I_j is linearly ordered or it consists of two incomparable elements for $j = 1, \ldots, a$.

4. Bound quivers of bounded stratified posets

Suppose that $I_{\varrho A} = (I, \varrho, A)$ is a principally bounded stratified poset with a bound matrix $A = [a_{pq}]$. We are going to construct a *bound quiver* (Q, Ω) (see [1, 5, 6]) such that

$$FI^*_{\varrho A} \cong F(Q, \Omega) := FQ/(\Omega)$$

and (Ω) is admissible. For this we first construct a quiver Q' having as vertices the elements of I^* and such that given $i, j \in I^*$ there is an edge $\beta_{ij}: i \to j$ (unique) in Q' if and only if $i \prec j$ and $[i, j] = \{i, j\}$. Let

$$\bar{Q}'(I^*) = Q'/\varrho$$

be the quiver obtained from Q' by identifying ϱ-equivalent vertices.

Let Ω' be the set consisting of the following relations in $\bar{Q}'(I^*)$:

(4.1) $\beta_{ii_1}\beta_{i_1i_2}\cdots\beta_{i_rj} - \beta_{ij_1}\beta_{j_1j_2}\cdots\beta_{j_sj}$;

(4.2) $a_{i_1j_1}\cdots a_{i_rj_r}\beta_{i_1j_1}\cdots\beta_{i_rj_r} - a_{t_0t_1}\cdots a_{t_rt_{r+1}}\beta_{t_0t_1}\beta_{t_1t_2}\cdots\beta_{t_{r-1}t_r}$

if $r \geqslant 1$ and $\beta_{i_pj_p}\varrho\beta_{t_{p-1}t_p}$ for $p = 1, \ldots, r$;

(4.3) $\beta_{i_1j_1}\beta_{i_2j_2}\cdots\beta_{i_rj_r}$ if there is no sequence $\beta_{t_0t_1}, \ldots, \beta_{t_{r-1}t_r}$ such that $\beta_{i_pj_p}\varrho\beta_{t_{p-1}t_p}$ for $p = 1, \ldots, r$.

PROPOSITION 4.4. *Let $I_{\varrho A}$ be a principally bounded poset and let $(\bar{Q}', \Omega') = (\bar{Q}'(I^*), \Omega')$. If ϱ is multiplicative (i.e. $(i, j)\varrho(s, t)$ and $(j, k)\varrho(t, r)$ imply $(i, k)\varrho(s, r)$) then there is an F-algebra isomorphism $F(\bar{Q}', \Omega') \cong F(I^*_{\varrho A})$.*

Proof. Let h: $FQ' \to R := F(I^*_{\varrho A})$ be an F-algebra map defined by

$$h(i) = \sum_{t\varrho i} e_t, \quad h(\beta_{ij}) = e_{ij} \quad \text{if } \tau(i, j) = 1,$$
$$= a_{ij}^{-1} v_{ij} \quad \text{if } \tau(i, j) \geqslant 2,$$

where $v_{ij} = \sum_{(p,q)\varrho(i,j)} d_{pq}e_{pq}$ with $d_{pq} = a_{pq}^{-1}\sum_{(s,t)\varrho(p,q)} a_{st}$. It is clear that $v_{ij} \in R$ and $v_{ij} = v_{st}$ provided $(i, j)\varrho(s, t)$. Moreover, one can show that $v_{ij}v_{jt}b_{ij} = v_{it}$ for some $b_{ij} \in F$ if $i \prec j \prec t$, and $v_{ij}v_{rp} = 0$ if there are no elements $s \prec u \prec t$ such that $(i, j)\varrho(s, u)$ and $(r, p)\varrho(u, t)$. It follows that h is surjective and a straightforward calculation shows that $\Omega' \subseteq \operatorname{Ker} h$. Hence h induces an algebra surjection $\bar{h}$: $F(\bar{Q}', \Omega') \to R$. Since a simple analysis shows that $\dim_F F(\bar{Q}', \Omega') \leqslant \dim_F R$, $\bar{h}$ is an isomorphism.

Note that in general (Ω) is not an admissible ideal in $F\bar{Q}'$ because some of its generators contains paths of length one. However, if $I_{\varrho A}$ is primitively bounded (i.e. $A = E = E(I^*, \varrho)$) and ϱ is multiplicative we define a new bound quiver

(4.5) $$(Q_{I_\varrho}, \Omega_{I_\varrho}) = (Q'/\sim, \Omega'/\sim)$$

of $I_{\varrho E}$ by taking for $Q'/\!\sim$ the residue quiver of Q' modulo the relation

$$\beta_{ij} \sim \beta_{st} \Leftrightarrow (i, j)\varrho(s, t).$$

We take for Ω_{I_ϱ} the set of relations induced by Ω'. It is clear that the vertices of Q_{I_ϱ} are ϱ-cosets $\bar{i}$ of elements $i \in I^*$ and the edges are $\sim$-cosets $\bar{\beta}_{ij}$ of β_{ij}. We identify $\bar{\beta}_{ij}$ with the ϱ-coset of the element $(i, j) \in \Delta I$ such that $[i, j] = \{i, j\}$. Ω_{I_ϱ} is obtained from (4.1), (4.3) by interchanging β_{ij} and $\bar{\beta}_{ij}$. Now it is easy to conclude the following.

Corollary 4.6. *If $I_{\varrho E}$ is primitively bounded, $E = E(I^*, \varrho)$ and ϱ is multiplicative then (Ω) is an admissible ideal in FQ_{I_ϱ} and there is an F-algebra isomorphism $F(Q_{I_\varrho}, \Omega_{I_\varrho}) \cong F(I^*_{\varrho E})$.*

Examples 4.7. (a) Let

$$I^*:\quad \begin{array}{ccccccc} 1 & \xrightarrow{a} & 2 & \xrightarrow{b} & 3 & & \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 4 & \xrightarrow{c} & 5 & \xrightarrow{d} & 6 & \xrightarrow{\xi} & * \end{array}$$

and let $a\varrho b$, $c\varrho d$, $1\varrho 2\varrho 3$, $4\varrho 5\varrho 6$, $\alpha\varrho\beta\varrho\gamma$ and $(1, 5)\varrho(2, 6)$. Then

$$FI^*_{\varrho E} = \begin{bmatrix} F & F & F & F & F & F & F \\ & F & F & 0 & F & F & F \\ & & F & 0 & 0 & F & F \\ & & & F & F & F & F \\ & & & & F & F & F \\ & & & & & F & F \\ & & & & & & F \end{bmatrix},$$

Q_{I_ϱ} is shown in Fig. 1 and $\Omega_{I_\varrho} = \{\bar{a}^3, \bar{c}^3, \bar{\alpha}\bar{c} - \bar{a}\bar{\alpha}\}$. Note that if we omit the

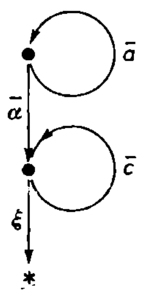


Fig. 1

relation $(1, 5)\varrho(2, 6)$ then ϱ is not multiplicative and Corollary 4.6 is not true in this case. Note also that $(\tilde{Q}, \Omega_{I_\varrho})$ with $\tilde{Q}$ depicted in Fig. 2 is the universal cover of $(Q_{I_\varrho}, \Omega_{I_\varrho})$ with the group $G = \mathbf{Z}$ [6].

