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LOCAL METHOD OF NONLINEAR ANALYSIS

A. D. BRUNO

Institute of Applied Mathematics,
Moscow, U.S.S.R.

0. Introduction

In three lectures I wish to explain the principal ideas of the local method. A more detailed exposition of the method is to be found in my book [5]. Publishing house "Springer-Verlag" will soon publish an English translation of it.

I shall explain the principal ideas of the local method taking for a single example the following problem: to study solutions of a system of ordinary differential equations

$$dX/dt = \Phi(X) \tag{1}$$

in a neighbourhood $\mathcal{U}$ of a fixed point $X = (x_1, \ldots, x_n) = 0$. But the method is useful in many other problems (see [5]–[9]). Here X and Φ are complex (in C^n) or real (in R^n). Functions $(\varphi_1(X), \ldots, \varphi_n(X)) = \Phi(X)$ are supposed to be analytic or sufficiently smooth in $\mathcal{U}$. I do not impose other restrictions. So we can consider any degenerate and resonance cases.

We look for a solution of the problem by means of construction of a special local change of coordinates such that the system (1) is integrable in the new coordinates. Only for comparatively simple cases such a change is possible in the whole neighbourhood $\mathcal{U}$. For complicated cases the neighbourhood $\mathcal{U}$ must be partitioned into several pieces and we introduce such local coordinates in each piece so that the system (1) becomes integrable. Generally speaking, the construction of the partition and change of variables are performed gradually, step by step. In each step we construct finer pieces and in each piece we introduce coordinates, in which the system (1) become simpler. In general, the system (1) become integrable in each piece after a finite number of such steps. Usually, for systems stemming from Mechanics or Physics or Astronomy, one or two such steps are sufficient.

Two ideas are fundamental for the local method:

1. *Normal form.* If a linear part of the system (1) is nonzero, then the system (1) can be transformed into a normal form. The normal form is easily reduced to a system of smaller order with zero linear part.

2. *Resolution of complicated singularity.* Let the linear part of $\Phi(X)$ be zero. Then considering Taylor series of $\Phi(X)$, we can construct several "short-cut systems"

$$dX/dt = \hat{\Phi}_j^{(d)}(X) \tag{2}$$

and we can partition the neighbourhood $\mathcal{U}$ into pieces $\mathcal{U}_j^{(d)}(\varepsilon)$ so that each piece is a distorted cone, originating from the fixed point. The short-cut system (2) is the first nontrivial approximation of (1) in $\mathcal{U}_j^{(d)}(\varepsilon)$.

This is made by means of construction of a polyhedron in the space of power exponents. Further, for a simple (in a sense) short-cut system (2) we can transform the system (1) to a normal form in the piece $\mathcal{U}_j^{(d)}(\varepsilon)$, that is, we make the system more simple in the piece. For a complicated short-cut system (2) we apply a so-called power transformation, that blows up the fixed point into a certain manifold $\mathcal{M}$ and the piece $\mathcal{U}_j^{(d)}(\varepsilon)$ into a neighbourhood $\mathcal{U}'$ of the manifold $\mathcal{M}$. Now, in $\mathcal{M}$ we must find all fixed points and investigate their neighbourhoods, which are pieces of $\mathcal{U}'$. The system is simpler there than the initial one, and we can continue our construction which leads to new simplifications. The method of resolution of singularity is similar to blowing up (multi sigma process) in algebraic geometry and originates from the Newton polygon.

Naturally, the local method displays two sides. The first one (algebraic) consists in finding the needed formal expansions. The second one is an interpretation of those series in terms of analytic or smooth functions or in terms of appropriate estimations of accuracy of approximate integration. In the present course we shall deal only with the first side, assuming that all series are convergent. However, this is not always so (see [1]–[5]).

1. Normal form

Let us consider a system (1), where $\Phi(X) = (\varphi_1(X), \ldots, \varphi_n(X))$ and all $\varphi_j(X)$ are power series, $\varphi_j(0) = 0$. Let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of the matrix $A = (\partial\varphi_i/\partial x_j)$ in $X = 0$.

The fixed point $X = 0$ is called *elementary* if there exists $\lambda_j \neq 0$. If all $\lambda_j = 0$ then it is called *nonelementary*. We introduce new coordinates Y,

$$x_i = \xi_i(Y), \quad \xi_i(0) = 0, \quad i = 1, \ldots, n, \tag{3}$$

so that the system (1) will have the simplest form

$$\dot{y}_i = \psi_i(Y), \quad i = 1, \ldots, n. \tag{4}$$

Here ξ_i and ψ_i are power series and the Jacobian $\det(\partial\xi_i/\partial y_j) \neq 0$ in $Y = 0$. That is the coordinate change (3) is invertible. Let us write the system (4) in the form

$$\dot{y}_i = y_i g_i(Y) = y_i \sum_{Q \in N_i} g_{iQ} Y^Q, \quad i = 1, \ldots, n \tag{4'}$$

where $Q = (q_1, \ldots, q_n)$, $Y^Q = y_1^{q_1} \ldots y_n^{q_n}$. Here $y_i g_i(Y)$ are series with nonnegative powers of variables, and

$$N_i = \{Q = (q_1, \ldots, q_n)\colon Q \in \mathbf{Z}^n, q_i \geqslant -1, \text{ other } q_j \geqslant 0\}.$$

Write $N = N_1 \cup \ldots \cup N_n$ and $\Lambda = (\lambda_1, \ldots, \lambda_n) = (g_{10}, \ldots, g_{n0})$.

THEOREM 1. *Let $X = 0$ be an elementary fixed point of the system* (1). *Then there exists a formal change of variables* (3), *such that in the system* (4') *all $g_{iQ} = 0$, if $\langle Q, \Lambda \rangle = q_1\lambda_1 + \ldots + q_n\lambda_n \neq 0$.*

That is the system (4') has only *resonant members* $y_i g_{iQ} Y^Q$, for which

$$\langle Q, \Lambda \rangle = 0. \tag{5}$$

Such a system (4) is said to be in *normal form.* If the initial system (1) is real, then the real values of the coordinates X correspond to complex values of the coordinates Y which satisfy the specific real conditions. The normal form preserves many properties of the initial system, such as symmetry, a Hamiltonian character, invertibility. Some of the coordinates x_j may be parameters. A small parameter x_j satisfies the equation $\dot{x}_j = 0$ and does not change in normalizing transformation. The corresponding $\lambda_j = 0$.

