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AN INTRODUCTION TO SHAPE THEORY FOR THE NON-SPECIALIST *

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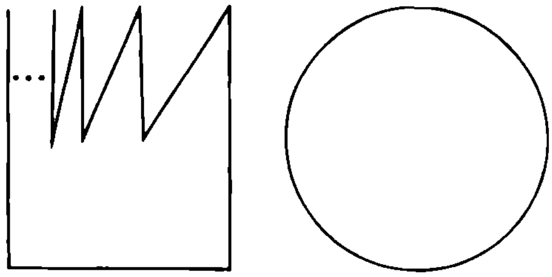
Although shape theory applies to general topological spaces, in this introduction we will only consider compact metric spaces. Moreover, we will concentrate on a few examples and techniques in an attempt to convey some of the basic ideas and yet keep the paper reasonably self-contained with respect to shape theory. To reduce the abstractness of the subject we deal with the relationship of shape theory and some geometric notions. More specifically, we consider its interaction with cell-like mappings and their variations.

Shape theory is like homotopy theory in that it studies the global properties of topological spaces. However, the approach used in homotopy theory is of such a nature that it yields interesting results only for spaces which behave well locally (like ANR's). On the other hand, the tools of shape theory are so designed that they yield interesting results in the case of bad local behavior (like that which occurs in metric compacta). Moreover, shape theory does not modify homotopy theory on ANR's, i.e., it agrees with homotopy theory on such spaces.

It should be mentioned that one can not ignore spaces with bad local properties since they arise in nice settings, for example, they show up as fibers of maps between spaces with good local properties. In an attempt to overcome such difficulties K. Borsuk [2] undertook the development of shape theory in 1968. One would expect shape theory to yield a classification of metric compacta, weaker than homotopy type but coinciding with it when applied to ANR's.

EXAMPLE 1. Let X denote the Warsaw circle $W \subset R^2$, and Y denote the unit circle $S^1 \subset R^2$.

* This paper is in final form and no version of it will be submitted for publication elsewhere.



Then there are maps $f: X \to Y$ which are essential (i.e., not homotopic to a constant map), but there is no essential map of Y into X since the image of Y in X would have to be a locally connected continuum and the only locally connected subcontinua of X are arcs or points. Since all maps $g: Y \to X$ are homotopic to a constant map, $fg \simeq 0$, and so they are not homotopic to the identity map on Y. Therefore, $Y \not\simeq X$ (i.e., they are of different homotopy type). Basically, there are not enough maps of Y into X due to local difficulties to get the two spaces to be of the same homotopy type.

Borsuk's idea to overcome this difficulty was to introduce a notion more general than mapping, called fundamental sequences. Borsuk was able to generalize mappings and yet maintain a great deal of the geometry inherent in the original notion. Roughly, these fundamental sequences are based on having spaces X, Y embedded in the Hilbert cube Q and considering maps of Q into itself which behave in a certain way on neighborhoods of X and Y. This turned out to be the basic notion which allowed Borsuk to construct a theory of shape to treat the global properties of metric compacta. In particular, one expects the Warsaw circle W to be in the same shape class as S^1 because of their global similarities (e.g., they both divide the plane into two components).

After hearing Borsuk [3] speak at the 1968 topology conference in Herceg-Novi, Yugoslavia, S. Mardešić and J. Segal [25] (or [27]) decided they could give a more categorical description of shape theory using ANR-systems which would also generalize the theory to the compact Hausdorff case. Now we give a brief description of this ANR-sequence approach to shape theory for compact metric spaces.

1. ANR-sequences

We consider ANR-sequences, i.e., $\boldsymbol{X} = \{X_n, p_{n,n+1}\}$, where X_n is a compact ANR for compact metric spaces and $p_{n,n+1}: X_{n+1} \to X_n$ is a continuous map for all $n \in N$ the positive integers. These ANR-sequences will be organized into equivalence classes so that one can use any representative to denote the class. In this paper we say that an ANR-sequence $\boldsymbol{X}$ is associated with a space X if $X = \lim \boldsymbol{X}$, i.e., X is the inverse limit of $\boldsymbol{X}$. This allows one to use any such sequence associated with X. Either X is described this way to begin with as in the case of the solenoids or one obtains such an ANR-sequence associated with X through some construction. Such an ANR-sequence associated with X always exists. A map of ANR-sequences $\boldsymbol{f}$: $\boldsymbol{X} \to \boldsymbol{Y}$ consists of an increasing function f: $N \to N$ and a collection of maps $\{f_n\}, f_n$: $X_{f(n)} \to Y_n$ such that the following diagram

$$\begin{tikzcd} X_{f(n)} \arrow[d, "f_n"'] \arrow[r, phantom, "\simeq" description, shift right=6] & X_{f(n+1)} \arrow[l, "p"'] \arrow[d, "f_{n+1}"] \\ Y_n & Y_{n+1} \arrow[l, "q"] \end{tikzcd} \tag{1}$$

commutes up to homotopy (where we delete subscripts from bonding maps) i.e., $f_n p_{f(n),f(n+1)} \simeq q_{n,n+1} f_{n+1}$.

