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PERMANENCE OF LOCAL PROPERTIES UNDER HYPERPLANE SECTIONS

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Introduction

The classical theorem of Bertini states that the "generic" hyperplane section of a non-singular variety X embedded in P^n_C is again non-singular and, whenever X is connected and $\dim X \geq 2$, irreducible. In this statement "generic" means that the theorem is true for a dense open subset of the space of all hyperplanes of P^n_C.

Some results similar to the classical theorem of Bertini have been proved also for other local properties P such as: normal (see [14], [8], [11], [5]), reduced (see [8], [11], [5]), R_s, S_r (see [8], [5]), weakly normal (in characteristic zero, see [4]); i.e., all these properties are preserved by a generic hyperplane section.

This seminar is a survey on this kind of results. We describe three different ways of proving Bertini type theorems for the properties above.

Section 1 concerns the first method, which can be used in the analytic case; it consists in deducing Bertini theorems from a Sard type theorem for the local property P and from the openess of the P-locus of a flat morphism. This method does not apply to weak normality, since for this property it is not known whether the P-locus of a flat morphism is open.

Later on in Section 2 we describe a method introduced by Flenner [8] and then used in [4], which consists in reducing the proof of the Bertini type theorem to the proof of an analogous statement for local rings. This method applies to projective varieties over an algebraically closed field of characteristic zero, but it cannot be applied to analytic subsets of C^n.

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The third method, described in Section 3, applies to projective varieties over an algebraically closed field of arbitrary characteristic and to local properties P satisfying some axioms, which do not hold for weak normality in positive characteristic (see [5]).

For the proofs of the results stated here we refer to [4], [5], [11].

1

In this section we shall use the following notations:

1. All topological spaces are assumed to have a countable topology.

2. Whenever $f: X \to Y$ is a holomorphic map between complex spaces, for each $y \in Y$ we denote $X_y := f^{-1}(y)$ with its natural complex structure.

3. According to Bourbaki [3] we say that a subset of a topological space is *meagre* if it is a countable union of *rare* subsets, where a subset A is rare if $\mathring{\bar{A}} = \emptyset$.

We shall call *fat* the complement of a meagre subset, that is, a subset which contains a countable intersection of dense open subsets. We recall that in a complete metric space, by Baire's theorem, every fat subset is dense.

We recall the well-known

THEOREM 1.1. (Sard). *For each differentiable map $f: X \to Y$ between differentiable manifolds, the image of the critical points has Lebesgue measure zero in Y.*

Remark 1.2. From Sard theorem one has in particular:

(S) If X, Y are complex manifolds and if f is holomorphic, then the image of critical points is even analytically meagre, that it is contained in a countable union of locally analytic subsets of Y of codimension $\geqslant 1$ (see [13], Satz 20).

(S′) If X is a complex manifold and $f: X \to \boldsymbol{C}$ is a holomorphic function, then $X_c := f^{-1}(c)$ is a complex manifold for all but countable many $c \in \boldsymbol{C}$.

It is easy to see that from (S′) one can deduce a generalization of the first part of the classical theorem of Bertini such as:

THEOREM 1.3. *Let $X \subset \boldsymbol{C}^n$, $\boldsymbol{P}^n_{\boldsymbol{C}}$ be a non-singular locally analytic subset. Then the "general" hyperplane section of X is non-singular (where "general" means that the set of hyperplanes, for which theorem holds, is a fat subset of the space of all hyperplanes of $\boldsymbol{C}^n$ or $\boldsymbol{P}^n$).*

Observing that the first part of the classical theorem of Bertini can be deduced from Sard theorem, one is induced to try to follow the same way of proof for other local properties.

DEFINITION 1.4. We say that a *local property P satisfies a Sard type theorem* (S″) if the following holds:

(S″) If X is a complex space, which verifies P and $f: X \to C$ is a holomorphic function, then $X_c := f^{-1}(c)$ verifies P for all but countable many $c \in C$.

Remark 1.5. The Sard type theorem (S″) holds for many local properties P such as: regular (for which it is nothing else than (S′)), normal, reduced, weakly normal (which is the same as "maximal" according to Fischer [7]); as you can see in [11], Th. I.4, I.12.

DEFINITION 1.6. We say that a *local property P satisfies* (A) if the following holds:

(A) Let $g: X \to Y$ be a flat morphism of complex spaces and let us assume that X has the property P. Then

$$P_f(X) := \{x \in X \mid X_{f(x)} \text{ is not } P \text{ at } x\}$$

is an analytic subset of X.