2. Power transformation

The notation (4'), as well as the definition of the normal form, can be given a simple geometric interpretation. With each coefficient $g_{iQ} \neq 0$ let us associate the lattice point $Q = (q_1, \ldots, q_n)$ in the n-dimensional affine space $\mathbf{R}_1^n$. The set of all such points will be denoted by $D(g_1, \ldots, g_n)$ or $D(G)$. The normal form differs from an arbitrary system (1) by the fact that the set $D(G)$, corresponding to the normal form, lies entirely in the subspace orthogonal to the vectors $\operatorname{Re}\Lambda$ and $\operatorname{Im}\Lambda$. This enables us to lower the order of the system in normal form by means of power transformations

$$z_i = y_1^{\alpha_{i1}} \ldots y_n^{\alpha_{in}}, \quad i = 1, \ldots, n, \tag{6}$$

with α_{ij} real and $\det(\alpha_{ij}) \neq 0$. Suppose that the power transformation (6) transform the system (4') into the system

$$\dot{z}_i = z_i g_i'(Z) = z_i \sum g_{iQ'}' Z^{Q'}, \quad i = 1, \ldots, n. \tag{7}$$

We shall prove that

$$D(G') = \alpha^{*-1} D(G), \tag{8}$$

where α is the matrix (α_{ij}) and α^* is the transpose of α.

Let us set

$$\ln Y = (\ln y_1, \dots, \ln y_n), \qquad \ln Z = (\ln z_1, \dots, \ln z_n).$$

In the vector form, the system (4′) can be written as follows:

$$(\ln^{\cdot} Y) = \sum G_Q \exp \langle Q, \ln Y \rangle, \tag{9}$$

where $G_Q = (g_{1Q}, \dots, g_{nQ})$. For the system (7), we get in the same way

$$(\ln^{\cdot} Z) = \sum G'_{Q'} \exp \langle Q', \ln Z \rangle, \tag{10}$$

and for transformation (6), $\ln Z = \alpha \ln Y$. The transformation (6) transforms the term Y^Q as follows

$$Y^Q = \exp \langle Q, \ln Y \rangle = \exp \langle Q, \alpha^{-1} \ln Z \rangle = \exp \langle \alpha^{-1*} Q, \ln Z \rangle = Z^{\alpha^{-1*}Q},$$

and similarly,

$$(\ln^{\cdot} Z) = \alpha (\ln^{\cdot} Y) = \sum \alpha G_Q Y^Q = \sum \alpha G_Q Z^{\alpha^{-1*}Q}.$$

Hence

$$\sum \alpha G_Q Z^{\alpha^{-1*}Q} = \sum G'_{Q'} Z^{Q'}.$$

Therefore

$$Q' = \alpha^{*-1} Q, \qquad G'_{Q'} = \alpha G_Q.$$

Thus the set $D(G')$ of the points Q' for which $G'_{Q'} \neq 0$ is obtained from $D(G)$ by the linear transformation (8).

THEOREM 2. *Let the system* (4′) *be in normal form and let m be the number of linearly independent vectors $Q \in N$ satisfying the condition* (5). *Then there exists a power transformation* (6) (*with α_{ij} integers and* $\det \alpha = \pm 1$) *reducing the normal form* (4′) *to the system*

$$\dot{z}_i = z_i g'_i(z_1, \dots, z_m), \qquad i = 1, \dots, n. \tag{11}$$

The first m equations of this system form a system of order m,

$$\dot{z}_i = z_i g'_i(z_1, \dots, z_m), \qquad i = 1, \dots, m, \tag{12}$$

and the remaining equations can be integrated by quadratures.

The system (12) does not have a linear part, so it is impossible to transform the system (12) into a normal form. If $\Lambda \neq 0$ in the system (4′), then $m < n$ and as a result of our transformations the system (1) of order n is reduced to the system (12) of order m. If $m = 0$ or $m = 1$, the system (12) is integrable. But if $m \geqslant 2$, it is nonintegrable in general, and for this case we must study a neighbourhood of a nonelementary fixed point.

As *examples* let us consider normal forms corresponding to two-dimensional systems (1). If $n = 2$, then the set N consists of the lattice points of either of the following three kinds: all lattice points of the first quadrant; the lattice points of the second quadrant lying on the half-line $q_1 = -1$, $q_2 \geqslant 1$; and the lattice points of the fourth quadrant lying on the half-line $q_2 = -1$, $q_1 \geqslant 1$ (see Figure 1). Furthermore, $\Lambda = (\lambda_1, \lambda_2)$. For $\lambda_2 \neq 0$ and $\lambda = \lambda_1/\lambda_2$ the equation (5) is equivalent to

(13) $$\lambda q_1 + q_2 = 0.$$

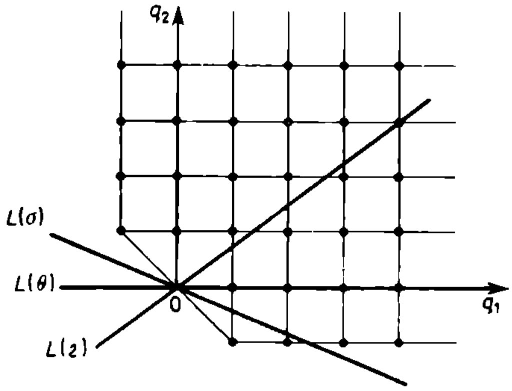

Fig. 1

When λ is real, equation (13) defines in the plane (q_1, q_2) a straight line L which is orthogonal to the vector $(\lambda, 1)$ and thus also to (λ_1, λ_2). The normal form is defined by those points of N which lie on the line L.

Case $\lambda = 0$, i.e. $\lambda_1 = 0$ and $\lambda_2 \neq 0$. The straight line L intersects N at the points $Q = (k, 0)$, where k is a nonnegative integer (see Figure 1, $L = L(\theta)$). Hence the normal form is

(14) $$\begin{aligned} \dot{y}_1 &= y_1 \sum_{k=1}^{\infty} g_{1(k,0)} y_1^k, \\ \dot{y}_2 &= y_2 \Big(\lambda_2 + \sum_{k=1}^{\infty} g_{2(k,0)} y_1^k\Big). \end{aligned}$$

System (14) can be written in the form

$$\begin{aligned} (\ln^{\cdot} y_1) &= \sum_{k=1}^{\infty} g_{1(k,0)} y_1^k, \\ (\ln^{\cdot} y_2) &= \lambda_2 + \sum_{k=1}^{\infty} g_{2(k,0)} y_1^k. \end{aligned}$$

The right-hand sides of this system do not depend on y_2. Therefore, finding $y_1(t)$ from the first equation, we obtain y_2 by quadratures. Here $m = 1$.