The identity map $\mathbf{1}$: $\boldsymbol{X} \to \boldsymbol{X}$ is given by $1(n) = n$, $1_n = \text{id}$. The composition of maps of sequences $\boldsymbol{f}$: $\boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{g}$: $\boldsymbol{Y} \to \boldsymbol{Z} = \{Z_n, r_{n,n+1}\}$ is the map $\boldsymbol{h} = \boldsymbol{gf}$: $\boldsymbol{X} \to \boldsymbol{Z}$ defined by $h = fg$: $N \to N$ and for h_n: $X_{h(n)} \to Z_n$ we take $g_n f_{g(n)}$. So the increasing function $h = f(g)$: $N \to N$ and the collection $\{h_n\}$ form a map of ANR-sequences $\boldsymbol{h}$: $\boldsymbol{X} \to \boldsymbol{Z}$.

Now we define homotopy for maps of sequences. $\boldsymbol{f}, \boldsymbol{g}$: $X \to Y$ are homotopic (written $\boldsymbol{f} \simeq \boldsymbol{g}$), if for every $n \in N$, there is an $n' \in N$ such that the following diagram commutes up to homotopy

$$\begin{tikzcd} & X_{n'} \arrow[dl, "p"'] \arrow[dr, "p"] \arrow[dd, phantom, "\simeq" description] & \\ X_{f(n)} \arrow[dr, "f_n"'] & & X_{g(n)} \arrow[dl, "g_n"] \\ & Y_n & \end{tikzcd} \tag{2}$$

This homotopy relation for maps of ANR-sequences is an equivalence relation and classifies all maps of ANR-sequences associated with X to those associated with Y. These classes are called the shape maps from X to Y, written f: $X \to Y$. A continuous map f: $X \to Y$ always determines a shape map f: $X \to Y$. The converse is not true in general. For example, consider a shape map f: $\boldsymbol{X} \to \boldsymbol{Y}$, i.e.,

$$\begin{tikzcd} X_{f(1)} \arrow[d, "f_1"'] & X_{f(2)} \arrow[l, "p"'] \arrow[d, "f_2"] & X_{f(3)} \arrow[l, "p"'] \arrow[d, "f_3"] & \cdots X \arrow[l] \\ Y_1 & Y_2 \arrow[l, "q"] & Y_3 \arrow[l, "q"] & \cdots Y \arrow[l] \end{tikzcd} \tag{3}$$

and note the squares in the diagram only commute up to homotopy (not exactly) so one does not expect to get a continuous map from X to Y but only a shape map from X to Y. We do have a special case though when Y is an ANR, then any shape map into Y is induced by a continuous map into Y.

In analogy with homotopy theory we define two spaces X and Y to be of the same shape ($\operatorname{Sh} X = \operatorname{Sh} Y$) if and only if there exist shape maps $\boldsymbol{f}$: $X \to Y$ and $\boldsymbol{g}$: $Y \to X$ such that (1) $\boldsymbol{fg} \simeq \mathbf{1}_Y$ and (2) $\boldsymbol{gf} \simeq \mathbf{1}_X$. In case (1) and (2) hold $\boldsymbol{f}$ is called a shape equivalence. Further we say that X is shape dominated by Y ($\operatorname{Sh} X \leqslant \operatorname{Sh} Y$) provided (2) holds. In this case $\boldsymbol{f}$ is called a shape domination. Note that in the special case that X and Y are ANR's if $\operatorname{Sh} X = \operatorname{Sh} Y$ then $X \simeq Y$.

EXAMPLE 2. In general X and Y may have the same shape but be of different homotopy type. Let X denote the Warsaw circle W, and Y denote the circle S^1. Then we consider the ANR-sequence $\boldsymbol{X} = \{X_n, p_{n,n+1}\}$ associated with X given by $X_n = S^1$ for each n and $p_{n,n+1}$: $X_{n+1} \to X_n$ is a properly chosen degree one map. We also use an ANR-sequence $\boldsymbol{Y} = \{Y_n, q_{n,n+1}\}$ associated with Y given by $Y_n = S^1$ for each n and $q_{n,n+1}$ is the identity map on S^1. For the shape maps $\boldsymbol{f}$: $\boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{g}$: $\boldsymbol{Y} \to \boldsymbol{X}$ we take $f_n = g_n =$ identity on S^1. Then $\boldsymbol{fg} \simeq \mathbf{1}_Y$ and $\boldsymbol{gf} \simeq \mathbf{1}_X$. To obtain the necessary commutativity up to homotopy in the diagrams below we recall certain well-known facts about the degree of maps from S^1 to itself. These are (1) $\deg(f(g)) = \deg g \cdot \deg f$, (2) if $\deg f = \deg g$, then $f \simeq g$, and (3) $\deg(\mathrm{id}) = 1$.

$$
\begin{tikzcd}
 & Y_{g(f(n))} \arrow[dl, "\mathrm{id}"'] \arrow[dr, "q"] & \\
Y_{g(f(n))} \arrow[dr, "f_n g_{f(n)}"'] \arrow[rr, phantom, "\simeq"] & & Y_n \arrow[dl, "\mathrm{id}"] \\
 & Y_n &
\end{tikzcd}
,\qquad
\begin{tikzcd}
 & X_{f(g(n))} \arrow[dl, "\mathrm{id}"'] \arrow[dr, "p"] & \\
X_{f(g(n))} \arrow[dr, "g_n f_{g(n)}"'] \arrow[rr, phantom, "\simeq"] & & X_n \arrow[dl, "\mathrm{id}"] \\
 & X_n &
\end{tikzcd}
\tag{4}
$$

So we have $\operatorname{Sh} X = \operatorname{Sh} Y$ but by Example 1 we know X and Y are of different homotopy type.