(b) Let I^* be the poset in (a) and let $1\varrho 4$, $2\varrho 5$, $3\varrho 6$, $a\varrho c$. Then

$$FI^*_{\varrho E} = \begin{bmatrix} F & F & F & F & F & F & F \\ & F & 0 & F & 0 & F & F \\ & & F & F & F & F & F \\ & & & F & 0 & F & F \\ & & & & F & F & F \\ & & & & & F & F \\ & & & & & & F \end{bmatrix},$$

Q_{I_ϱ} is shown in Fig. 3 and

$$\Omega_{I_\varrho}: \begin{cases} \alpha\bar{a} = \bar{a}\beta, \ \beta\bar{d} = \bar{b}\gamma, \\ \alpha^2 = \beta^2 = \gamma^2 = 0, \\ \beta\bar{b} = \bar{d}\gamma = \bar{b}\xi = 0. \end{cases}$$

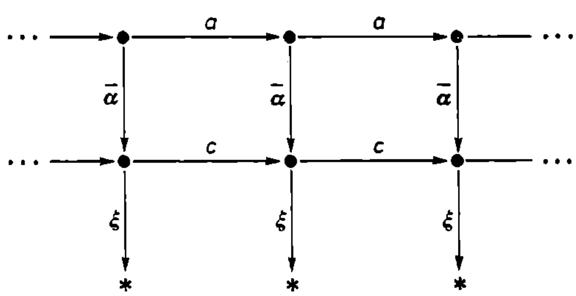


Fig. 2

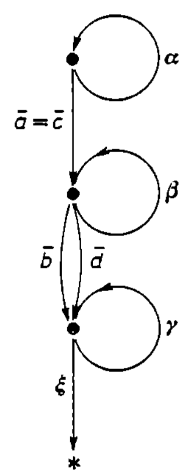


Fig. 3

The bound quiver $(\tilde{Q}, \tilde{\Omega})$ with $\tilde{Q}$ shown in Fig. 4 and $\tilde{\Omega} = \Omega_{I_o} \cup \{\bar{d}\gamma, \beta\bar{b}\}$ is a cover of $(Q_{I_\varrho}, \Omega_{I_\varrho})$.

(c) Let (I^*, ϱ) be the poset of Fig. 5 with $1\varrho 1'$, $2\varrho 2'$, $3\varrho 3'$, $\alpha\varrho\alpha'$, $\beta\varrho\beta'$ and $(\alpha\beta)\varrho(\alpha'\beta')$. Then the bound quiver (Q, Ω) of $I^*_{\varrho E}$ has Q as in Fig. 6,

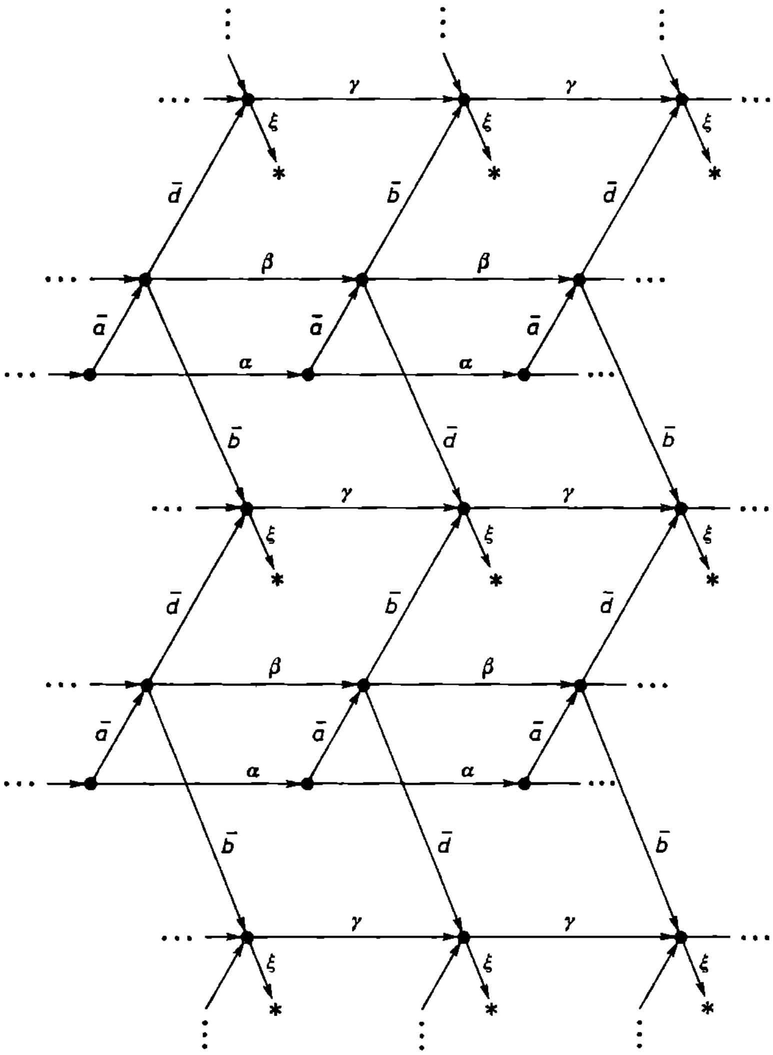


Fig. 4

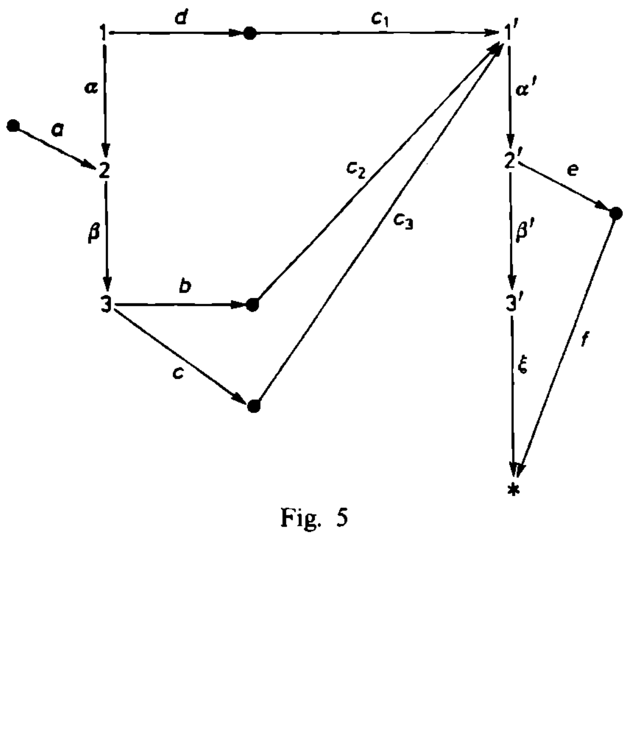


Fig. 5

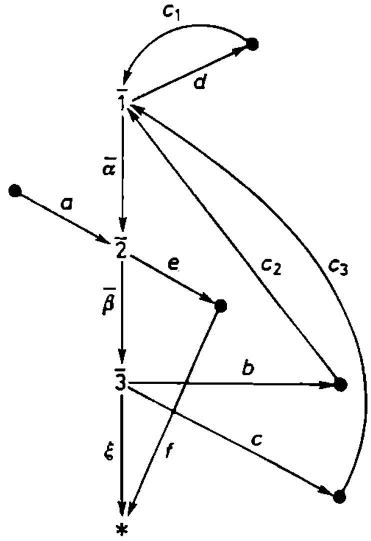


Fig. 6

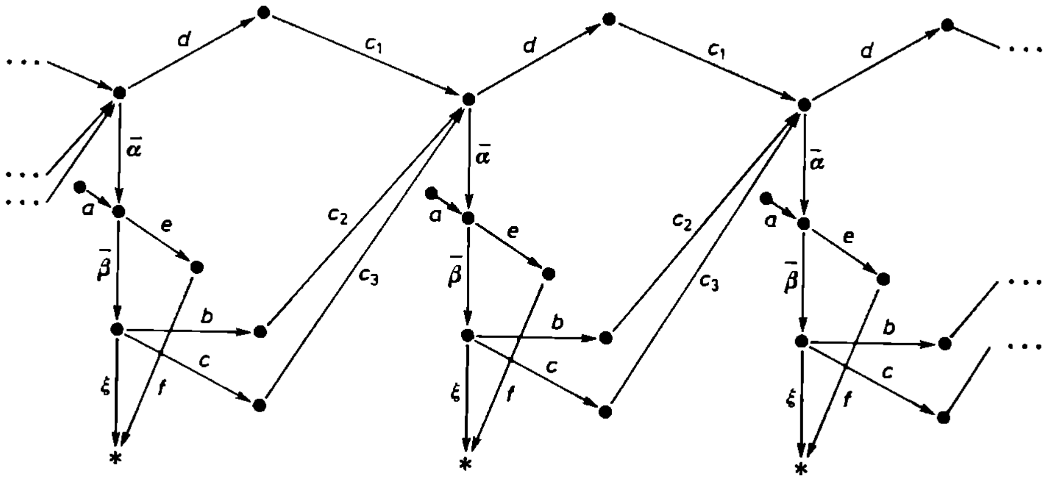


Fig. 7

$$\Omega: \begin{cases} c_1 d = ae = 0, \ c_1 \bar{\alpha}\bar{\beta} b = c_1 \bar{\alpha}\bar{\beta} c = 0, \\ c_i h c_j = 0 \text{ for every path } h \text{ and } i, j \geqslant 1, \\ \bar{\alpha}\bar{\beta} b c_2 = d c_1, \ c c_3 = b c_2, \\ ef = \bar{\beta}\xi, \end{cases}$$

and $(\tilde{Q}, \Omega)$ is the universal cover of (Q, Ω), where $\tilde{Q}$ is shown in Fig. 7.