Case $\lambda < 0$. The stright line L passes through the first and the third quadrants (see Figure 1, $L = L(\iota)$). In the third quadrant there are no points of N, but in the first quadrant every lattice point belongs to N. If λ is irrational, equation (13) has only the trivial solution $q_1 = q_2 = 0$ and the normal form is

$$\dot{y}_1 = \lambda_1 y_1, \quad \dot{y}_2 = \lambda_2 y_2.$$

Here $m = 0$. If $\lambda = -1$, i.e. $\lambda_1 = -\lambda_2$, the stright line L intersects N at the points of the form $Q = (k, k)$, where k is a nonnegative integer. The normal form is then

$$\begin{aligned} \dot{y}_1 &= y_1\Big(\lambda_1 + \sum_{k=1}^{\infty} g_{1(k,k)} y_1^k y_2^k\Big), \\ \dot{y}_2 &= y_2\Big(\lambda_2 + \sum_{k=1}^{\infty} g_{2(k,k)} y_1^k y_2^k\Big). \end{aligned} \tag{15}$$

The transformation $z_1 = y_1 y_2$, $z_2 = y_2$ reduces (15) to the system

$$\begin{aligned} \dot{z}_1 &= z_1 \sum_{k=1}^{\infty} \big(g_{1(k,k)} + g_{2(k,k)}\big) z_1^k, \\ \dot{z}_2 &= z_2\Big(\lambda_2 + \sum_{k=1}^{\infty} g_{2(k,k)} z_1^k\Big), \end{aligned}$$

which is analogous to (14). Here $m = 1$.

The case $\lambda = -r/s$, where r and s are relatively prime, can be discussed analogously to the previous case (see [1, § 0, Section III]); here $m = 1$.

3. Generalization of the normal form

Let $R_1, R_2, \ldots, R_l$ be vectors in $\boldsymbol{R}^n$ such that, for a certain vector T,

$$\langle R_i, T \rangle < 0, \quad i = 1, \ldots, l.$$

Then the set

$$V = \{Q\colon Q = \beta_1 R_1 + \ldots + \beta_l R_l,\ \beta_i \geqslant 0\}$$

is a *polyhedral convex cone*. The series

$$f = \sum f_Q X^Q \tag{16}$$

is called a *series of class* $\mathscr{V}$, if all $Q \in V \cap \boldsymbol{Z}^n$.

Let us consider the system

$$(\ln^{\cdot} x_i) = f_i(X) = \sum f_{iQ} X^Q, \quad i = 1, \ldots, n, \tag{17}$$

where f_i are series of class $\mathscr{V}$. Denote $\Lambda = (f_{10}, \ldots, f_{n0}) = F_0$. If the series f_i are convergent, then they converge in a set

$$\mathscr{U}_V(\varepsilon) = \{X\colon |X|^{R_i} \leqslant \varepsilon,\ i = 1, \ldots, l\} \tag{18}$$

where $\varepsilon > 0$ is small enough.

THEOREM 3. *In the system* (17), *let f_i be series of class $\mathscr{V}$. Then there exists a formal change of variables* (3), *where $y_i^{-1}\xi_i(Y)$ are series of class $\mathscr{V}$, which transforms the system* (17) *into the normal form*

$$\dot{y}_i = y_i g_i(Y) = y_i \sum g_{iQ} Y^Q, \quad i = 1, \ldots, n, \tag{18'}$$

where g_i are series of class $\mathscr{V}$ and $g_{iQ} = 0$ if $\langle Q, \Lambda\rangle \neq 0$.

Remark. If equation (5) has no solution $Q \in \mathbf{Z}^n$ inside the cone V, then in the normal form (18′) $g_{iQ} = f_{iQ}$ for $\langle Q, \Lambda\rangle = 0$.

Let us consider in detail the structure of a set (18), which is the set of convergence of a series of class $\mathscr{V}$. At first, let $R_1 = E_1, \ldots, R_n = E_n$ be the unit coordinate vectors and $l = n$. Then the cone V is equal to $\mathbf{R}^n_+$, i.e. it coincides with the first quadrant, octant and so on. The set $\mathscr{U}_V(\varepsilon) = \{X\colon$ all $|x_i| \leqslant \varepsilon\}$. Write

$$U_V(\varepsilon) = \{P\colon P = \ln|X|,\ X \in \mathscr{U}_V(\varepsilon)\}.$$

Then in our case $U_V(\varepsilon) = \{P\colon$ all $p_i \leqslant \ln\varepsilon\}$. In the case of arbitrary vectors $R_1, \ldots, R_l$,

$$U_V(\varepsilon) = \{P\colon \langle R_i, P\rangle \leqslant \ln\varepsilon,\ i = 1, \ldots, l\}.$$

For $n = 2$ and $\varepsilon < 1$ the sets V, $U_V(\varepsilon)$ and $\mathscr{U}_V(\varepsilon)$ are shown in Figures 2, 3 and 4 respectively (there $R_1 = (2, -1)$ and $R_2 = (-1, 2)$).

THEOREM 4. *Let the system* (18′) *be in generalized normal form and let m be the number of linearly independent vectors $Q \in V \cap \mathbf{Z}^n$ satisfying the condition* (5). *Then the statement of Theorem* 2 *is valid.*

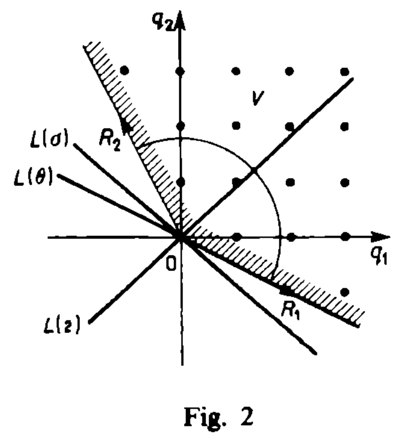

Fig. 2

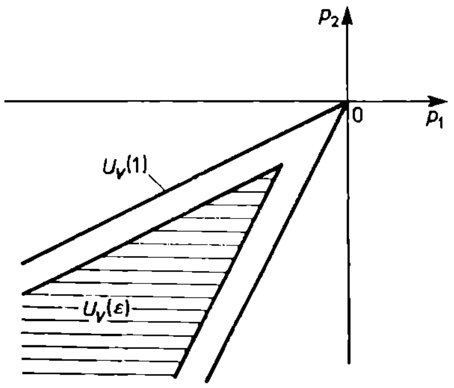

Fig. 3

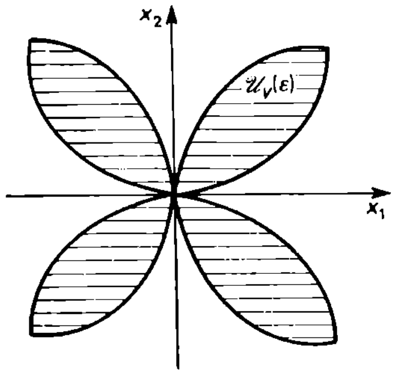

Fig. 4

4. Partition into pieces

At first I shall explain some intuitive ideas and after that I shall present a rigorous theory. If we have a power series $\sum_{k=0}^{\infty} f_k x^k$ of one variable x, then the first approximation of it as $x \to 0$ is the monomial $f_l x^l$, where f_l is the first nonzero coefficient of the series. A power series (16) of several variables $X = (x_1, \ldots, x_n)$ can admit several distinct first approximations as $X \to 0$ depending of the way in which X tends to 0. Let us consider a curve

$$(19) \qquad x_i = c_i \tau^{p_i}, \quad p_i < 0,\ c_i = \text{const} \neq 0,\ i = 1, \ldots, n,$$
$$\tau \to +\infty,\ P = (p_1, \ldots, p_n),\ C = (c_1, \ldots, c_n).$$