Remark 1.7. Condition (A) holds for $P =$ regular, reduced, normal (see [1]), Gorenstein (see [2]) and, if $\dim Y = 1$, for $P =$ weakly normal (see [11], (I.14)).

Then Theorem 1.3 can be generalized to the following:

THEOREM 1.8. *Let X be a complex space satisfying a local property P for which* (S″) *and* (A) *hold. Let L be a holomorphic line bundle on X and $V \subset \Gamma(X, L)$ a finite dimensional linear subspace, which generates L.*

Then there exists a fat subset $M \subset V$ such that for each $s \in M$ the zero set $\{s = 0\}$ is a complex space satisfying P.

Proof. Let $\{K_i\}_{i \in N}$ be a countable covering of X such that for each $i \in N$, K_i is compact and it is contained in an open subset U_i of X on which there exists $G_i \in V$ without zeros on U_i. For each $i \in N$ let

$$M_i := \{s \in V \mid Z := \{s = 0\} \text{ is a 1-codimensional analytic subset of } X, \text{ which has the property } P \text{ in each point of } Z \cap K_i\}.$$

We put $M := \bigcap_{i \in N} M_i$, so we have to prove that each M_i is an open dense subset of V and this follows by adapting the argument of [11] (II.5), since P verifies (S″) and (A). □

COROLLARY 1.9. *Let $X \subset C^n$ be a locally analytic subset for which a locally property P, verifying* (S″) *and* (A), *holds. Then there exists a fat subset M of the space $\mathscr{H}$ of all hyperplanes in C^n, such that for every $H \in M$, $X \cap H$ has the property P.*

When X is compact, M is not only fat, but it is the complement of a proper algebraic subset. In fact it holds:

COROLLARY 1.10. *Let X be a compact complex space for which a local property P, verifying* (S″) *and* (A), *holds. Let L be a holomorphic line bundle on X, $V \subset \Gamma(X, L)$ a linear subspace, which generates L and such that every $s \in V - \{0\}$ is not identically zero on any irreducible component of X.*

Then there exists a proper algebraic subset $A \subset V$ such that for each $s \in V - A$ the zero set $\{s = 0\}$ has the property P.

Proof. One can adapt the argument of [11] (II.7). □

COROLLARY 1.11. *Let $X \subset \mathbf{P}^n_{\mathbf{C}}$ be a closed subvariety, let P be a local property verifying* (S″) *and* (A). *Let V be a finite dimensional linear system on X. Then the general element of V considered as a subvariety of X has the property P, but perhaps at the base points of V and at the points of X which are not P.*

COROLLARY 1.12. *The generic hyperplane section of a complex subvariety X of $\mathbf{P}^n_{\mathbf{C}}$, for which a property P verifying* (S″) *and* (A) *holds, is again P.*

Remark 1.13. By Remarks 1.5 and 1.7, Theorem 1.8 and its Corollaries 1.9, 1.10, 1.11, 1.12 hold for P = regular (and one obtains, in particular, the classical theorem of Bertini), normal (and in this case Corollary 1.12 is the theorem of Seidenberg [14]), reduced.

Since we do not known whether the property P = weakly normal verifies (A), it is not clear if Theorem 1.8 and its corollaries hold for this property.

2

In this section the field k is always algebraically closed.

PROPOSITION 2.1. (see [4] (2.1); [8] (5.1)). *Let k be a field, $X \subset \mathbf{P}^n_k$ a projective variety and $Y \subset X$ a closed subset of X. Let $Y^+ \subset X^+ \subset \mathbf{A}^{n+1}_k$ be the corresponding affine cones; put $R := \mathcal{O}_{X^+, v}$ (where v is the vertex) and let I be the ideal of Y^+ in R (if $Y = \emptyset$ we agree that $Y^+ = \{v\}$). Let P be a local property which is preserved by polynomials and fractions and which descends by faithful flatness.*

Then the following are equivalent:

(i) $X - Y$ *is* P;

(ii) $X^+ - Y^+$ *is* P;

(iii) $\operatorname{spec} R - V(I)$ *is* P.