(d) Let (I^*, ϱ): $\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \xrightarrow{c} \bullet \xrightarrow{a'} \bullet \xrightarrow{b'} \bullet \xrightarrow{\xi} *$, $a\varrho a'$, $b\varrho b'$. Note that ϱ is not multiplicative and therefore we cannot apply Corollary 4.6. However, one can show that the algebra

$$FI^*_{\varrho E} = \begin{bmatrix} F & F & F & F & F & F & F \\ 0 & F & 0 & F & 0 & F & F \\ 0 & F & F & F & F & F & F \\ 0 & 0 & 0 & F & 0 & F & F \\ 0 & F & 0 & F & F & F & F \\ 0 & 0 & 0 & 0 & 0 & F & F \\ 0 & 0 & 0 & 0 & 0 & 0 & F \end{bmatrix},$$

is isomorphic to $F(Q, \Omega)$, where Q is shown in Fig. 8 and $\Omega = \{u\xi,\ c\bar{a}\bar{b}c,\ \bar{a}\bar{b}c - uc,\ cu\}$. The universal cover of (Q, Ω) has the form $(\tilde{Q}, \Omega)$, where Q is illustrated in Fig. 9.

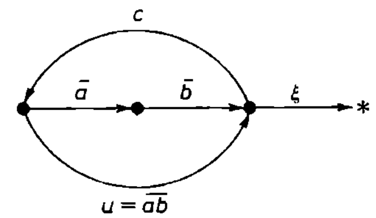


Fig. 8

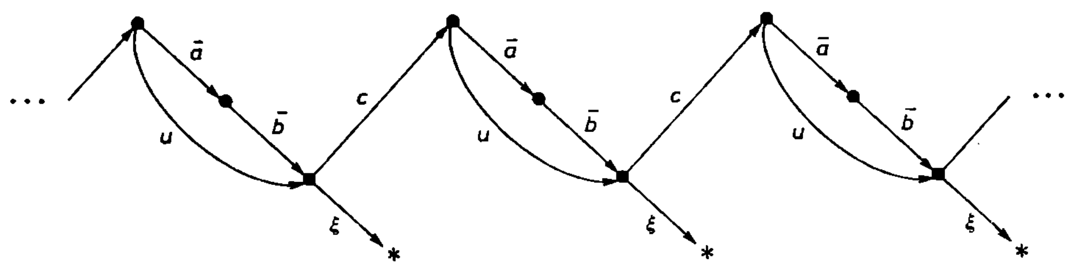


Fig. 9

Remark 4.8. Suppose that (Q, Ω) is a bound quiver and let G be an automorphism group acting freely on (Q, Ω). Suppose that Q contains a finite full subquiver I which is a poset (i.e. Ω restricted to I consists of full commutativity relations) with a unique maximal element and such that $Q = \bigcup_{g \in G} gI$. Then the action of G restricted to I defines a stratification $\varrho = \varrho_G$ on I such that:

(i) ϱ is multiplicative.

(ii) There is no relation $(i, j)\varrho(i, k)$, $(i, j)\varrho(t, j)$, $k \neq j$, $i \neq t$, in ΔI.

(iii) The algebra $F(\bar{Q}, \bar{\Omega})$ with $(\bar{Q}, \bar{\Omega}) = (Q, \Omega)/G$ is a right peak algebra isomorphic to $F(I^*, \varrho_G, {}'E(I^*, \varrho_G))$.

It would be interesting to know if any primitively stratified poset I with the properties (i) and (ii) can be obtained in this way.

By Lemma 1.3 the condition (ii) is satisfied if (I, ϱ) is of finite type.

The following definition is useful in studying stratified posets.

DEFINITION 4.9. A pair (i, j) with $i \prec j$ is called a *rib* of (I, ϱ) if $\tau(i, j) \geqslant 2$ [9]. A *rib skeleton* of (I, ϱ) is a full stratified subposet $\mathbf{rsk}(I, \varrho)$ of (I, ϱ) such that if (i, j) is a rib and $(i, j)\varrho(s, t)$ then $i, j, s, t \in \mathbf{rsk}(I, \varrho)$.

It is clear that $\mathbf{rsk}(I, \varrho)$ is empty if and only if (I, ϱ) is simply stratified.

Throughout this paper we fix the disjoint union decomposition

$$\mathbf{rsk}(I, \varrho) = \mathfrak{R}_1 \dot\cup \ldots \dot\cup \mathfrak{R}_h, \tag{4.10}$$

where $\mathfrak{R}_1, \ldots, \mathfrak{R}_h$ are the connected components of $\mathbf{rsk}(I, \varrho)$ with respect to the equivalence relation generated by

$$i \,\overline{\tau}\, j \Leftrightarrow \text{either } (i, j) \text{ or } (j, i) \text{ is a rib}.$$

The number $\boldsymbol{h} = \boldsymbol{h}(I, \varrho)$ is called the *rib complexity* of (I, ϱ).

By a *rib path* from i to j in (I, ϱ) we mean a formal composition $\beta_1 \ldots \beta_s$ of ribs such that β_1 starts from i, β_s ends at j and the start point of β_{j+1} is the end point of β_j for $j = 1, \ldots, s-1$.

LEMMA 4.11. *Suppose that (I, ϱ) is a stratified poset with the following properties*:

(i) $\tau(i, j) \leqslant 2$ *and* $\tau(i) \leqslant 2$ *for all* $i \in I$ *and* $(i, j) \in \Delta I$.

(ii) *There is no relation* $(i, j)\varrho(i, t)$ *and* $(s, r)\varrho(u, r)$ *with* $j \neq t$, $s \neq u$.

Then:

(a) *If $\beta_1 \ldots \beta_s$ is a rib path starting from i (resp. ending at i) and $i\varrho i'$ then there is a unique rib path $\beta'_1 \ldots \beta'_s$ starting from i' (resp. ending at i') such that $\beta_j \varrho \beta'_j$ for all j.*

(b) *For every $\mathfrak{R}_t$ in the decomposition* (4.10) *there is a unique $\mathfrak{R}_u$, $u \neq t$, such that there is a poset isomorphism σ: $\mathfrak{R}_t \to \mathfrak{R}_u$ with the property*

$$\text{If } \tau(t, j) = 2, \text{ then } (t, j)\varrho(\sigma(t), \sigma(j)). \tag{4.12}$$

(*We shall write $\mathfrak{R}'_t$ instead of $\mathfrak{R}_u$ and i' instead of $\sigma(i)$ for $i \in \mathfrak{R}_t$.*)

(c) *If a primitively bounded poset $I_{\varrho E}$ is of finite type (resp. of tame type) then the posets $\mathfrak{R}_1, \ldots, \mathfrak{R}_h$ are linearly ordered (resp. are of width $\leqslant 2$).*

Proof. (a) is left to the reader.

(b) We proceed by induction on the cardinality of $J = \mathbf{rsk}(I, \varrho)$. Choose a minimal element i in $\mathfrak{R}_t$ and $i' \neq i$ such that $i\varrho i'$. We claim that if $i' \in \mathfrak{R}_u$ then $u \neq t$ and there is a poset isomorphism σ: $\mathfrak{R}_t \to \mathfrak{R}_u$ satisfying (4.12) and such that $\sigma(i) = i'$. By our assumption there is a rib (i, j) in $\mathfrak{R}_t$. It follows from (a) that i' is minimal in $\mathfrak{R}_u$ and there is a unique rib (i', j') in $\mathfrak{R}_u$ such that $(i, j)\varrho(i', j')$. Consider the poset $J' = J - \{i\}$. If j is minimal in the component $\mathfrak{R}'_t = \mathfrak{R}_t - \{i\}$ of J' then by the inductive assumption there is a unique poset isomorphism $\sigma'\colon \mathfrak{R}'_t \to \mathfrak{R}'_u = \mathfrak{R}_u - \{i'\}$ satisfying (4.12) and such that $\sigma'(j) = j'$. If j is not minimal in $\mathfrak{R}'_t$ we can choose $k \in \mathfrak{R}'_t$ minimal and such that there is a rib path $\beta_1 \ldots \beta_s$ from k to j. By (a) there is a unique rib path $\beta'_1 \ldots \beta'_s$ from some k' to j' (in $\mathfrak{R}'_u$) such that $\beta_r \varrho \beta'_r$ for all r. Now again, by the inductive assumption, there is a unique poset isomorphism $\sigma'\colon \mathfrak{R}'_t \to \mathfrak{R}'_u$ satisfying (4.12) and such that $\sigma'(k) = k'$. It follows that $\sigma'(j) = j'$ like in the first case. Now we extend σ' to the required poset isomorphism $\sigma\colon \mathfrak{R}_t \to \mathfrak{R}_u$ by putting $\sigma(i) = i'$, and (b) follows.