We have, at the points of this

$$f(X) = \sum f_Q C^Q \tau^{\langle Q, P\rangle}.$$

The first approximation of this series is the sum of those members $f_Q X^Q$ for which the value of $\langle Q, P\rangle$ is greatest, since $\tau \to \infty$. Let $r = \max \langle Q, P\rangle$ over all Q with $f_Q \neq 0$. Then the first approximation of the series (16) along a curve (19) is

$$\hat{f}_P(X) = \sum_{\langle Q, P\rangle = r} f_Q X^Q.$$

It is called the *short-cut of the series* f along vector order P. For distinct vectors P, the short-cuts of f can be distinct. For instance, if

$$(20) \qquad f = x_1^3 + x_2^3 - x_1 x_2 + x_1 x_2^2,$$

then $Q_1 = (3, 0)$, $Q_2 = (0, 3)$, $Q_3 = (1, 1)$, $Q_4 = (1, 2)$. Let $P = -(1, 1)$, then $\langle Q_1, P\rangle = -3 = \langle Q_2, P\rangle = \langle Q_4, P\rangle$, $\langle Q_3, P\rangle = -2$. So, short-cut $\hat{f} = -x_1 x_2$. If $P = -(1, 2)$, then $\langle Q_1, P\rangle = -3 = \langle Q_3, P\rangle$ and $\langle Q_2, P\rangle = -6$, $\langle Q_4, P\rangle = -5$. So short-cut $\hat{f} = x_1^3 - x_1 x_2$.

Now I shall give the rigorous theory. Let D be a set of points $Q = (q_1, \ldots, q_n)$ in the n-dimensional real space $\boldsymbol{R}_1^n$ and let $\boldsymbol{R}_2^n$ be the dual space. We consider the following problem: For each $P \in \boldsymbol{R}_2^n$, find a subset D_P of D such that

$$\langle Q', P \rangle = r \quad \text{for all } Q' \in D_P,$$

$$\langle Q, P \rangle < r \quad \text{for all } Q \in D \setminus D_P,$$

where $r = r(P) = \max_{Q \in D} \langle Q, P \rangle$.

To solve the problem, note that, for a fixed vector P, the equation

(21) $$\langle Q, P \rangle = c_0 = \text{const}$$

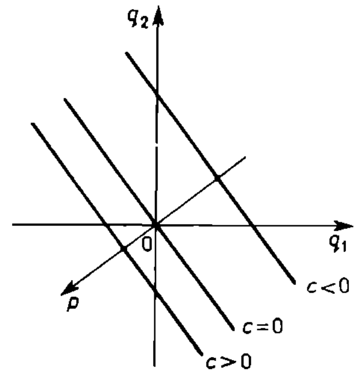

Fig. 5

defines a hyperplane H in the space $\boldsymbol{R}_1^n$. The hyperplane H divides the space $\boldsymbol{R}_1^n$ into two halfspaces: the positive one $H^{(+)} = \{Q\colon \langle Q, P \rangle > c_0\}$ and the negative one $H^{(-)} = \{Q\colon \langle Q, P \rangle \leqslant c_0\}$ (see Figure 5). A hyperplane (21) is called *supporting* for the set D, if its positive halfspace $H^{(+)}$ contains no points of D, and for each hyperplane $\langle Q, P \rangle = c < c_0$ its positive halfspace contains points of D. Denote by H_P the supporting hyperplane corresponding to the vector P. Evidently, $D_P = H_P \cap D$.

To describe the sets D_P for distinct vectors P, let us consider the convex span Δ of the set D

$$\Delta = \{Q\colon Q = \delta_1 Q_1 + \ldots + \delta_n Q_n,\ Q_i \in D,\ \delta_i \geqslant 0,\ \textstyle\sum \delta_i = 1\}.$$

Denote by Γ the intersection of all negative halfspaces $H_P^{(-)}$ of the set D. The set Γ is the closure of the set Δ. The intersection of Γ with a supporting hyperplane H_P is called a *face* of Γ. The boundary $\partial\Gamma$ of the closed set Γ consists of faces of various dimensions. Thus, a zero-dimensional face is a vertex, a one-dimensional face is an edge, etc. We shall denote the faces by $\Gamma_j^{(d)}$, where d indicates dimension and j is the successive number. Write $D_j^{(d)}$

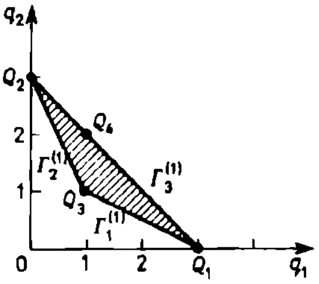

Fig. 6

$= D \cap \Gamma_j^{(d)}$. If $\Gamma_j^{(d)} = H_P \cap \Gamma$, then evidently $D_P = D_j^{(d)}$. This means that all boundary subsets D_P are subsets $D_j^{(d)}$ lying in $\Gamma_j^{(d)}$.

For example, if $D = \{Q_1, Q_2, Q_3, Q_4\}$ as in (20), then Γ is a triangle, the points Q_1, Q_2, Q_3 are its vertices. Let $\Gamma_1^{(0)} = Q_1$, $\Gamma_2^{(0)} = Q_3$, $\Gamma_3^{(0)} = Q_2$. Denote by $\Gamma_1^{(1)}$, $\Gamma_2^{(1)}$, and $\Gamma_3^{(1)}$ the edges of the triangle Γ as shown in Figure 6. Then the sets $D_j^{(d)}$ are

$$D_1^{(0)} = Q_1, \qquad D_2^{(0)} = Q_3, \qquad D_3^{(0)} = Q_2,$$
$$D_1^{(1)} = \{Q_1, Q_2\}, \quad D_2^{(1)} = \{Q_3, Q_2\}, \quad D_3^{(1)} = \{Q_2, Q_4, Q_1\}.$$

Let us fix a face $\Gamma_j^{(d)}$ and discribe the set $U_j^{(d)}$ of such vectors P for which $H_P \cap \Gamma = \Gamma_j^{(d)}$. The set

$$U_j^{(d)} = \left\{P\colon \begin{array}{l} \langle Q', P\rangle = \langle Q'', P\rangle,\ Q', Q'' \in \Gamma_j^{(d)}, \\ \langle Q', P\rangle > \langle Q, P\rangle,\ Q \in \Gamma \setminus \Gamma_j^{(d)} \end{array}\right\}$$