In particular, X is P if and only if $\operatorname{spec} R - \mathfrak{m}$ *is P, where* $\mathfrak{m}$ *is the maximal ideal of R.*

Remark 2.2. Proposition 2.1 holds, for example, for P = regular, R_s, S_r, normal, reduced (see [12], Cor. 2 p. 154 and (21.E) p. 156); P = weakly normal (see [10], (II.1), (III.2), (IV.2)) and for all these properties it allows us to study whether P holds for X by studying whether P holds for the local ring R. In particular, we can deduce a Bertini type theorem for X from a Bertini type theorem for local rings as follows:

THEOREM 2.3. *Let k be a field of characteristic zero, $(A, \mathfrak{m})$ a local excellent k-algebra, P one of the local properties of Remark 2.2. Let $x_1, \ldots, x_n \in \mathfrak{m}$, let $\lambda = (\lambda_1, \ldots, \lambda_n) \in k^n$; put $x_\lambda := \sum \lambda_i x_i$ and $Z_\lambda := (\operatorname{spec}(A) - V(x_1, \ldots, x_n)) \cap P(A) \cap V(x_\lambda)$(*).*

Then there is a non-empty open set $U \subset k^n$ such that if $\lambda \in U$ and $\mathfrak{p} \in Z_\lambda$, then $A_\mathfrak{p}/x_\lambda A_\mathfrak{p}$ is P.

Proof. See [4], (1.8), for P = weakly normal; [8], (4.1), for P = regular; [8], (4.2), for $P = R_s$, S_r; [8], (4.3), for P = reduced, normal. □

THEOREM 2.4. *Let k be a field of characteristic zero and let X be a closed subvariety of $\mathbf{P}_k^r$. Let $F_0, \ldots, F_r$ be forms of the same degree in $k[x_0, \ldots, x_r]$; put $Y := V(F_0, \ldots, F_n) \cap X$ and for every $\lambda = (\lambda_0, \ldots, \lambda_n) \in k^{n+1}$ let F_λ the hypersurface $\sum \lambda_i F_i = 0$.*

Then for every property P of Remark 2.2 there is a non-empty open subset $U \subset k^{n+1}$ such that $(P(X-Y)) \cap F_\lambda \subseteq P(X \cap F_\lambda)$(*) for all $\lambda \in U$.

Proof. It is an easy consequence of Proposition 2.1 and Theorem 2.3. □

COROLLARY 2.5. *Let P be one of the properties of Remark 2.2 and let X be a projective variety over a field of characteristic zero. If X satisfies P, then the general hyperplane section of X satisfies P.*

Remark 2.6. We do not known whether Theorem 2.3 is true if k is a field of positive characteristic; therefore in this way we have the Bertini type theorems like Theorem 2.4 and Corollary 2.5, only for projective varieties over a field of characteristic zero.

3

Let P be a local property of Noetherian schemes, satisfying the following axioms:

(A 1) Whenever ϕ: $Y \to Z$ is a flat morphism with regular fibers and Z is P, then Y is P too.

(A 2) Let ϕ: $Y \to S$ be a morphism of finite type, where Y is excellent

(*) $P(A) := \{\mathfrak{p} \in \operatorname{spec} A \mid A_\mathfrak{p}$ has the property $P\}$.
$P(X-Y) := \{x \in X - Y \mid \mathcal{O}_{X,x}$ has the property $P\}$.

and S integral with generic point η; if Y_η is geometrically P (i.e., $Y_\eta \otimes_{k(\eta)} K$ is P for every field extension $K/k(\eta)$), then there exists an open neighbourhood U of η in S such that Y_s is geometrically P for each $s \in U$.

(A 3) P is open on schemes of finite type over a field.

Remark 3.1. It is known that axioms (A 1), (A 2), (A 3) are satisfied by properties: regular, normal, reduced, R_s, S_r (see [12], (21.E), p. 156; [6], n. 28, (9.9.5), n. 24, (7.8.6)).

Axioms (A 2) and (A 3) holds also for weak normality (see [5], Th. 2; [10], (IV.3)), but this property does not satisfy (A 1) if $\operatorname{char} k > 0$ (see [5], Section III).

THEOREM 3.2. *Let X be a scheme of finite type over an algebraically closed field k, let ϕ: $X \to \boldsymbol{P}_k^n$ be a morphism with separably generated (not necessary algebraic) residue field extensions. Suppose X has a local property P satisfying* (A 1) *and* (A 2). *Then there exists a non-empty open subset U of $(\boldsymbol{P}_k^n)^*$ such that $\phi^{-1}(H)$ has the property P for each hyperplane $H \in U$.*

Proof. Let $\boldsymbol{P} := \boldsymbol{P}_k^n$ and let Z be the reduced subscheme of $\boldsymbol{P} \times \boldsymbol{P}^*$ whose set of closed points is $\{(x, H) \in \boldsymbol{P} \times \boldsymbol{P}^* \mid x \in H\}$. Let us consider the commutative diagram

$$\begin{tikzcd}
X \times_{\boldsymbol{P}} Z \arrow[rr, "\sigma"] \arrow[dr, "\varrho"] \arrow[dd] & & Z \arrow[dl, "\pi"'] \arrow[d] \\
& \boldsymbol{P}^* & \boldsymbol{P} \times \boldsymbol{P}^* \arrow[l] \arrow[d] \\
X \arrow[rr, "\phi"] & & \boldsymbol{P}
\end{tikzcd}$$

where σ, π are the canonical projections.