EXAMPLE 3. Here we describe two circle-like continua of different shape. Let X denote the dyadic solenoid, i.e., X is the inverse limit of a inverse sequence of circles $X_n = S^1$ and all bonding maps $p_{n,n+1}$ are maps of degree 2. Let Y denote S^1 as described in Example 2. Then any shape map $\boldsymbol{g}$: $Y \to X$ must have the property that $\deg g_n \neq 1$ (due the commutativity up to homotopy in the diagram below). Then relation (2) of the definition of the homotopy of two maps of ANR-sequences can not hold, since if it did we would have $f_n(g_n) \simeq 1_Y$ so $\deg f_n \cdot \deg g_n = 1$ which is impossible. Therefore we have that X and Y are of different shape.

$$
\begin{tikzcd}
X_n \arrow[d, phantom, ""] & X_{n+1} \arrow[l, "p"'] \\
Y_{g(n)} \arrow[u, "g_n"] \arrow[ur, phantom, "\simeq"] & Y_{g(n+1)} \arrow[l, "q"] \arrow[u, "g_{n+1}"']
\end{tikzcd}
\tag{5}
$$

2. Pro-groups

Various functors of algebraic topology such as Čech homology or cohomology are shape invariants. In addition to such classical invariants, it is possible to describe new continuous functors called the shape groups by taking inverse limits of inverse sequences of homotopy groups. Furthermore, if one does not pass to the limit in this situation, one obtains the homotopy pro-groups which are an even more delicate shape invariant.

For every ANR-sequence $\boldsymbol{X} = \{X_n, p_{n,n+1}\}$ one can define the homology pro-groups. These are inverse sequences of groups $H_m(\boldsymbol{X}) = \{H_m(X_n), p_{n,n+1*}\}$, objects the category of pro-groups. Here we are taking the integers as the coefficient groups but one could use any abelian group. If $\boldsymbol{X}$ and $\boldsymbol{X}'$ are two ANR-sequences associated with X, then $H_m(\boldsymbol{X})$ and $H_m(\boldsymbol{X}')$ are naturally isomorphic pro-groups, i.e., they are isomorphic objects of pro-Group. Therefore, one can define the homology pro-groups of a compact metric space X as the homology pro-groups of an associated ANR-sequence $\boldsymbol{X}$ since they are determined up to isomorphism in pro-Group. Clearly, isomorphic pro-groups have isomorphic inverse limits but the converse is not always true. For example, consider the pro-group $\boldsymbol{G} = \{G_n, \varphi_{n,n+1}\}$ where each G_n is a copy of the integers Z and each $\varphi_{n,n+1}$ is the homomorphism determined by multiplication by 2. Then $\boldsymbol{G}$ is not isomorphic to the zero pro-group $\{0\}$ although they both have as their inverse limit the zero group.

The inverse limit of the homology pro-group $H_m(\boldsymbol{X})$ is the usual Čech homology group $\check{H}_m(X)$. Homology pro-groups are finer invariants than the Čech homology groups. For example, $\check{H}_1$ of the dyadic solenoid is zero but the corresponding pro-group is nontrivial (as shown in the previous paragraph). In a similar way one defines the mth homotopy pro-groups $\pi_m(\boldsymbol{X}, \boldsymbol{x})$

$$\text{pro-}\pi_m(X, x) = \pi_m(\boldsymbol{X}, \boldsymbol{x}) = \{\pi_m(X_n, x_n), p_{n,n+1\,\#}\}$$

and the mth shape groups

$$\check{\pi}_m(X, x) = \lim_n \{\pi_m(X_n, x_n), p_{n,n+1\,\#}\}$$

(which depends on the choice of base point).

3. Whitehead theorem in shape theory

Recall the classical Whitehead theorem from homotopy theory: If a map $f\colon (X, *) \to (Y, *)$ of connected CW-complexes induces isomorphisms of all homotopy groups $f_{n\#}\colon \pi_n(X, *) \to \pi_n(Y, *)$ is an isomorphism for all n, then f is a homotopy equivalence. The importance of this theorem arises from the fact that it uses algebraic information to obtain homotopy information. The next example shows that without the restriction that X and Y be CW-complexes the theorem is no longer true.

EXAMPLE 4. Let X denote the Warsaw circle $W \subset R^2$, $Y = *$, and f the only map from X to Y. Since X fails to be locally connected on the limit segment, it is seen that every map $(S^n, *) \to (X, *)$ is inessential, i.e., $\pi_n(W, *) = 0 = \pi_n(*, *)$. Consequently, $f_{n\#}$ is an isomorphism for all $n = 0, 1, 2, \ldots$ Nevertheless, f fails to be a homotopy equivalence since X is not contractible. This can be seen by noting that the Čech homology group $\check{H}_1(X) = Z \neq 0$.

The following is a shape version of the Whitehead theorem (see [29], [23], [28], and [12]). Notice that the spaces are now only required to be continua and that the homotopy pro-groups have replaced the homotopy groups.