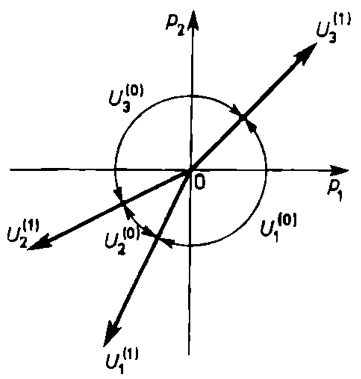

Fig. 7

is a convex cone and is called the *normal cone of the face* $\Gamma_j^{(d)}$. In our example $U_j^{(0)}$ are sectors and $U_j^{(1)}$ are rays (see Figure 7). So

$$U_1^{(1)} = \{P\colon\ p_2 = 2p_1 < 0\}, \qquad U_2^{(1)} = \{P\colon\ 2p_2 = p_1 < 0\},$$
$$U_2^{(0)} = \{P\colon\ 2p_1 < p_2,\ 2p_2 < p_1\}.$$

Let us further introduce the cone $V_j^{(d)}$ normal to $U_j^{(d)}$, that is

$$V_j^{(d)} = \{Q\colon \langle Q, P\rangle \leqslant 0,\ P \in U_j^{(d)}\},$$

we call it the *tangent cone of the face* $\Gamma_j^{(d)}$. In our example $V_j^{(0)}$ are sectors and $V_j^{(1)}$ are halfplanes (see Figure 8). So

$$V_1^{(1)} = \{Q\colon q_1 + 2q_2 \geqslant 0\}, \quad V_2^{(1)} = \{Q\colon 2q_1 + q_2 \geqslant 0\},$$

$$V_2^{(0)} = \{Q\colon q_1 + 2q_2 \geqslant 0,\ 2q_1 + q_2 \geqslant 0\}.$$

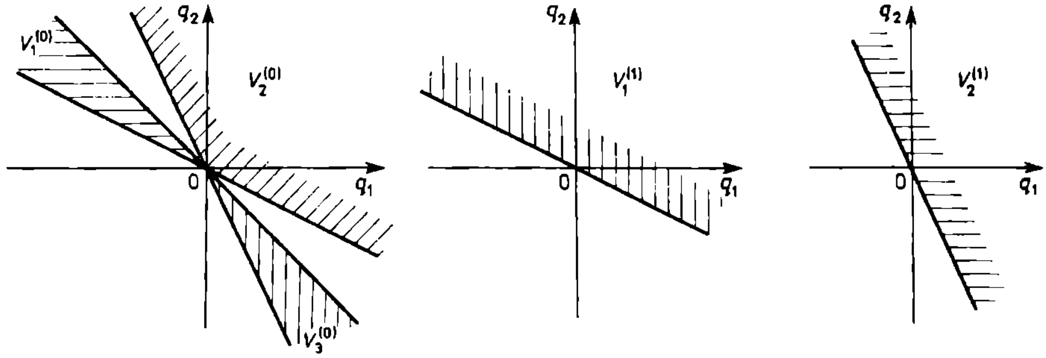

Fig. 8

For many sets D the set Γ is a polyhedron and its boundary $\partial\Gamma$ consists of a finite number of faces $\Gamma_j^{(d)}$. This is true, e.g., if the set D has only finite by many points.

Let $\Gamma_j^{(d)}$ be a face, then its normal cone $U_j^{(d)}$ can be given as follows,

$$U_j^{(d)} = \left\{P\colon \begin{array}{l} \langle R_i, P\rangle = 0,\ i = 1, \ldots, k, \\ \langle R_i, P\rangle < 0,\ i = k+1, \ldots, l \end{array}\right\} \tag{22}$$

where R_i are some vectors, which are simply computed from the set D. Then the tangent cone is

$$V_j^{(d)} = \{Q\colon \gamma_1 R_1 + \ldots + \gamma_k R_k + \delta_{k+1} R_{k+1} + \ldots + \delta_l R_l,\ \delta_i \geqslant 0\}.$$

Thus, to find all boundary subsets $D_j^{(d)}$ and their normal cones $U_j^{(d)}$, we must consider the polyhedron Γ and find all its faces $\Gamma_j^{(d)}$ and for each face $\Gamma_j^{(d)}$ find its normal cone $U_j^{(d)}$. We can do this performing linear operations on vectors $Q \in D$. Such computations are carried out in linear programming.

Together with the cone (22) let us consider the set

$$U_j^{(d)}(\varepsilon) = \left\{P\colon \begin{array}{ll} \ln\varepsilon \leqslant \langle R_i, P\rangle \leqslant -\ln\varepsilon, & i = 1, \ldots, k, \\ \langle R_i, P\rangle \leqslant \ln\varepsilon, & i = k+1, \ldots, l \end{array}\right\} \tag{23}$$

for small positive $\varepsilon \leqslant 1$. If $\varepsilon = 1$, then $U_j^{(d)}(1)$ is the closure of $U_j^{(d)}$. Finally let us put $P = \ln X$ and consider the set $U_j^{(d)}$ in coordinates X:

$$\mathscr{U}_j^{(d)}(\varepsilon) = \left\{X: \begin{array}{ll} \varepsilon \leqslant |X|^{R_i} \leqslant \varepsilon^{-1}, & i = 1, \ldots, k, \\ |X|^{R_i} \leqslant \varepsilon, & i = k+1, \ldots, l \end{array}\right\}, \tag{24}$$

Figures 9 and 10 show the sets $U_1^{(1)}(\varepsilon)$, $U_2^{(1)}(\varepsilon)$, $U_2^{(0)}(\varepsilon)$ and $\mathscr{U}_1^{(1)}(\varepsilon)$, $\mathscr{U}_2^{(1)}(\varepsilon)$, $\mathscr{U}_2^{(0)}(\varepsilon)$ respectively for our example.

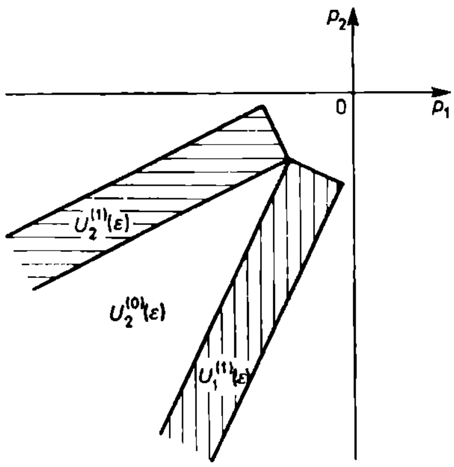

Fig. 9

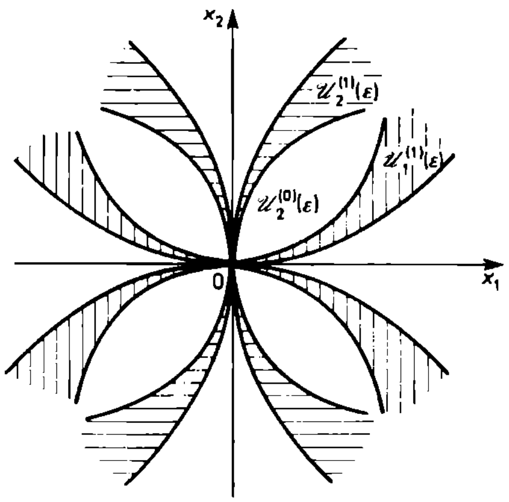

Fig. 10

For a sum (16), the set

$$D = D(f) = \{Q: f_Q \neq 0\}$$

is called its *support*. The closure of the convex span of D is called the *Newton polyhedron* $\Gamma = \Gamma(f)$ of the series f. To each face $\Gamma_j^{(d)}$ of the polyhedron Γ there corresponds a short-cut

$$\hat{f}_j^{(d)} = \sum f_Q X^Q, \quad \text{where } Q \in D_j^{(d)}.$$

This sum is a short-cut for each vector order $P \in U_j^{(d)}$. Denote by $\mathscr{V}_j^{(d)}$ the class of those power series (16) whose supports lie in the tangent cone $V_j^{(d)}$. If $d = 0$, then the set $U_j^{(0)}(\varepsilon)$ is a set of those values of $P = \ln|X|$, for which a series of class $\mathscr{V}_d^{(0)}$ can converge.