(c) Let $i_1, \ldots, i_s$ be pairwise incomparable elements in $\mathfrak{R}_t$ with respect to the partial order $\prec$. It follows from (a) that $\mathfrak{R}'_t = \mathfrak{R}_u$ contains pairwise incomparable elements $i'_1, \ldots, i'_s$ such that $i_r \varrho i'_r$. Moreover, $\mathfrak{R}_t \cap \mathfrak{R}'_t$ is empty. Consider the subposet $K = \{i_1, \ldots, i_s, i'_1, \ldots, i'_s\}$ of I with the stratification induced by ϱ. Then K_ϱ is simply stratified and we consider the narrow ring extension $R_1 = FK^*_\varrho \subseteq R'_1 = FK^*$ (in the sense of [21]) as well as the corresponding pairs of adjoint functors

$$\operatorname{mod}_{sp}(R)/[P_*] \cong \operatorname{adj}(R) \underset{L_K}{\overset{r}{\rightleftarrows}} \operatorname{adj}(R_1) \underset{\mathfrak{z}}{\overset{\mathfrak{J}}{\rightleftarrows}} \operatorname{adj}(R'_1) \cong K\text{-sp}/[P_*],$$

where P_* is the unique simple right ideal in R_1 and in R, r is the restriction, L_K is a fully faithful embedding [17; 1.14], [20; 2.22], $\mathfrak{z}$ is the forgetful functor and $\mathfrak{J}$ is the blowing-up functor [21; 3.2]. Let $\mathscr{C} = \{P, P'\}$, where $P = P_{i_1} + \ldots + P_{i_s}$, $P' = P_{i'_1} + \ldots + P_{i'_s} \subseteq E(P_*)$ are indecomposable R'_1-modules with $P_r = e_r R'_1$. Let $\operatorname{adj}(R_1 \nearrow \mathscr{C})$ be the full subcategory of $\operatorname{adj}(R_1)$ consisting of modules X such that $\mathfrak{J}(X)$ is a direct sum of copies of P and P'. Then the adjustment functor $\operatorname{ad}^{\mathscr{C}}_{\mathfrak{a}}\colon \operatorname{adj}(R_1 \nearrow \mathscr{C}) \to \operatorname{adj}(T)$ with respect to $\mathfrak{a} = J(FK)$ is a representation equivalence, where T is a hereditary F-algebra of the type shown in Fig. 10 (see [21; 3.27 and 5.17]). Since obviously T is of finite type iff $s = 1$, and T is of tame type if $s = 2$, (c) follows and the proof is complete.

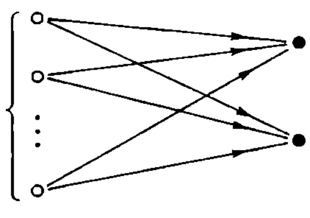


Fig. 10

In [12] Nazarova and Roiter study the case when $\boldsymbol{h}(I_\varrho) = 2$, $\mathbf{rsk}(I_\varrho) = \mathfrak{R}_1 \cup \mathfrak{R}'_1$ and $\mathfrak{R}_1$ is linearly ordered. In the next section we shall show how the covering technique works in this case. Before we do it let us consider a special situtation in a more general case when $\boldsymbol{h}(I_\varrho) = 2$ and the width $\boldsymbol{w}(\mathfrak{R}_1)$ of $\mathfrak{R}_1$ is two.

Proposition 4.13. *Let I_ϱ be a stratified poset such that $\mathbf{rsk}(I_\varrho) = I_\varrho$, ϱ is* *multiplicative,* $\mathbf{h}(I_\varrho) = 2$, $\mathbf{rsk}(I_\varrho) = \mathfrak{R}_1 \cup \mathfrak{R}'_1$, $\mathbf{w}(\mathfrak{R}_1) = 2$ *and* $p \prec q$ *for all* $p \in \mathfrak{R}_1$ *and* $q \in \mathfrak{R}'_1$. *Then the following conditions are equivalent*:

(a) $I_{\varrho E} = (I, \varrho, E)$ *is of tame type, where* $E = E(I, \varrho)$.
(b) I *does not contain as a full subposet the poset* $(1, 2) = \{\bullet, \bullet\!\!\to\!\!\bullet\}$.
(c) $\mathfrak{R}_1$ *is a full subposet of a garland* $\mathfrak{G}$ (see Fig. 11).

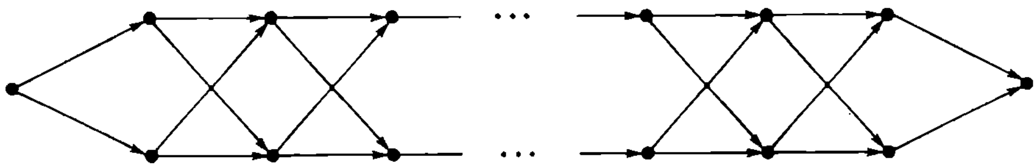


Fig. 11

Proof. (b)$\Leftrightarrow$(c) is obvious.

(a)$\Rightarrow$(b). Assume to the contrary that $(1, 2)$ is contained in I. Without loss of generality we can suppose that $\mathfrak{R}_1 = \mathfrak{R}'_1 = \{\bullet, \bullet\!\!\to\!\!\bullet\}$. It follows from Corollary 4.6 that $F(I^*_{\varrho E})$ is the path algebra of the quiver $(Q_{I_\varrho}, \Omega_{I_\varrho})$, where Q_{I_ϱ} is shown in Fig. 12 and

$$\Omega_{I_\varrho}\colon \quad \begin{cases} c_i a c_j = c_i c_j = 0, \\ c_3 \eta = c_4 a \xi, \\ c_2 \eta = c_1 a \xi. \end{cases}$$

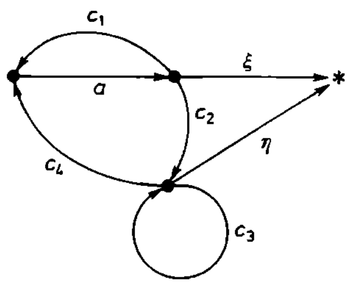


Fig. 12

It is easy to see that the quiver $\tilde{Q}$ of Fig. 13 with the relations Ω_{I_ϱ} above is a universal cover of $(Q_{I_\varrho}, \Omega_{I_\varrho})$ with infinite cyclic group. One can check that every indecomposable finite-dimensional socle projective representation of $(\tilde{Q}, \Omega_{I_\varrho})$ has a support of the form (D, Ω_{I_ϱ}) because $\operatorname{rad} P_2 \cong P_* \oplus E$, $\operatorname{rad} P_3 \cong E$, where E is the injective envelope of a simple projective representation. Furthermore, $\operatorname{mod}_{\mathrm{sp}}(D, \Omega_{I_\varrho})$ has a cofinite subcategory equivalent to the socle projective representations of the quiver of Fig. 14 (see Lemma 4.14 below) and therefore it is of wild type. Consequently, $\operatorname{mod}_{\mathrm{sp}}(\tilde{Q}, \Omega_{I_\varrho})$ is of wild type and according to Theorem 0.2, $\operatorname{mod}_{\mathrm{sp}}(FI^*_{\varrho E})$ is of wild type, which is a contradiction.