WHITEHEAD THEOREM (Shape theory version). *If a shape map f: $(X, *) \to (Y, *)$ of continua induces isomorphisms of all the homotopy pro-groups*

$$f_{n\#}\colon \text{pro-}\pi_n(X, *) \to \text{pro-}\pi_n(Y, *)$$

for $n \leqslant m$ and $\max(\operatorname{ddim} X, \operatorname{ddim} Y) \leqslant m$, *then f is a shape equivalence.*

EXAMPLE 5. In Example 2 we showed that the Warsaw circle X and the circle Y were of the same shape. Here we use the Whitehead theorem to obtain a shape equivalence between them. Define $f = \{f_n\}$ by taking each f_n to be a map of degree 1 of S^1 into itself. Then by applying the π_1 functor to the diagram in Example 2 we get the following diagram with pro-π_1 of the spaces represented horizontially and the vertical arrows representing the isomorphisms $f_{n\#}$. Then the shape version of the Whitehead theorem implies that f is a shape equivalence.

$$\begin{tikzcd}
\pi_1(X_1, x_1) \arrow[d, "f_{1\#}"'] \arrow[dr, phantom, "="] & \pi_2(X_2, x_2) \arrow[l, "p_\#"'] \arrow[d, "f_{2\#}"] & \cdots \arrow[l] \\
\pi_1(Y_1, y_1) & \pi_2(Y_2, y_2) \arrow[l, "q_\#"] & \cdots \arrow[l]
\end{tikzcd} \tag{6}$$

Note that a dimension restriction has been imposed. This involves the deformation dimension which can be defined as follows. The deformation dimension of X is less than or equal to m (written $\operatorname{ddim} X \leqslant m$) if and only if every map f: $X \to P$, a polyhedron, is homotopic to map g: $X \to P$ with $g(X) \subset P^{(m)}$, the m-skeleton of P. The theorem fails without a dimension restriction as is shown in the following example.

EXAMPLE 6 (The Kahn continuum). This example depends on the work of Adams [1] or Toda and is based on a map A of the rth suspension of a ceratin compact polyhedron Y (of the form $S^k \cup_q B^{k+1}$) to Y, A: $\Sigma^r Y \to Y$. The crucial property of this map is that for any positive integer s, the composition

$$A \circ \Sigma^r A \circ \ldots \circ \Sigma^{r(s-1)} A\colon \Sigma^{rs} Y \to Y$$

is an essential map. Let K denote the inverse limit of the following inverse sequence.

$$Y \leftarrow \Sigma^r Y \leftarrow \Sigma^{2r} Y \leftarrow \dots$$

with the nth bonding map given by $\Sigma^{(n-1)r} A$. It follows that pro-$\pi_n(K, *) = 0$, for all n, but K is not of trivial shape.

4. Movability

In [4] Borsuk introduced a far-reaching generalization of ANR's, called movability. The name comes from a geometric interpretation of Borsuk's original definition. After its restatement in the ANR-system approach [26] it became apparent that this is a categorical notion. So although the definition which follows is for spaces and maps it applies more generally (see [27, p. 164]).

DEFINITION OF MOVABILITY. A compactum X is *movable* if there exists an ANR-sequence $X = \{X_n, p_{n,n+1}\}$ associated with X such that for every positive integer n, there exists $n' \geqslant n$ such that for all $n'' \geqslant n$, there is a map $r\colon X_{n'} \to X_{n''}$ satisfying $p_{nn''} r \simeq p_{nn'}$.

EXAMPLE 7. Recall from Example 3 the description of the dyadic solenoid X as the inverse limit of a sequence of circles with bonding maps of degree 2. Applying the H_1 functor to this sequence, we get pro-$H_1(X)$ as the following inverse sequence

$$Z \leftarrow Z \leftarrow Z \leftarrow \dots$$

with each bonding homomorphism given by multipication by 2. Then pro-$H_1(X)$ is not movable as a pro-group (and so X is not movable as a space). To see this suppose otherwise and take $n = 1$ in the definition. Then for $n'' = n'+1$ we would have a homomorphism $r_*\colon H_1(S^1) \to H_1(S^1)$. So we would have $p_{nn''} r_* = p_{nn'}$. Thus $2^{n''-1} r_*(1) = 2^{n'-1}$ which implies that $r_*(1)$ is not an integer, a contradiction.

In the case of movable spaces one can pass to the limits of the pro-homotopy groups in the Whitehead theorem and obtain the following movable version of the theorem.

MOVABLE VERSION OF THE WHITEHEAD THEOREM. *Let $f\colon (X, *) \to (Y, *)$ be a shape map of movable continua. If f induces isomorphism of all shape groups*

$$f_\#\colon \check{\pi}_n(X, *) \to \check{\pi}_n(Y, *)$$

for $n \leqslant m$ and $\max(\operatorname{ddim} X, \operatorname{ddim} Y) \leqslant m$, *then f is a shape equivalence.*

5. Some geometric manifestations of shape theory

T. A. Chapman [5] or [7] has given an geometric characterization of shape in terms of infinite dimensional topology. The setting will be in the Hilbert cube Q. We need the notion of Z-set due to R. D. Anderson but we take an alternate definition due to H. Toruńczyk [32]. A Z-set in Q is a closed subset $X \subset Q$ such that for any integer n, map $f\colon I^n \to Q$ and $\varepsilon > 0$, there exists a map $g\colon I^n \to Q - X$ satisfying $\operatorname{dist}(f, g) < \varepsilon$. Every metric compactum embeds in Q as a Z-set.