5. A nonelementary fixed point

Let us consider the system

$$\dot{x}_i = x_i f_i(X) = x_i \sum f_{iQ} X^Q, \qquad i = 1, \ldots, n, \tag{25}$$

where $x_i f_i(X)$ are analytic functions and the point $X = 0$ is a nonelementary fixed point, i.e., all eigenvalues $\lambda_i = 0$. For each vector $Q \in N$ we introduce its vectorial coefficient $F_Q = (f_{1Q}, \ldots, f_{nQ})$. Let $D(F)$ be the support of the series $F = \sum F_Q X^Q$, i.e.,

$$D = D(F) = \{Q\colon\ F_Q \neq 0\}.$$

Let $\Gamma = \Gamma(F)$ be the closure of the convex span of D. In general, Γ is a convex polyhedron (the Newton polyhedron of the system (25)). Its boundary $\partial\Gamma$ consists of faces $\Gamma_j^{(d)}$. Since the system (25) and the series $F(X)$ are studied only in a neighbourhood of the point $X = 0$, it is sufficient to find only all those faces $\Gamma_j^{(d)}$ whose normal cone $U_j^{(d)}$ contains a vector $P < 0$.

So far, we have associated with a set D several geometrical objects. Namely, given the system (25), we have defined:

1) In the space $\boldsymbol{R}_1^n = \{Q\}$: the support $D(F)$, the Newton polyhedron $\Gamma = \Gamma(F)$, its faces $\Gamma_j^{(d)}$ and their tangent cones $V_j^{(d)}$ and sets $D_j^{(d)} = D \cap \Gamma_j^{(d)}$.

2) In the dual space $\boldsymbol{R}_2^n = \{P\}$: the normal cones $U_j^{(d)}$ to the faces $\Gamma_j^{(d)}$ and the sets $U_j^{(d)}(\varepsilon)$.

3) In the space of points X: the sets $\mathscr{U}_j^{(d)}(\varepsilon)$, corresponding to the sets $U_j^{(d)}(\varepsilon)$ for $\ln X = P$. And for each $\Gamma_j^{(d)}$ we have in $\mathscr{U}_j^{(d)}(\varepsilon)$ a short-cut system

$$(\ln^{\cdot} X) = \hat{F}(X) = \sum F_Q X^Q, \qquad Q \in D_j^{(d)}.$$

On the other hand we have two kinds of transformations:

1) The first one is the power transformation $\ln X' = \alpha \ln X$. They lead to linear transformations in the spaces $\boldsymbol{R}_1^n = \{Q\}$ and $\boldsymbol{R}_2^n = \{P\}$:

$$Q' = \alpha^{*-1} Q,$$

$$P' = \alpha P.$$

The scalar product is preserved:

$$\langle Q', P'\rangle = \langle \alpha^{*-1} Q, \alpha P\rangle = \langle Q, P\rangle.$$

So, $\boldsymbol{R}_1^n$ and $\boldsymbol{R}_2^n$ are dual spaces. Hence, our objects D, Γ, $\Gamma_j^{(d)}$, $V_j^{(d)}$, $D_j^{(d)}$ in $\boldsymbol{R}_1^n$ and $U_j^{(d)}$, $U_j^{(d)}(\varepsilon)$ in $\boldsymbol{R}_2^n$ are transformed into other objects D', Γ' and so on, in such a way that linear relations between them are preserved. And thus, the polyhedron Γ is transformed into the polyhedron Γ', its face $\Gamma_j^{(d)}$ – into a face $\Gamma_j^{(d)'}$ of Γ' and the normal cone $U_j^{(d)}$ of $\Gamma_j^{(d)}$ is transformed into the normal cone $U_j^{(d)'}$ of $\Gamma_j^{(d)'}$. This means that all our constructions are commute with power transformations.

2) The second kind of transformation is a change of time,

$$dt' = X^{\tilde{Q}}\,dt. \tag{26}$$

In $\boldsymbol{R}_1^n$ this transformation induces parallel translation

$$Q' = Q - \tilde{Q}$$

for sets D, Γ, $\Gamma_j^{(d)}$, $D_j^{(d)}$, because

$$d \ln X/dt' = \sum F_Q\, X^{Q-\tilde{Q}}.$$

But the tangent cones $V_j^{(d)}$ are not changed.

We thus have a nontrivial geometry: linear objects in the dual spaces $\boldsymbol{R}_1^n$ and $\boldsymbol{R}_2^n$ and a group of linear transformations between the objects.

As a result of our constructions, we have obtained a partition of a neighbourhood of $X = 0$ in the X-space into several pieces $\mathcal{U}_j^{(d)}(\varepsilon)$. For each positive $\varepsilon \leqslant 1$, the union of the sets $\mathcal{U}_j^{(d)}(\varepsilon)$ fills a neighbourhood of $X = 0$. Now, in each set $\mathcal{U}_j^{(d)}(\varepsilon)$ we shall introduce new coordinates in which the system will be simpler. Simplification will be achieved differently, according to the value of $d = 0, 1, 2, \ldots$ Let us consider several cases.

1) $d = 0$. To study the system (25) in the set $\mathcal{U}_j^{(0)}(\varepsilon)$, corresponding to the vertex $\Gamma_j^{(0)} = \tilde{Q}$, we apply a change of time (26). Then the vertex goes to point $Q' = 0$, $F_{\tilde{Q}} = \Lambda$ and we can apply Theorem 3 on the generalization of the normal form. A coordinate change $X \to Y$ of class $\mathcal{V}_j^{(0)}$ transforms the system (25) to normal form and a power transformation considered in Theorem 4 reduces the normal form to a system of order $m < n$. For that system we must again study a neighbourhood of a nonelementary fixed point, but now the order is $m < n$. So, we must again construct the Newton polyhedron and partition a neighbourhood of zero into several pieces $\mathcal{U}_k^{(d)'}(\varepsilon)$, corresponding to a partition of the set $\mathcal{U}_j^{(0)}(\varepsilon)$ into pieces $\mathcal{U}_{jk}^{(0)(d)}(\varepsilon)$. In each piece we apply its own simplification and so on, until we arrive at an integrable system.