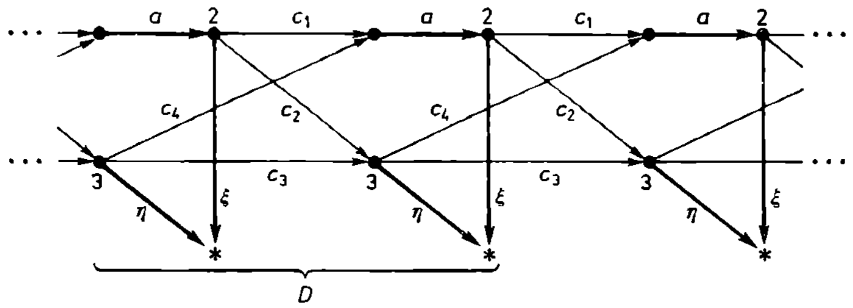


Fig. 13

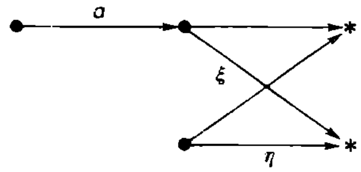


Fig. 14

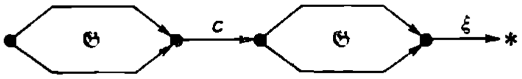


Fig. 15

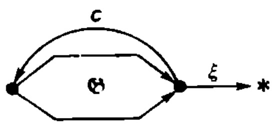


Fig. 16

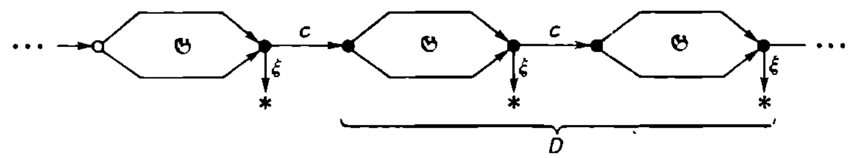


Fig. 17

(c)$\Rightarrow$(a). Without loss of generality we can suppose that $\mathfrak{R}_1 = \mathfrak{R}'_1 = \mathfrak{G}$ and σ: $\mathfrak{R}_1 \to \mathfrak{R}'_1$ in (4.11.b) is the identity map. Then I^* has the form as in Fig. 15 and according to Corollary 4.6 there is an algebra isomorphism $FI^*_{\varrho E} \cong F(Q, \Omega)$, where Q is shown in Fig. 16 and $\Omega = \Omega_{I_e}$ consists of the commutativity relations (4.1) induced by those in I^* and the zero relations (4.3). It is easy to check that the quiver $\tilde{Q}$ of Fig. 17 with relations Ω above is a universal cover of (Q, Ω). By Lemma 4.14 below indecomposables in

$\mathrm{mod}_{sp} F(\tilde{Q}, \Omega)$ have supports of the form (D, Ω) and $\mathrm{mod}_{sp} F(D, \Omega)$ has a cofinal subcategory equivalent to $\mathrm{mod}_{sp}(FU)$, where U is the poset of Fig. 18

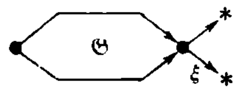


Fig. 18

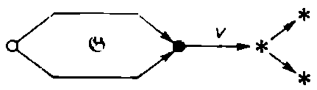


Fig. 19

with two maximal elements. If $\hat{U}$ is the poset of Fig. 19 then according to [20; Prop. 2.24] and [22; Theorem 4.12] we have the diagram

$$\begin{array}{ll} \mathrm{mod}_{sp}(FU) \xhookrightarrow{TL} \mathrm{mod}_{ic}(F\hat{U})_B & \\ \qquad\qquad\qquad\quad \downarrow{\scriptstyle \Theta_B} & \\ \qquad\qquad\qquad \mathrm{adj}_B^{FG}(F\hat{U}) \xrightarrow[\simeq]{\delta_v} \mathrm{mod}_{sp}(FN^*) \cong N\text{-sp}, & \end{array}$$

where $B = F(\hat{U} - \mathfrak{G})$, $N = (\mathfrak{G} - \{\bullet\}) \cup \{\bullet, \bullet\}$ and Θ_B is a full dense functor vanishing only on a finite set of indecomposables [20]. Since according to [7], N is tame, it follows that $\mathrm{mod}_{sp}(FU)$ is tame and according to Theorem 0.2, $\mathrm{mod}_{sp}(FI^*_{QE})$ is tame as required.

Let us finish this section by the following multipeak splitting lemma (cf. [18; Section 8]) which we use in the next section.

LEMMA 4.14. *Let*

$$R = \begin{bmatrix} A & {}_AM_B & 0 \\ 0 & B & {}_BN_C \\ 0 & 0 & C \end{bmatrix}$$

be a right multipeak artinian ring. Suppose that $M_B \cong E_B(P_*)^t$, *where* $t > 0$ *and* $E_B(P_*)$ *is the* B*-injective envelope of a simple right ideal* $P_* = e_* B$. *Then every indecomposable module in* $\mathrm{mod}_{sp}(R)$ *belongs to the image of one of the following fully faithful functors*:

$$\mathrm{mod}_{sp}\begin{bmatrix} A & M' \\ 0 & F \end{bmatrix} \xrightarrow{L} \mathrm{mod}_{sp}(R) \xleftarrow{T} \mathrm{mod}_{sp}(S),$$

where $S = [\begin{smallmatrix} B & N \\ 0 & C \end{smallmatrix}]$, $M' = Me_*$, $F = e_* B e_*$, $T(Y_S) = (0, Y_S)$ *and* $L(X_A, F', f) = (X_A, E_B(P_*)^t, f)$ [16; p. 539].

Proof. Let $X_R = (X'_A, X''_S, g)$ be an indecomposable module in $\mathrm{mod}_{sp}(R)$ with $X'_A \neq 0$. Note that according to our assumption the right S-module

$(M_B, 0)$ is isomorphic to $E_S(P_*)^t$. Hence if the A-homomorphism

$$\bar{g}\colon X'_A \to \mathrm{Hom}_S((M, 0)_S, X''_S) \cong \mathrm{Hom}_S(E_S(P_*)^t, X''_S)$$

adjoint to g is nonzero then X''_S being socle projective is isomorphic to a direct sum of copies of $E_S(P_*)$ and therefore X_R belongs to the image of $\boldsymbol{L}$. If $\bar{g}$ is zero then X''_S is zero and again X_R belongs to the image of $\boldsymbol{L}$. Thus the lemma is proved.

5. On bipartite completed posets and coverings

We are going to show how the covering technique can be applied in the study of representations of bipartite completed posets in the sense of [12].

In our terminology a *bipartite completed poset* in the sense of [12] is a primitively bounded stratified poset

$$\tilde{I} = (I, \varrho, E), \qquad E = E(I, \varrho),$$

such that I is the disjoint union of two subposets P, P' and the following conditions are satisfied:

(B1) $\boldsymbol{h}(I, \varrho) = 2$ and $\mathbf{rsk}(\tilde{I}) = \mathring{P} \cup \mathring{P}'$, where $\mathring{P} = \{s_1 \prec \ldots \prec s_m\} = \mathfrak{R}_1 \subseteq P$ and $\mathring{P}' = \{s'_1 \prec \ldots \prec s'_m\} = \mathfrak{R}_2 \subseteq P'$ are linearly ordered full subposets of I such that $[s_i, s_j] = \{s_i, s_{i+1}, \ldots, s_j\}$ and $[s'_i, s'_j] = \{s'_i, s'_{i+1}, \ldots, s'_j\}$ whenever $(s_i, s_j)\varrho(s'_i, s'_j)$.

(B2) $(s_1, s_2)\varrho(s'_1, s'_2), \ldots, (s_{m-1}, s_m)\varrho(s'_{m-1}, s'_m)$ (usually there are more relations in ΔI). The relations $s_1\varrho s'_1, \ldots, s_m\varrho s'_m$ are the only nontrivial ϱ-relations in I.

(B3) $p \prec q$ for all $p \in P$ and $q \in P'$.

Given $j \leqslant m$ we put $\mu(j) = t$ if $(s_j, s_t)\varrho(s'_j, s'_t)$ and there is no relation $(s_j, s_{t+1})\varrho(s'_j, s'_{t+1})$.