Chapman's Complement Theorem. *Let X, $Y \subset Q$ be two Z-sets. Then X and Y have the same shape iff $Q-X$ and $Q-Y$ are homeomorphic.*

There are a number of finite dimensional versions of this theorem (see for example [6], [17] and [33]).

Borsuk also introduced an m-dimensional stratification of movability, called *m-movability*.

Definition of m-movability. An ANR-sequence $X = \{X_n, p_{n,n+1}\}$ is *m-movable* iff for each $n \in N$, there is an $n' \geqslant n$ such that for all $n'' \geqslant n$ and any map of an m-dimensional polyhedron into $X_{n'}$ $f\colon K \to X_{n'}$, there is a map $g\colon K \to X_{n''}$ with

$$p_{nn''}\, g \simeq p_{nn'} f,$$

that is, the following diagram commutes up to homotopy.

$$\begin{tikzcd} & K \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X_{n'} \arrow[dr, "p"'] & \simeq & X_{n''} \arrow[dl, "p"] \\ & X_n & \end{tikzcd} \tag{7}$$

Then a metric compactum X is said to be n-movable if it has an n-movable ANR-sequence associated with it.

We need some notions from pro-groups which are useful here (see [27]). One is the Mittag–Leffler condition (ML). It essentially says that the images under the bonding homomorphisms stabilize in G_n. Every movable pro-group is ML, but not conversely. The pro-group described in Example 7 is not ML.

Definition of the Mittag-Leffler condition. A pro-group $G = \{G_n, \varphi_{n,n+1}\}$ satisfies the *Mittag–Leffler condition* (or is ML) if for any $n \in N$, there is an $n' \geqslant n$ such that for any $n'' \geqslant n'$ we have

$$\varphi_{nn''}(G_{n''}) = \varphi_{nn'}(G_{n'}).$$

The other notion is stability.

DEFINITION OF STABILITY. A pro-group G is *stable* if it is isomorphic in pro-Group to a group. (The stability of a pro-group implies its movability but not conversely.)

In shape theory the question of the pointed versus unpointed version of results is a serious problem (see [16] or [27]).

LEMMA. *A pointed continuum X is pointed 1-movable iff $(X, *)$ is 1-movable for some $* \in X$.*

THEOREM 1. *Let $(X, *)$ and $(Y, *)$ be pointed continua. If* $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ *and $(X, *)$ is 1-movable, then* $\mathrm{Sh}(X, *) = \mathrm{Sh}(Y, *)$.

COROLLARY. *A continuum X is pointed 1-movable iff $(X, *)$ is 1-movable for some $* \in X$.*

An interesting problem in shape theory is when does a shape class have a nice representative, for example, when does it have a locally connected member? J. Krasinkiewicz [20] obtained the following characterization for this case.

THEOREM 2. *A continuum X has the shape of a locally connected continuum iff X is pointed 1-movable.*

EXAMPLE 8. The dyadic solenoid X does not have the shape of any locally connected continuum since pro-$\pi_1(X, *)$ is not ML.

S. Ferry [15] generalized Theorem 2 obtaining the following algebraic characterization of metric continua having the shape of a LC^m continuum (homotopy locally connected up to dimension m).

THEOREM 3. *A continuum X has the shape of a* LC^m *continuum iff* pro-$\pi_n(X)$ *is stable for $0 \leqslant n \leqslant m$ and is* ML *for $n = m+1$.*

6. Cell-like mappings and shape theory

A map $f\colon X \to Y$ between metric spaces is called a cell-like map if, for all y in Y, $\mathrm{Sh}(f^{-1}(y)) = \mathrm{Sh}(\text{point})$, i.e., is of trivial shape. The class of cell-like maps is of central importance in geometric topology. Between ANR's and, in particular, manifolds the importance of cell-like maps is seen from their role in the work of L. C. Siebenmann [30], R. D. Edwards. and J. E. West [35]. However, J. L. Taylor's example [31] of a cell-like map from the Kahn space (Example 6) onto the Hilbert cube which is not a shape equivalence showed the need to limit cell-like maps in this more general setting. G. Kozlowski [19] did this by introducing the notion of hereditary shape equivalence. A mapping $f\colon X \to Y$ of X onto Y is a hereditary shape equivalence iff $f|f^{-1}(C)\colon f^{-1}(C) \to C$ is a shape equivalence for all closed subsets C of X. By restricting C to the points of X one sees that hereditary shape equivalences are cell-like maps. Hereditary shape equivalences behave well with respect to quotients and agree with cell-like maps on spaces which have a strong local structure (e.g., cell-like maps are hereditary shape equivalences when they map between ANR's or when the range has finite dimension). Kozlowski used this shape theoretic notion to give the following elegant characterization of the cell-like images of ANR's.

THEOREM 4. *If f: $X \to Y$ is a cell-like map and X is an* ANR, *then Y is an* ANR *iff f is a hereditary shape equivalence.*

J. E. West [36] has proven that every compact ANR Y is the image of a compact Q-manifold X under a cell-like map. A cell-like map of metric compacta induces an isomorphism of all homotopy pro-groups of the spaces.