2) $d = 1$. To study the system (25) in the set $\mathcal{U}_j^{(1)}(\varepsilon)$, corresponding to an edge $\Gamma_j^{(1)}$, we must use a power transformation, which transforms the edge $\Gamma_j^{(1)}$ of Γ into an edge $\Gamma_j^{(1)'}$ of Γ' parallel to the coordinate axis q^n. Then the set $\mathcal{U}_j^{(1)}(\varepsilon)$ is transformed into a set $\mathcal{U}'(\varepsilon)$, which is a neighbourhood of the portion of y_n-axis: $\varepsilon \leqslant |y_n| \leqslant \varepsilon^{-1}$. To study the system in that neighbourhood, we must reduce the system relative to the greatest powers of $y_1, \ldots, y_{n-1}$, i.e. change the time. Then we must find all fixed points $y_n = y_n^0 \neq 0, \infty$ in the axis y_n. For each of these fixed points we take a neighbourhood of it. The remaining part of the set $\mathcal{U}'(\varepsilon)$ does not contain fixed points, and

the system is integrable in it. Thus, the set $\mathscr{U}'(\varepsilon)$ is split into several pieces. In the piece corresponding to a fixed point $y_1 = \ldots = y_{n-1} = 0$, $y_n = y_n^0$, we must study solutions of the transformed system (in coordinates Y). It is the same problem from which we have started, but now singularity is more simple than in the initial system (25), and we can continue the process of resolution of singularity.

3) $d \geqslant 2$. Let vectors $R^1, \ldots, R^d \in Z^n$ constitute a basis of the face $\Gamma_j^{(d)}$ and let vectors $S_1, \ldots, S_{n-d} \in Z^n$ complement it to a basis of R_1^n. Then the power transformation

$$y_i = X^{S_i}, \quad i = 1, \ldots, n-d,$$
$$y_{n-d+i} = X^{R_i}, \quad i = 1, \ldots, d$$

transforms the face $\Gamma_j^{(d)}$ into a face $\Gamma_j^{(d)\prime}$ parallel to the coordinate plane of $q'_{n-d+1}, \ldots, q'_n$. Then we must reduce the system in Y with respect to the greatest powers of $y_1, \ldots, y_{n-d}$, i.e. change the time. The set $\mathscr{U}_j^{(d)}(\varepsilon)$ is transformed into a set $\mathscr{U}'(\varepsilon)$, which is a neighbourhood of the part of the coordinate plane of $y_{n-d+1}, \ldots, y_n$:

$$\varepsilon \leqslant |y_{n-d+i}| \leqslant \varepsilon^{-1}, \quad i = 1, \ldots, d.$$

This portion of the plane is split into pieces so that each piece has at most one fixed point. Now, we must inspect a neighbourhood of each fixed point $y_1 = \ldots = y_{n-d} = 0$, $y_{n-d+i} = y^0_{n-d+i} \neq 0, \infty$, $i = 1, \ldots, d$. Here singularity is more simple than in the initial system (25).

Thus we can resolve singularity, reducing the study of solutions in the set $\mathscr{U}_j^{(d)}(\varepsilon)$ with $d > 0$ to the study of solutions in neighbourhoods of several fixed points. In a neighbourhood of an elementary fixed point and in a set $\mathscr{U}_j^{(0)}(\varepsilon)$ we can apply Theorems 1 or 3 about the normal form. Together with Theorems 2 and 4, this gives a system of smaller order. And thus, every step simplifies complicated singularity and lowers the order of the system. After several such steps a neighbourhood of the fixed point $X = 0$, for the system (25), will be divided into a finite number of sets $\mathscr{U}_{j_1 \ldots j_k}^{(d_1)\ldots(d_k)}(\varepsilon)$, where $j_1, \ldots, j_k$ indicate additional divisions. And in each set we shall have variables, in which the system will be integrable.

The local method reduces the study of an essentially nonlinear problem to the study of some linear objects in the space of power exponents and in its dual space, and computations in this method are in most cases also linear.

6. An example

Let us consider the system

$$\dot{x}_1 = x_2 + x_1^2 - x_1 x_2 = x_1(x_1^{-1} x_2 + x_1 - x_2),$$
$$\dot{x}_2 = 3x_1^2 + 3x_1 x_2 - 5x_2^2 = x_2(3x_1^2 x_2^{-1} + 3x_1 - 5x_2).$$

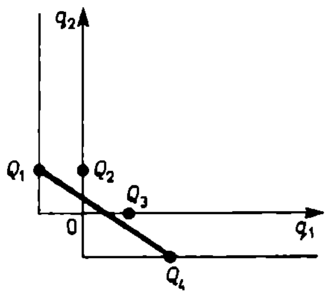

Fig. 11

Here

$$Q_1 = (-1, 1), \qquad Q_2 = (1, 0),$$
$$Q_3 = (0, 1), \qquad Q_4 = (2, -1),$$
$$F_{Q_1} = (1, 0), \qquad F_{Q_2} = (1, 3),$$
$$F_{Q_3} = (-1, -5), \qquad F_{Q_4} = (0, 3).$$

Γ is a triangle (see Figure 11). Its boundary $\partial\Gamma$ consists of three vertices and three edges. But only two vertices $\Gamma_1^{(0)} = Q_4$ and $\Gamma_2^{(0)} = Q_1$ and one edge $\Gamma_1^{(1)}$ have a negative vector $P < 0$ in their normal cones $U_j^{(d)}$. To the vertices $\Gamma_1^{(0)}$ and $\Gamma_2^{(0)}$ there correspond the sets $\mathscr{U}_1^{(0)}(\varepsilon)$ and $\mathscr{U}_2^{(0)}(\varepsilon)$ (see Figure 12). By Theorem 3 and the Remark following it, in the set $\mathscr{U}_1^{(0)}(\varepsilon)$ the system has normal form

$$dy_1/dt' = 0, \quad dy_2/dt' = 3y_1^2.$$

Its integral curves are $y_1 = \text{const}$. Since the normalizing transformation is $x_i = y_i + \ldots$, the integral curves in the set $\mathscr{U}_1^{(0)}(\varepsilon)$ are $x_1 \approx \text{const}$ (see Figure 12). Analogously, in the set $\mathscr{U}_2^{(0)}(\varepsilon)$ the integral curves are $x_2 \approx \text{const}$. The set $\mathscr{U}_1^{(1)}(\varepsilon)$ is bounded by the inequalities $\varepsilon \leqslant |X|^R \leqslant \varepsilon^{-1}$, where $R = Q_1 - Q_4 = (-3, 2)$. Let us find a such vector $S = (s_1, s_2)$ that $s_1 r_2 - s_2 r_1 = 1$. We can take $S = (2, -1)$ and apply the power transformation