It is easy to see that the category $\mathrm{rep}(\tilde{I})$ defined in [12] is equivalent to $\mathscr{V}(\boldsymbol{K}(\tilde{I})_F)$, where $\boldsymbol{K}(\tilde{I})_F$ is the vector space category (2.14) of $\tilde{I}$. It follows from Corollary 3.8 that the representation types of $\tilde{I}$ and of $\mathrm{mod}_{\mathrm{sp}}(F\tilde{I}^*)$ coincide. In order to determine that type we are going to find a suitable bound quiver (Q, Ω) of the algebra $F\tilde{I}^*$ and calculate its universal covering $(\tilde{Q}, \tilde{\Omega}) \to (Q, \Omega)$. We shall show that the fundamental group of (Q, Ω) [6] is infinite cyclic, the support of any indecomposable module in $\mathrm{mod}_{\mathrm{sp}}(F(\tilde{Q}, \tilde{\Omega}))$ is a finite 2-peak bound quiver $\tilde{I}^{++}$ and the categories $\mathrm{mod}_{\mathrm{sp}}(F\tilde{I}^{++})$ and $\mathrm{mod}_{\mathrm{sp}}(F\tilde{I}^*)$ have the same representation type.

Let us introduce some notation. A pair (i, j) in ΔI will be called *short* (and denoted by β_{ij}) if $[i, j] = \{i, j\}$. We call $(i, j) \in \Delta I$ *ϱ-extremal* (and denote it by a_{ij}) if $\tau(i, j) = 1$, whereas $\tau(i, r) \geqslant 2$ and $\tau(r, j) \geqslant 2$ for all $r \in I$ such that $i \prec r \prec j$.

DEFINITION 5.1. *The bound quiver* (Q, Ω) *of a bipartite completed poset* $\tilde{I}$ is defined as follows. The set of vertices of Q is the set

$$Q_0 = \{\bar{1}, \ldots, \bar{n}, \bar{*}\} = \{I_1, \ldots, I_a, \bar{*}\}$$

of ϱ-cosets $\bar{i}$ of elements $i \in I^*$. We write $\bar{i} = i$ if $\tau(i) = 1$. If $i \prec j$ and (i, j) is short then the ϱ-coset $\bar{\beta}_{ij}\colon \bar{i} \to \bar{j}$ of $\beta_{ij} = (i, j)$ is a unique edge from $\bar{i}$ to $\bar{j}$. We write $\bar{\beta}_{ij} = \beta_{ij}$ if $\tau(i, j) = 1$. If $i \prec j$, $i, j \in \mathring{P}$ and (i, j) is ϱ-extremal then $a_{ij} = (i, j)\colon \bar{i} \to \bar{j}$ is a unique edge from $\bar{i}$ to $\bar{j}$. There are no more edges in Q_1. We put $a_{s_i} = \bar{\beta}_{s_i s_{i+1}}\colon \bar{s}_i \to \bar{s}_{i+1}$, for $s_i \in \mathring{P}$.

The set Ω consists of the following relations:

(a) $\bar{\beta}_{ii_1}\bar{\beta}_{i_1 i_2}\ldots\bar{\beta}_{i_r j} - \bar{\beta}_{ij_1}\bar{\beta}_{j_1 j_2}\ldots\bar{\beta}_{j_s j}$ if $i \prec i_1 \prec \ldots \prec i_r \prec j, i \prec j_1 \prec \ldots \prec j_s \prec j$ and neither $\{i, i_1, \ldots, i_r, j\}$ nor $\{i, j_1, \ldots, j_s, j\}$ is contained in one of the sets $\mathring{P}$, $\mathring{P}'$.

(b) $\bar{\beta}_{i_1 j_1}\ldots\bar{\beta}_{i_r j_r}$ if there is no sequence $\beta_{t_0 t_1}, \ldots, \beta_{t_{r-1} t_r}$ such that $\beta_{i_p j_p} \varrho \beta_{t_{p-1} t_p}$ for $p = 1, \ldots, r$.

(c) $a_{st} a_t a_{t+1} \ldots a_{r-1} \beta_{rj} - a_s a_{s+1} \ldots a_{r-1} \beta_{rj}$, $s, t, r \in \mathring{P}$, $j \notin \mathring{P}$;
$a_{st} a_t a_{t+1} \ldots a_{r-1} a_{rq} - a_s a_{s+1} \ldots a_{r-1} a_{rq}$;
$a_{st} \beta_{tj} - a_s a_{s+1} \ldots a_{t-1} \beta_{tj}$, $t, s \in \mathring{P}$, $j \notin \mathring{P}$;
$a_{st} a_{tq} - a_s a_{s+1} \ldots a_{t-1} a_{tq}$;
$a_{st} a_t a_{t+1} \ldots a_{r-1} \beta_{r'j}$, $r' \in \mathring{P}'$, $s, t, r \in \mathring{P}$, $j \notin \mathring{P}'$;
$a_{st} \beta_{t'j}$, $t' \in \mathring{P}'$, $s, t \in \mathring{P}$, $j \notin \mathring{P}'$.

(d) $\beta_{jt} a_t a_{t+1} \ldots a_{r-1} a_{rq} - \beta_{jt} a_t a_{t+1} \ldots a_{q-1}$, $t, r, q \in \mathring{P}$, $j \notin \mathring{P}$;
$\beta_{jt} a_{tq} - \beta_{jt} a_t a_{t+1} \ldots a_{q-1}$, $t, q \in \mathring{P}$, $j \notin \mathring{P}$;
$\beta_{jt'} a_t a_{t+1} \ldots a_{r-1} a_{rq}$, $t' \in \mathring{P}'$, $t, r, q \in \mathring{P}$, $j \notin \mathring{P}'$;
$\beta_{jt'} a_{tq}$, $t' \in \mathring{P}'$, $t, q \in \mathring{P}$, $j \notin \mathring{P}'$.

(e) $a_{st} a_t a_{t+1} \ldots a_{r-1} - a_s a_{s+1} \ldots a_{q-1} a_{qr}$ for all a_{st}, a_{qr} with $t, q \in [s, r]$.

Here we write indices $k+i$, $k-i$ instead of s_{k+i}, s_{k-i} for $k = t, r, s, q$.

Consider the F-algebra homomorphism

$$g\colon FQ \to F\tilde{I} = FI^*_{\varrho E} \tag{5.2}$$

given by

$$g(j) = e_j + e_{j'} \quad \text{for } j \in \mathring{P}, \qquad g(\beta_{ij}) = e_{ij} \quad \text{if } \tau(i,j) = 1,$$
$$= e_j \quad \text{otherwise}; \qquad g(a_{s_j}) = e_{s_j s_{j+1}} + e_{s'_j s'_{j+1}} \quad \text{if } s_j \in \mathring{P},$$
$$g(a_{ij}) = e_{ij},$$

where $e_1, \ldots, e_n, e_*$ are the standard matrix idempotents in FI^* and $e_{ij} \in FI^*$ is the matrix with 1 in the (i, j) place and zeros elsewhere.

It is easy to check that g is surjective and $g(\Omega) = 0$. Since a straightforward calculation shows that $\dim F(Q, \Omega) \leqslant \dim F\tilde{I}^*$, we get

LEMMA 5.3. *If $\tilde{I} = I_{gE}$ is a bipartite completed poset with $I = P \cup P'$ as above and if (Q, Ω) is a bound quiver of $\tilde{I}$ then g induces an algebra isomorphism $F(Q, \Omega) \cong F\tilde{I}^*$.*

Now to any bipartite completed poset $\tilde{I}$ we associate a *two-peak bound quiver*

$$\tilde{I}^{++} = (Q^{++}, \Omega^{++}) \tag{5.4}$$

as follows. Denote by $p_1, \ldots, p_r$ and $q_1, \ldots, q_s$ the maximal elements in P and minimal in P' respectively, and put $c_{ij} = \bar{\beta}_{p_i q_j}$. Let (Q, Ω) be the bound quiver of $\tilde{I}$. First we construct a new bound quiver (Q^+, Ω^+) by removing in (Q, Ω) the edges c_{ij}, $i = 1, \ldots, r$, $j = 1, \ldots, s$. Next we form $Q_0^{++} = Q_0^+ \cup \{+\}$ by attaching to $Q_0^+ = Q_0$ a new vertex $+$. Further we form Q_1^{++} by enlarging Q_1^+ by new edges $\beta_{p_j+}\colon \bar{p}_j \to +$, $j = 1, \ldots, r$. Finally, we take for Ω^{++} the set Ω^+ enlarged by the following two groups of relations:

1° All commutativity relations ending with $+$.

2° The relations (c)–(e) in Definition 5.1 involving also the edges β_{p_j+}, $j = 1, \ldots, r$.