THEOREM 5. *If X, Y are both finite dimensional metric compacta, then a cell-like map f: $X \to Y$ is a shape equivalence.*

EXAMPLE 9 (Taylor's map). Here we describe Taylor's cell-like map f: $K \to Q$ of the Kahn space K (Example 6) onto the Hilbert cube which fails to be a shape equivalence. We use the notation of Example 6 and observe that the mth suspension $\Sigma^m Y$ of a compactum Y can be regarded as the space obtained from $I^m \times Y$ by identifying $\{s\} \times Y$ to a point for each $s \in \partial I^m$. Hence there is a surjection f_m: $\Sigma^m Y \to I^m$ with $f_m^{-1}(s)$ a point if $s \in \partial I^m$ and with $f_m^{-1}(s)$ homeomorphic to Y if $s \in I^m - \partial I^m$.

Let $p_{n,n+1}$: $I^{(n+1)r} = I^{nr} \times I^r \to I^{nr}$ denote the first projection and let $q_{n,n+1} = \Sigma^{nr} A$. Then

$$f_{nr} q_{n,n+1} = p_{n,n+1} f_{(n+1)r}.$$

Hence, the maps f_{nr}, $r = 1, 2, \ldots$, induce a map

$$f\colon K \to Q = \lim(I^{nr}, p_{n,n+1}).$$

If $s = (s_n) \in Q$, i.e., $s_n \in I_{nr}$ and $p_{n,n+1}(s_{n+1}) = s_n$, n a positive integer, then

$$f^{-1}(s) = \lim(f_{nr}^{-1}(s_n), h_{n,n+1}),$$

where $h_{n,n+1}$ is the map given by restricting $q_{n,n+1}$ to $f_{(n+1)r}^{-1}(s_{n+1})$.

In order to show that f is a cell-like map, it suffices to show that each $h_{n,n+1}$ is null-homotopic. If $s_{n+1} \in \partial I^{(n+1)r}$, then $f_{(n+1)r}^{-1}(s_{n+1})$ is a point. If $s_{n+1} \in \operatorname{Int}(I^{(n+1)r})$, then $q_{n,n+1} | f_{(n+1)r}^{-1} p_{n,n+1}^{-1}(s_n)$ is just the map A: $\Sigma^r Y \to Y$, and $h_{n,n+1}$ is also A restricted to $f_r^{-1}(t)$ restricted to some $t \in I^r$. The restriction of A to $f_r^{-1}(\gamma)$, where γ is an arc joining t to ∂I_r, yields a homotopy connecting $A | f_r^{-1}(t)$ to a constant map, hence, a homotopy connecting $h_{n,n+1}$ to a constant map. Since K is not of trivial shape and Q is, we have that f is not a shape equivalence.

A problem of current interest is whether or not there exists a cell-like map which raises dimension. This is related to the classical dimension theory problem (due to P. S. Aleksandrov) of whether there exists an infinite dimensional compactum with finite cohomological dimension. R. D. Edwards and J. J. Walsh [34] have shown that these problems are equivalent. G. Kozlowski [19] has shown that the dimension raising cell-like map problem can be formulated in shape theory as follows: Suppose that X is a finite dimensional compactum and f: $X \to Y$ is a cell-like map, then is f a shape equivalence?

7. Fibrations and shape theory

Recall the homotopy lifting property and the notion of Hurewicz fibration from homotopy theory. A map p: $E \to B$ has the homotopy lifting property (HLP) fo a space X provided given maps h, H in the commutative diagram

$$\begin{array}{ccc} X\times 0 & \xrightarrow{h} & E \\ \cap & & \downarrow p \\ X\times I & \xrightarrow[H]{} & B \end{array} \tag{8}$$

there is a map $\tilde{H}$: $X \times I \to E$ which extends h such that $p\tilde{H} = H$. Then a mapping p: $E \to B$ is a (Hurewicz) fibration if it has the HLP for all spaces.

Motivated by the work of Lacher [21, 22] and Kozlowski [18] on cell-like mappings, D. Coram and P. F. Duvall [9] have considered the following approximate homotopy lifting property (AHLP):

A map p: $E \to B$ between metric spaces has the AHLP with respect to a class $\mathscr{X}$ of topological spaces provided every $\varepsilon > 0$ admits a $\delta > 0$ such that each map h: $X \to E$, $X \in \mathscr{X}$ and for each homotopy H: $X \times I \to B$ with distance

$$d(ph, H_0) \leqslant \delta \tag{9}$$

there exists a homotopy $\tilde{H}$: $X \times I \to B$ satisfying

$$d(\tilde{H}_0, h) < \varepsilon \tag{10}$$

and

$$d(p\tilde{H}, H) < \varepsilon. \tag{11}$$

An approximate fibration p: $E \to B$ is a map between compact ANR's which has the AHLP with respect to the class of all topological spaces. In case E and B are ANR's one can restate the definition of AHLP by replacing (9) and (10) by the equalities $ph = H_0$ and $\tilde{H}_0 = h$. Coram and Duvall have also shown that in the definition of approximate fibration one can replace the class $\mathscr{X}$ of all topological spaces by the class of compact polyhedra. This and earlier results of Lacher prove that a cell-like map between compact ANR's is an approximate fibration.