$$y_1 = x_1^2 x_2^{-1}, \quad y_2 = x_1^{-3} x_2^2.$$

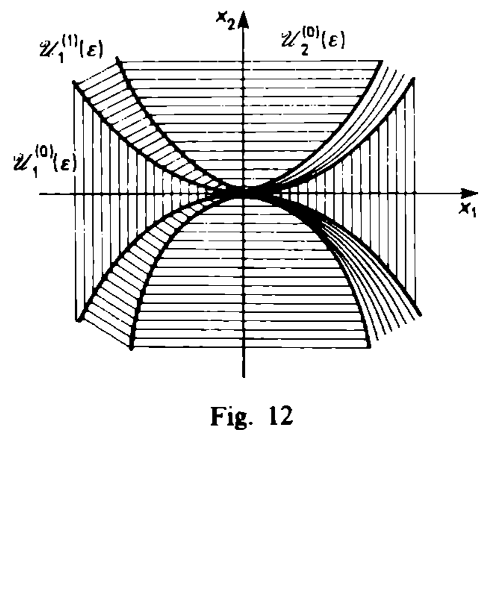

Fig. 12

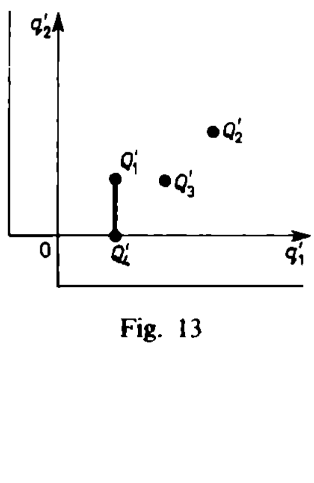

Fig. 13

The inverse transformation is

$$x_1 = y_1^2 y_2, \quad x_2 = y_1^3 y_2^2.$$

The original system is carried into the system

$$(\ln^{\cdot} y_1) = 2y_1 y_2 - y_1^2 y_2 + 3y_1^3 y_2^2 - 3y_1,$$
$$(\ln^{\cdot} y_2) = -3y_1 y_2 + 3y_1^2 y_2 - 7y_1^3 y_2^2 + 6y_1;$$

Figure 13 shows its support. After substitution $dt' = y_1\,dt$ we get the system

$$dy_1/dt' = 2y_1 y_2 - y_1^2 y_2 + 3y_1^3 y_2^2 - 3y_1,$$
$$dy_2/dt' = -3y_2^2 + 3y_1 y_2^2 - 7y_1^2 y_2^3 + 6y_2.$$

Here the axis y_2 is an integral curve; the fixed points $y_1 = 0$ and $y_2 = y_2^0 \neq 0$ in it are found from the equation

$$-3y_2^2 + 6y_2 = 0.$$

Thus we have only one fixed point $y_2^0 = 2$. Through any other point $y_1 = 0$, $y_2 \neq y_2^0$, 0 there passes only one integral curve, namely, the axis y_2. To inspect a neighbourhood of the fixpoint $y_1 = 0$, $y_2 = 2$ we apply the parallel translation

$$y_2 = y_2^0 + z_2 = 2 + z_2.$$

Then we obtain the system

$$dy_1/dt' = y_1 + 2y_1 z_2 - 2y_1^2 - y_1^2 z_2 + 3y_1^3 (2+z_2)^2,$$
$$dz_2/dt' = 12y_1 - 6z_2 - 3z_2^2 + 12y_1 z_2 + y_1 z_2^2 - 7y_1^2 (2+z_2)^3.$$

The matrix of the linear part is

$$A = \begin{pmatrix} 1 & 0 \\ 12 & -6 \end{pmatrix}$$

Its eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -6$. The fixed point is a saddle-point, and only two integral curves pass through it: one is the axis z_2 and the other is tangent to the eigenvector B_2 of the matrix A. We find $B_2 = (7, 12)$ or $z_2 = \frac{12}{7} y_1$. So the unique integral curve crossing the y_2 axis is

$$z_2 = \tfrac{12}{7} y_1 + \ldots, \quad \text{i.e. } y_2 = 2 + \tfrac{12}{7} y_1 + \ldots$$

It consists of two halfbranches F_1' ($y_1 > 0$) and F_2' ($y_1 < 0$) (see Figure 14). In variables X they are

$$x_1 = 2y_1^2 + \tfrac{12}{7} y_1^3 + \ldots,$$
$$x_2 = 4y_1^3 + \tfrac{16}{7} y_1^4 + \ldots$$

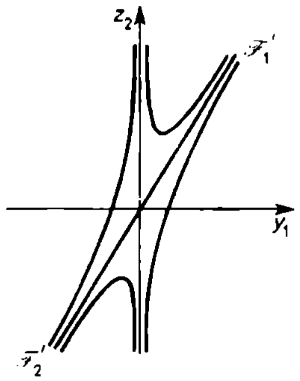

Fig. 14

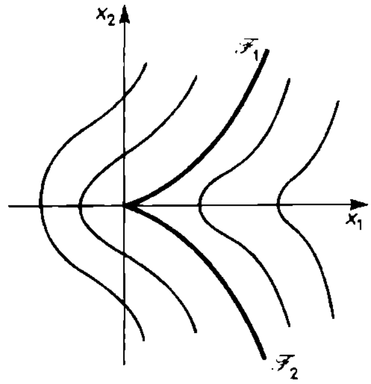

Fig. 15

It is the unique integral curve coming to the fixed point $X = 0$, it consists of two halfbranches F_1 and F_2 (see Figures 12 and 15). In the set $\mathcal{U}_1^{(1)}(\varepsilon)$ other integral curves are situated as shown in Figure 12. By piecing together portions of integral curves, found in different sets $\mathcal{U}_j^{(d)}(\varepsilon)$, we get a phase portrait shown in Figure 15.

References

[1] A. D. Brjuno, *Analytical form of differential equations*, Trans. Moscow Math. Soc. 25 (1971), 131–288; 26 (1972), 199–239.

[2] –, *Integral analytic sets*, Soviet Math. Dokl. 16 (1975), 224–228.

[3] –, *Analytical integral sets*, in *VII Internat. Konf. über nichtlineare Swingungen, Berlin, 1977*, B. 1.1, 119–127.

[4] –, *Formal and analytical integral sets*, in *Proc. Internat. Congr. Mathematicians, Helsinki 1978*, vol. 2, 807–810.

[5] A. D. Bruno, *Local Method of Nonlinear Analysis of Differential Equations*, Nauka, Moscow 1979 (in Russian).

[6] –, *Power asymptotics of solutions of nonlinear systems*, Izv. Akad. Nauk SSSR Ser. Mat. 29 (1965), 329–364 (in Russian).

[7] –, *On power asymptotics of solutions of nonlinear-systems*, Preprint No. 51, Inst. Appl. Math., Moscow 1973 (in Russian).

[8] –, *On periodic flybys of the Moon*, Celestial Mech. 24 (1981), 255–268.

[9] A. Soleev, *Determining the branches of analytic curve, and Newton polyhedra*, Soviet Math. Dokl. 27 (1983), 252–254.

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