It is clear that Q^{++} has no oriented cycles, has two maximal vertices $*$ and $+$, and $F(\tilde{I}^{++})$ is a right two-peak algebra [17].

Note that in Example 4.7(c) the bound quiver $\tilde{I}^{++}$ has the form shown in Fig. 20 with $dc_1 = \bar{\alpha}\bar{\beta}cc_3$, $bc_2 = cc_3$, $ef = \bar{\beta}\xi$, $ae = 0$, and in Example 4.7(d) it has the form of Fig. 21 with $u\xi = 0$, $uc = \bar{a}\bar{b}c$.

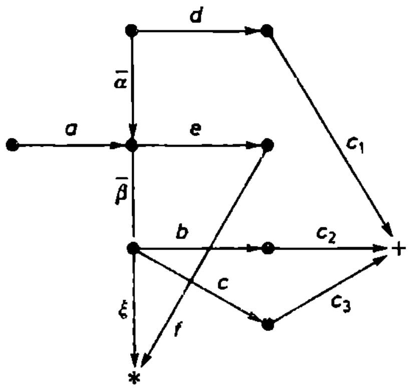


Fig. 20

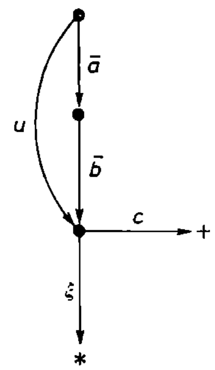


Fig. 21

Now we are able to prove the main result of this section.

THEOREM 5.5. *Let $\tilde{I}$ be a bipartite completed poset and let F be an algebraically closed field. Then $F\tilde{I}^{++}$ is a right two-peak algebra and there is an additive functor*

$$\tilde{f}_\lambda\colon \operatorname{mod}_{sp}(F\tilde{I}^{++}) \to \operatorname{mod}_{sp}(F\tilde{I}^*)$$

which establishes a one-to-one correspondence between the nonsimple indecom-

posable modules in $\operatorname{mod}_{\mathrm{sp}}(F\tilde{I}^{++})$ *and the nonsimple indecomposable modules in* $\operatorname{mod}_{\mathrm{sp}}(F\tilde{I}^{*})$. *The category* $\operatorname{mod}_{\mathrm{sp}}(F\tilde{I}^{++})$ *is of tame (resp. wild) type if and only if* $\operatorname{mod}_{\mathrm{sp}}(F\tilde{I}^{*})$ *is of tame (resp. wild) type.*

In order to prove the theorem we construct the universal cover $(\tilde{Q}, \tilde{\Omega})$ [6] of the bound quiver (Q, Ω) of $\tilde{I}$. It has the form

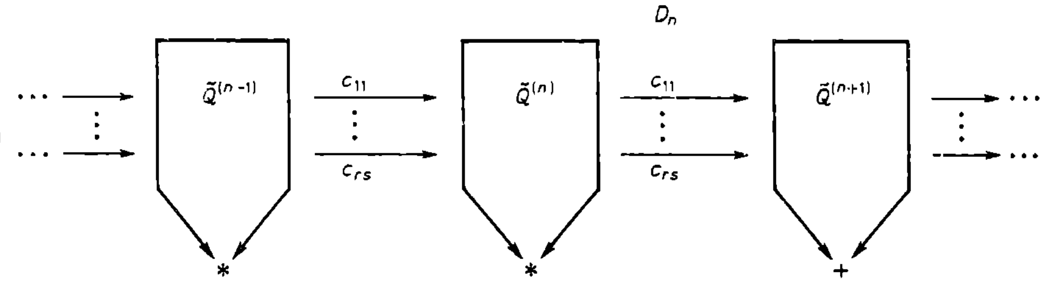
(5.6) 

Fig. 22

where $\tilde{Q}^{(n)} = Q^{+}$ for all $n \in \mathbf{Z}$, $c_{ji} = \beta_{p_j q_i}$: $\bar{p}_j^{(n)} \to \bar{q}_i^{(n+1)}$ and $\bar{p}_j^{(t)}$, $\bar{q}_i^{(t)}$ denote the vertices $\bar{p}_j$ and $\bar{q}_i$ in the quiver $\tilde{Q}^{(t)} = Q^{+}$. We take for $\tilde{\Omega}$ the set Ω. Since $I = P \cup P'$ is a splitting decomposition, the radical of any indecomposable projective P_j in $F(\tilde{Q}, \tilde{\Omega})$, $j = \bar{p}_1^{(n)}, \ldots, \bar{p}_r^{(n)}$, is isomorphic to the injective envelope $E(P_{+})$ of the simple projective P_{+} corresponding to the peak $+$ in $\tilde{Q}^{(n+1)}$. Moreover, it is easy to see that the full bound subquiver of $(\tilde{Q}, \tilde{\Omega})$ with the set of vertices $\tilde{Q}^{(n)} \cup \{+\}$ is isomorphic to $\tilde{I}^{++}$. Now by applying Lemma 4.14 we conclude that every indecomposable module in $\operatorname{mod}_{\mathrm{sp}} F(\tilde{Q}, \tilde{\Omega})$ is isomorphic up to shift to a module in the image of the natural fully faithful embedding [17; (1.14)]

$$T\colon \operatorname{mod}_{\mathrm{sp}}(F\tilde{I}^{++}) \to \operatorname{mod}_{\mathrm{sp}} F(\tilde{Q}, \tilde{\Omega}).$$

Thus we are in the situation of Theorem 0.2 and therefore Theorem 5.5 follows if we take for $\tilde{f}_{\lambda}$ the composed functor $f_{\lambda} T$, because we know from Lemma 5.3 that $F(Q, \Omega) \cong F\tilde{I}^{*}$.

The discussion above also yields

PROPOSITION 5.7. *Let* $\tilde{I}$ *be a primitively bounded stratified poset with* $I = P \cup P'$. *Suppose that* $\tilde{I}$ *satisfies the conditions* **(B1)**, **(B2)** *and the following one*:

(B3)$'$ *There is no relation* $q \prec p$ *with* $q \in P'$ *and* $p \in P$.

If $(\tilde{Q}, \tilde{\Omega})$ *is the bound quiver of* $\tilde{I}$ *then* $F\tilde{I}^{*} \cong F(Q, \Omega)$ *and the bound quiver* $(\tilde{Q}, \tilde{\Omega})$ *presented in* (5.6) *is a cover of* (Q, Ω) *with infinite cyclic group acting by shifts.*

If in addition the condition **(B3)** *is satisfied then every indecomposable module in* $\operatorname{mod}_{\mathrm{sp}} F(\tilde{Q}, \tilde{\Omega})$ *has a support of the form* $(\tilde{Q}^{(n)} \cup \{+\}, \tilde{\Omega})$ *which is* *isomorphic to $\tilde{I}^{++}$. The bound quiver algebra $F\tilde{I}^{++}$ is a simply connected right two-peak algebra.*

One of the main consequences of Theorem 5.5 is the following alternative solution of a problem solved in [12].

COROLLARY 5.8. *A bipartite completed poset $\tilde{I}$ is of finite type if and only if $\tilde{I}^{++}$ does not contain the critical quivers of Weichert [25]. Moreover, $\tilde{I}$ is faithful if and only if $F\tilde{I}^{++}$ is sp-sincere.*

Proof. Since $\tilde{I}^{++}$ is simply connected and $F\tilde{I}^{++}$ has the separation property for the radicals of indecomposable projective right ideals, $\operatorname{mod}_{sp}(F\tilde{I}^{++})$ has a preprojective component as well as a preinjective component (cf. [18; Section 11]). Then in view of Theorem 5.5, Lemma 5.3 and the results in [25] the first part of the corollary follows. The remaining part follows from the fact that the restrictions of $\mathbf{cdn}\, \tilde{f}_\lambda(Y)$ and $\mathbf{cdn}(Y)$ to the nonpeak parts coincide (see (3.0)).

Remarks. 5.9. If $F\tilde{I}^{++}$ is sp-representation-finite then similarly to [18; Section 11] we can construct all indecomposable modules in $\operatorname{mod}_{sp}(F\tilde{I}^{++})$ starting from hereditary projective modules and the radicals of indecomposable projective modules. Since the functor $\tilde{f}_\lambda$ in Theorem 5.5 is induced by the push-down functor (0.1), we get a constructive procedure for determining indecomposable modules in $\operatorname{mod}_{sp}(F\tilde{I}^*)$ as well as the Auslander–Reiten quiver of $\operatorname{mod}_{sp}(F\tilde{I}^*)$.