Coram and Duvall also show that approximate fibrations have several shape theoretic properties analogous to the corresponding homotopic theoretic properties of fibrations. In particular, for an arbitrary base point $*$ in E, an approximate fibration p: $E \to B$ induces an isomorphism of shape groups $\check{\pi}_k(p)$: $\check{\pi}_k(E, F, *) \to \pi_k(B, *)$ for all k where $F = p^{-1}(*)$ is the fiber of p over $*$. As a consequence one obtains an exact sequence of groups

$$\ldots \to \check{\pi}_k(F, *) \xrightarrow{i_\#} \pi_k(E, *) \xrightarrow{p_\#} \pi_k(B, *) \to \check{\pi}_{k-1}(F, *) \to \ldots \tag{12}$$

where i: $(F, *) \to (E, *)$ is the inclusion map.

Another important fact established by Coram and Duvall is that the fibers of an approximate fibration are FANR's, i.e., compacta shape dominated by compact polyhedra. If, in addition, B is connected, then the fibers are all of the same shape.

Coram and Duvall [10] have also obtained a criterion to decide whether a map between compact ANR's is an approximate fibration. They call a map p: $E \to B$ k-movable provided for each $b \in B$ and each neighborhood U_0 of the fiber $F_b = p^{-1}(b)$ there are neighborhoods U and V of F_b, $V \subset U \subset U_0$, such that for any $c \in B$ with fiber $F_c = p^{-1}(c) \subset V$ and for any $x \in F_c$ the natural homomorphism $\check{\pi}_i(F_c, x) \to \pi_i(U, x)$, $i \leqslant k$, is an isomorphism onto the image of the homomorphism $\pi_i(V, x) \to \pi_i(U, x)$, which is induced by the inclusion $V \subset U$. Then approximate fibrations p: $E \to B$ are characterized as maps which are k-movable for all k.

In [13] J. Dydak and J. Segal introduce n-stable and homology n-stable maps as generalizations of approximate fibrations. The n-stable maps are close to approximate fibrations (they are both based on homotopy groups) while the homology n-stable maps are obtained by considering homology groups with integer coefficients rather than homotopy groups. Intuitively, a map is homology n-stable if the Čech homology groups of its point inverses are locally constant up to dimension n. The homology n-stable maps are more general than the approximate fibrations. For example, let X be a Poincaré homology sphere (which is not a sphere) with a 3-simplex deleted. Then the quotient map p: $E^3 \to E^3/X$ is homology n-stable for all n, but is not an approximate fibration since point inverses are not of the same shape.

Definition of homology n-stable maps. A map f: $X \to Y$ is called *homology n-stable* if for any $\tilde{U} \in N(f, y)$ there exists $\tilde{V} \in N(f|\tilde{U}, y)$ such that for all $z \in V$

a) the natural homomorphism $\check{H}_k N(f, z) \to H_k(\tilde{V})$ is a monomorphism for all $k < n$,

b) the image of $H_k(\tilde{V}) \to H_k(\tilde{U})$ is equal to the image of $\check{H}_k N(f, z) \to H_k(\tilde{U})$ for all $k \leqslant n$.

Here $\tilde{A}$ denotes $f^{-1}(A)$ for $A \subset Y$ and $N(f, y)$ denotes the inverse system $\{\tilde{U}$: $y \in \operatorname{Int} U\}$ bounded with inclusions. The notion of an n-stable map is defined analogously using the appropriate homotopy groups in place of the homology groups and the two conditions are required to be valid for any base point in $f^{-1}(y)$.

Then as a generalization of theorems of Smale, Dugundji, Coram Duvall and Daverman–Walsh on when a decomposition space is LC^n, Dydak and Segal obtain the following result.

THEOREM 6. *Suppose that $f\colon X \to Y$ is a homology n-stable* map$(n \geqslant 0)$ *such that $\check{H}_0 N(f, y) = Z$, $N(f, y)$ is nearly 1-movable and homology $(n+1)$-stable for each $y \in Y$. If Y is metrizable and complete, then Y is homotopy locally connected up to dimension $n+1$.*

Let $X = \{X_\alpha, i_{\alpha\beta}, A\}$ be an inverse system of topological spaces bonded with inclusions. X is called nearly 1-movable if for each $\alpha \in A$ there exists a $\beta > \alpha$ such that for each loop $f\colon S^1 \to X_\beta$ and for each $\gamma \in A$ there exists an extension

$$\tilde{f}\colon B^2 - \bigcup_{i=1}^{r} \operatorname{Int} B_i^2 \to X_\alpha$$

of f with $(\operatorname{Int} B_i^2) \cap (\operatorname{Int} B_j^2) = \emptyset$ for $i \neq j$ and $\tilde{f}(\partial B_i^2) \subset X_\gamma$ for $i \leqslant r$. An inverse system of pointed topological spaces $\{U_\alpha, p_{\alpha\beta}, A\}$ is called n-stable if $\{\pi_k(U_\alpha), \pi_k(p_{\alpha\beta}), A\}$ is stable for $k < N$ and satisfies the Mittag–Leffler condition for $k = n$. By replacing homotopy groups by homology groups one gets the notion of homology n-stability of an inverse system.