For each hyperplane $H \subset \boldsymbol{P}$ considered as an element of $\boldsymbol{P}^*$, the preimage $\phi^{-1}(H)$ is a fiber of the morphism ϱ, hence we can apply (A 2) and we have to prove that the generic fiber of ϱ (i.e., $X \times_{\boldsymbol{P}} Z_\eta$ where η is the generic point of $\boldsymbol{P}^*$) is geometrically P as $k(\eta)$-scheme, that is $(X \times_{\boldsymbol{P}} Z_\eta) \otimes_{k(\eta)} K$, is P for every field extension $K/k(\eta)$.

We can consider the flat morphism $(X \times_{\boldsymbol{P}} Z_\eta) \otimes_{k(\eta)} K \xrightarrow{q} X$ and, since X has the property P, we can apply (A 1) and the problem is reduced to prove that the fibers of q are regular. This is rather technical and follows by our assumptions on residue field extension, by adapting an argument of J. P. Jouanolou (see [5], Lemma 2).

COROLLARY 3.3. *Let X be an algebraic variety over an algebraically closed field k and let S be a finite dimensional linear system on X. Assume that the rational map $X \cdots\to \boldsymbol{P}^n$ corresponding to S induces (whenever defined) separably generated field extensions.*

Let P be a property satisfying (A 1), (A 2) *and* (A 3). *Then the general element of S, considered as a subscheme of X has the property P but perhaps at the base points of S and at the points of X which are not P.*

COROLLARY 3.4. *Let $X \subset \mathbf{P}^n_k$ $(k = \bar{k})$ be a closed subscheme and let P be a local property satisfying* (A 1) *and* (A 2).

Then if X is P, the general hyperplane section of X is P. Moreover, if P satisfies (A 3), *then the P-locus of X is preserved by the general hyperplane section, that is $P(V \cap H) \supseteq P(V) \cap H$ for the general hyperplane H of $\mathbf{P}^n$.*

Remark 3.5. Since weak normality does not satisfy axiom (A 1) in positive characteristic, it remains open the problem whether a Bertini type theorem like Corollary 3.4 holds for weakly normal varieties over a field of positive characteristic. At the same time, we do not known whether a Bertini type theorem like Corollary 1.9 holds for weakly normal locally analytic subsets of $\mathbf{C}^n$.

References

[1] C. Bănică, *Le lieu réduit et le lieu normal d'un morphisme*, in *Proc. of the Romanian–Finnish Seminar on Complex Analysis, Bucharest, 1976*, Lecture Notes in Math. 743, Springer-Verlag, Berlin 1979, 389–398.

[2] C. Bănică and M. Stoia, *Gorenstein points of a flat morphism of complex spaces*, Revue Roumaine Math. Pures Appl. 26 (1981), 687–690.

[3] N. Bourbaki, *Topologie générale*, Hermann, Paris 1964.

[4] C. Cumino, S. Greco and M. Manaresi, *Bertini theorems for weak normality*, Compositio Math. 48 (1983), 351–362.

[5] —, —, —, *An axiomatic approach to the second theorem of Bertini*, J. Algebra 98 (1986), 171–182.

[6] A. Grothendieck, *Eléments de géométrie algébrique*, Inst. Hautes Études Sci. Publ. Math. 24, 28, Bures-sur-Yvette 1965, 1966.

[7] G. Fischer, *Complex Analytic Geometry*, Lecture Notes in Math. 538, Springer-Verlag, Berlin 1977.

[8] H. Flenner, *Die Sätze von Bertini für lokale Ringe*, Math. Ann. 229 (1977), 97–111.

[9] J.-P. Jouanolou, *Théorèmes de Bertini et applications*, Progr. Math. 42, Birkhäuser, Boston 1983.

[10] M. Manaresi, *Some properties of weakly normal varieties*, Nagoya Math. J. 77 (1980), 61–74.

[11] —, *Sard and Bertini type theorems for complex spaces*, Ann. Mat. Pura Appl. (4) 131 (1982), 265–279.

[12] H. Matsumura, *Commutative Algebra*, 2nd ed., Benjamin-Cummings, Reading 1980.

[13] R. Remmert, *Holomorphe und meromorphe Abbildungen komplexer Räume*, Math. Ann. 133 (1957), 328–370.

[14] A. Seidenberg, *The hyperplane sections of normal varieties*, Trans. Amer. Math. Soc. 50 (1941), 357–386.

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