Note also that in the case where the relation ϱ in $\tilde{I}$ is multiplicative $\tilde{I}^{++}$ is a poset with zero relations and $\tilde{I}^{++}$ has exactly two maximal elements. Therefore indecomposable modules in $\operatorname{mod}_{sp}(F\tilde{I}^{++})$ can be determined like in the proof of Proposition 4.13 by the method presented in [22; 3.10 and 5.2].

5.10. It is not difficult to check that if (Q, Ω) is the bound quiver of a bipartite completed poset $\tilde{I}$ then $F\tilde{I}^* \cong F(Q, \Omega)$ is isomorphic to the trivial extension $B \ltimes {}_BN_B$, where $B = F(Q^+, \Omega^+)$ and ${}_BN_B$ is the B-B-bimodule "generated" by the edges $\beta_{p_i q_j}$, $i = 1, \dots, r$, $j = 1, \dots, s$, in Q (see (5.4)). Furthermore, the repetitive infinite-dimensional algebra (cf. [24, p. 321])

$$\mathbf{r}(B, N) = \begin{bmatrix} \ddots & \ddots & & 0 & \\ & B_{-1} & N & & \\ & & B_0 & N & \\ & & & B_1 & \ddots \\ & 0 & & & \ddots \end{bmatrix}$$

with $B_n = B$ for $n = 0, \pm 1, \pm 2, \dots$, is isomorphic to the algebra $F(\tilde{Q}, \tilde{\Omega})$, where $(\tilde{Q}, \tilde{\Omega})$ is the universal cover (5.6) of (Q, Ω).

5.11. Suppose that $I_{\varrho E}$ is a primitively bounded poset and suppose that $R = FI^*_{\varrho E}$ is an sp-representation-finite bound quiver algebra $F(Q, \Omega)$ of some

(Q, Ω). If Γ_R^{sp} denotes the Auslander–Reiten translation quiver of $\mathrm{mod}_{\mathrm{sp}}(R)$ then similarly to [5] one can define a group epimorphism

$$\varkappa\colon \Pi(Q, \Omega) \to \Pi(\Gamma_R^{\mathrm{sp}})$$

from the fundamental group of (Q, Ω) to the fundamental group of Γ_R^{sp}. It would be interesting to determine the group $\operatorname{Ker}\varkappa$ of sp-constraints.

Note that one can prove socle projective analogues of the results in [6; Sections 3 and 4].

5.12. It would be interesting to give necessary and sufficient conditions for a bound quiver (Q, Ω) to be the bound quiver of a primitively bounded stratified poset $I_{\varrho E}$ as well as a simple construction of $I_{\varrho E}$ in terms of (Q, Ω)). Moreover, it would be interesting to give a characterization of vector space categories $\boldsymbol{K}_F$ equivalent to vector space categories of the form $\boldsymbol{K}(I_{\varrho\omega})$, where $I_{\varrho\omega}$ is a bounded stratified poset (see (2.14)).

Recall that $\boldsymbol{K}_F$ is equivalent to the vector space category $\boldsymbol{K}(I)_F$ of a finite poset I (nonstratified) if and only if $\boldsymbol{K}$ has a finite number of isoclasses of indecomposable objects $X_1, \ldots, X_n$, $F_j = \boldsymbol{K}(X_j, X_j)$ is a division ring and $\dim_{F_j}|X_j| = \dim|X_j|_F = 1$ for $j = 1, \ldots, n$ [23; Corollary 2.4].

5.13. It would be interesting to characterize all bound matrices A of a stratified poset I_ϱ such that the algebras $FI_{\varrho A}^*$ and $FI_{\varrho E}^*$ are isomorphic.

5.14. If I_ϱ is a simply stratified poset such that I-sp is of finite type then the overring adjustment method [21; Remark 5.17] allows us to determine indecomposable modules in $\mathrm{mod}_{\mathrm{sp}}(FI_{\varrho E}^*)$ and the Auslander–Reiten quiver of $\mathrm{mod}_{\mathrm{sp}}(FI_{\varrho E}^*)$. The case $w(I) = 2$ is studied in detail in [21; Theorem 5.14] by reducing the problem to a special biserial case. This shows that sometimes the application of the overring adjustment is easier than the covering technique.

5.15. Let $\tilde{I}$ be a bipartite completed poset and let $(\tilde{Q}, \tilde{\Omega})$ be the cover (5.6) of the bound quiver (Q, Ω) of $\tilde{I}$. Let $R = F\tilde{I} \cong F(Q, \Omega)$, $\tilde{R} = F(\tilde{Q}, \tilde{\Omega})$ and $B = F\tilde{I}^{++}$, where $\tilde{I}^{++}$ is the bound quiver (5.4). If follows from the proof of Lemma 4.14 that $\Gamma_{\tilde{R}}^{\mathrm{sp}}$ has the form of Fig. 23 (cf. [24; Remark 1] where Γ_n^{sp} is

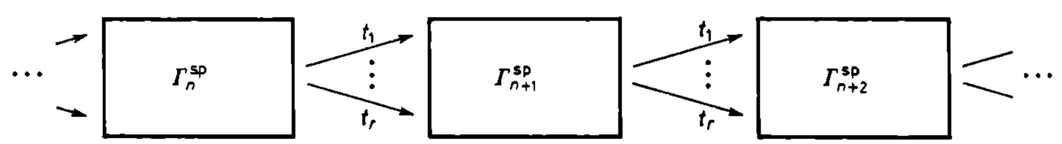


Fig. 23

obtained from Γ_B^{sp} by removing the vertex $e_* B$, and t_j is a unique edge from the injective envelope $E(e_+ B)$ of $e_+ B$ to $e_{p_j} B$ for $j = 1, \ldots, r$ (see (5.4)). Hence in view of Theorem 5.5, Proposition 5.7 and Theorem 0.2 we conclude that

$$\Gamma_R^{\mathrm{sp}} \cong \Gamma_{\tilde{R}}^{\mathrm{sp}}/\mathbf{Z} \cong \Gamma_B^{\mathrm{sp}}/\mathbf{Z} \tag{5.16}$$

is obtained from Γ_B^{sp} by identifying the modules $e_* B$ and $E(e_+ B)$.

Recall from the proof of Corollary 5.8 that Γ_B^{sp} has a preprojective

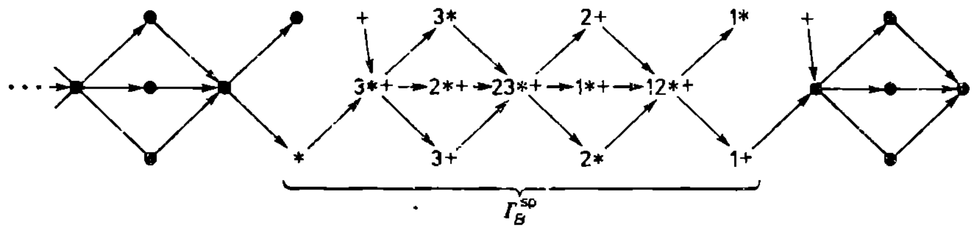


Fig. 24

component which can be simply constructed as for posets [18; Section 11] and coincides with Γ_B^{sp} in the case where B is sp-representation-finite. This gives a simple procedure for constructing Γ_R^{sp}.

Let us describe it for $\tilde{I}$ in Example 4.7 (d). If we denote by 1, 2, 3 the thick points in Fig. 21 then Γ_R^{sp} is shown in Fig. 24 where instead of a module X we write the nonzero coordinates of $\mathbf{cdn}(X)$ in the form $1^{t_1}2^{t_2}3^{t_3}*^{t_*}+^{t_+}$ (see (3.0)).

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Added in proof (February 1990). 1. During the preparation of this paper we used the preliminary Russian version of [12], where completed posets were defined as in [9–11] and as in our Definition 1.6. The published version of [12] contains a new definition of a completed poset, which is different from the previous one and close to our Definition 1.2 of stratified posets.

2. The results of Sections 4 and 5 are developed by the author in the paper *On the representation type of stratified posets*, to appear in C. R. Acad. Sci. Paris, 1990. The concept of a bipartite poset is there generalized and a characterization of bipartite posets of finite type is given in terms of an associated integral quadratic form; also, a list of 48 minimal bipartite posets of infinite type is presented.