For a more detailed discussion of approximate fibrations the reader is referred to Coram [8]. We now wish to consider the generalization of the notion of approximate fibration to that of shape fibration due to Mardešić and Rushing [24]. Here E and B are allowed to be arbitrary metric compacta instead of being required to be ANR's. For every map $p\colon X \to Y$ there exist ANR-expansions $\boldsymbol{p}\colon \boldsymbol{E} \to \boldsymbol{B}$, i.e., inverse sequences of compact ANR's $\boldsymbol{E} = \{E_i, q_{ii'}\}$, $\boldsymbol{B} = \{B_i, r_{ii'}\}$ with $\lim \boldsymbol{E} = E$, $\lim \boldsymbol{B} = B$, and sequences of maps $p_i\colon E_i \to B_i$ such that for each i the diagram

$$\begin{tikzcd} E_i \arrow[d, "p_i"'] \arrow[dr, phantom, "="] & E_{i+1} \arrow[l, "q_{ii+1}"'] \arrow[d, "p_{i+1}"] \\ B_i & B_{i+1} \arrow[l, "r_{ii+1}"] \end{tikzcd} \tag{13}$$

commutes and the maps p_i induce p, i.e., $p = \lim \boldsymbol{p}$.

We say that $\boldsymbol{p}$ has the AHLP with respect to a class of spaces $\mathcal{X}$ provided for each i and each $\varepsilon > 0$ there is a $j \geqslant i$ and a $\delta > 0$ such that whenever X belongs to $\mathcal{X}$ and $h\colon X \to E_j$, $H\colon X \times I \to B_j$ satisfy

$$d(H_0, p_j h) \leqslant \delta, \tag{14}$$

then there is an $\tilde{H}$: $X \times I \to E_i$ such that

$$d(\tilde{H}_{i0}, q_{ij}h) < \varepsilon \tag{15}$$

and

$$d(p_i \tilde{H}, r_{ij}H) < \varepsilon. \tag{16}$$

If $\boldsymbol{p}$: $\boldsymbol{E} \to \boldsymbol{B}$ and $\boldsymbol{p}'$: $\boldsymbol{E}' \to \boldsymbol{B}'$ are two ANR-expansions of the same map p: $E \to B$ and $\boldsymbol{p}$ has the AHLP with respect to X, then so does $\boldsymbol{p}'$. This justifies the following definition: a shape fibration p: $E \to B$ is a map between metric compacta such that p admits an ANR-expansion $\boldsymbol{p}$ having the AHLP with respect to all topological spaces. Then one can show that a map p: $E \to B$ between compact ANR's is an approximate fibration if and only if it is a shape fibration.

Mardešić and Rushing have also shown that a cell-like map between finite-dimensional metric compacta is a shape fibration. On the other hand the Taylor map (of the Kahn space) is a cell-like map which fails to be a shape fibration.

A shape fibration p: $E \to B$ induces an isomorphism of homotopy pro-groups

$$p_*: \text{pro-}\pi_k(E, F, e) \to \text{pro-}\pi_k(B, b), \tag{17}$$

where $b \in B$, $F = p^{-1}(b)$ and $e \in F$. One also has an exact sequence of homotopy pro-groups

$$\ldots \to \text{pro-}\pi_k(F, e) \to \text{pro-}\pi_k(E, e) \to \text{pro-}\pi_k(B, b) \to \text{pro-}\pi_{k-1}(F, e) \to \ldots \tag{18}$$

If E and B are ANR's, then p is an approximate fibration and the sequence specializes to the sequence of Coram and Duvall.

An important property of shape fibrations is the fact that the pullback p': $E' \to B'$ of a shape fibration p: $E \to B$ determined by an arbitrary map f: $B' \to B$ between metric compacta is again a shape fibration.

In [11] Coram, Mardešić and Toruńczyk establish the following result analogous to Kozlowski's Theorem 4 for maps which are not cell-like.

THEOREM 7. *If X is a compact* ANR *and if f: $X \to Y$ is a shape fibration and a shape domination onto a metric compactum, then Y is an* ANR.

Recall from E. Fadell [14] the corresponding situation in homotopy theory. Namely, if X is a compact ANR and if f: $X \to Y$ is a (Hurewicz) fibration which is a homotopy domination onto a metric compactum Y, then Y is an ANR.

Finally, we mention the notion of a movable map f: $X \to Y$ between separable metric spaces recently introduced by T. Yagasaki [37]. Let f be a proper map, M an ANR which contains X, and p: $Y \times M \to Y$ the projection map. The map f is said to be *movable* if each neighborhood U of f^{-1}

$= \bigcup \{y \times f^{-1}(y): y \in Y\}$ in $Y \times M$ contains a neighborhood V of f^{-1} such that for each neighborhood W of f^{-1} in V, there exists a homotopy $H: V \times [0, 1] \to U$ with $H_0 = \mathrm{id}$, $H_1(V) \subset W$, $pH_t = p$ $(0 \leqslant t \leqslant 1)$. Futhermore, he shows that if f is a movable map and X is an ANR, then so is Y. Yagasaki in [38] shows that movable maps are shape fibrations. However, there are shape fibrations which are not movable.

There are several relevant areas of shape theory which we did not touch upon, especially strong shape. This topic as well as some others are treated elsewhere in this volume. As a general reference the reader is referred to [27].

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