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SCHUBERT VARIETIES AND STANDARD MONOMIAL THEORY

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Let G be a semisimple, simply connected Chevalley group over a field k. Let T be a maximal (k-split) torus and B a Borel subgroup, $B \supset T$. Let Q be a parabolic subgroup, $Q \supseteq B$. Let W (resp. W_Q) be the Weyl group of G (resp. Q). For $w \in W/W_Q$, let e_w be the point and $X(w)(=\overline{BwQ} \,(\mathrm{mod}\, Q))$ the Schubert variety in G/Q associated to w. In the series G/P I–V (cf. [23], [15], [12], [13], [17]), a standard monomial theory has been developed for Schubert varieties in G/Q, G classical (and for certain Q's if Q is exceptional). This theory is a generalization of the classical Hodge–Young theory (cf. [3], [4]) which gives bases for the homogeneous coordinate rings of Schubert varieties in the grassmannians. To be more precise, standard monomial theory consists in the construction of explicit bases for $H^0(X(x), L)$, L being an ample line bundle on G/Q. The results of [17] give the theory for classical groups. Subsequently the theory was completed for G_2 and E_6 (cf. [7], [14]). In the meantime, the question arose whether this theory could be generalized to Schubert varieties in the infinite-dimensional flag manifolds associated to Kac–Moody groups (cf. [6], [24], [25]). Towards an affirmative answer to this question, the author in collaboration with Seshadri has developed the theory for $\widehat{\mathrm{SL}}_2$ (cf. [18], [19]). Towards completing the theory (both in the finite-dimensional and infinite-dimensional cases), the author has arrived at a conjecture (cf. § 2). This conjecture has been verified to hold in all cases where the theory has been developed. Using this conjecture, the author has been able to complete the theory for F_4 and E_7 (cf. [11]). Thus in the finite-dimensional case, the theory is now complete except for E_8. Among several important geometric and representation-theoretic consequences of standard monomial theory is the determination of singular loci of Schubert varieties (cf. [16], [20]). In this survey, we describe in an explicit way the singular loci of Schubert varieties for groups of rank 2 (using [14], [16], [7]) and also for Schubert varieties in G/P, G classical and P being given as follows:

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A_n: P is any maximal parabolic subgroup,
B_n, C_n: $P = P_{\hat{\alpha}}$, $\alpha = \alpha_1$ or α_n,
D_n: $P = P_{\hat{\alpha}}$, $\alpha = \alpha_1$, α_{n-1} or α_n
(here the indexing of the simple roots is as in [1]).

The sections are arranged as follows. In § 1, we recall results from [7], [14], [17]; in § 2, we state the conjecture mentioned above and § 3 deals with singular locus.

§ 1. Preliminaries

A basis for $H^0(X(w), L)$. Let G, B, T, Q, W, etc. be as in the introduction. Let $(,)$ be a W-invariant inner product on $X(T) \otimes \mathbf{Q}$, where $X(T) = \mathrm{Hom}_{\mathrm{alg.gp}}(T, \mathbf{G}_m)$. Let R (resp. R^+) be the set of roots (resp. positive roots) of G relative to T (resp. B).

NOTATION 1.1. For a fundamental weight ω, let

$$m(\omega) = \max \left\{ (\omega, \alpha^v) = 2\frac{(\omega, \alpha)}{(\alpha, \alpha)},\ \alpha \in R^+ \right\}.$$

In [14], it is shown that if P is a maximal parabolic subgroup such that the associated fundamental weight ω satisfies $(\omega, \alpha^v) \leqslant 3$, $\alpha \in R^+$, then $H^0(G/P, L)$ (where L is the ample generator of $\mathrm{Pic}(G/P)$) has a basis indexed by "admissible quadruples" of elements of W/W_P. Using this and the notion of standard Young diagrams on a Schubert variety $X(w)$ in G/Q (cf. [17], [14]), one obtains a basis for $H^0(X(w), L_{\boldsymbol{a}})$ consisting of standard monomials of type $\boldsymbol{a}$ on $X(w)$, where $\boldsymbol{a}$ is given as follows: if $Q = \bigcap_{i=1}^r P_i$ (P_i's being maximal parabolic subgroups) and L_i is the ample generator of $\mathrm{Pic}(G/P_i)$, then $L_{\boldsymbol{a}} = \bigotimes_{i=1}^r L_i^{a_i}$. (For details, see [14]; if one considers ω such that $m_\omega = 2$, say for example, G is classical, then for the details of the above result, one may refer to [17]). Since for § 3, we need results for G where G is classical or of type G_2, we recall below the necessary results for these cases.

DEFINITION 1.2 (cf. [17]). A maximal parabolic subgroup P is said to be of *classical type* if the associated fundamental weight ω satisfies the condition $(\omega, \alpha^v) \leqslant 2$, $\alpha \in R^+$.

DEFINITION 1.3 (cf. [17]). Let P be a maximal parabolic subgroup of classical type. A pair (τ, φ) of Weyl group elements in W^P (= the set of minimal representatives of W/W_P) is called an *admissible pair* if either

(1) $\tau = \varphi$ (in which case, we call it a trivial admissible pair) or
(2) there exists a sequence

$$X(\tau) = X(\tau_0) \supset X(\tau_1) \supset \ldots \supset X(\tau_r) = X(\varphi)$$

such that $X(\tau_i)$ is a Schubert divisor in $X(\tau_{i-1})$, say $\tau_i = \tau_{i-1} s_{\beta_i}$ for some $\beta_i \in R^+$, with $(\omega, \beta_i^v) = 2$.

We recall (cf. [17]) the following:

THEOREM 1.4. *Let P be a maximal parabolic subgroup of classical type and L the ample generator of* $\mathrm{Pic}(G/P)$. *Then there exists a basis $\{p(\tau, \varphi)\}$ of $H^0(G/P, L)$ indexed by admissible pairs such that*

(1) *$p(\tau, \varphi)$ is a weight vector of weight $-\frac{1}{2}(\tau(\omega)+\varphi(\omega))$*
(2) *For $X(w)$ in G/P, $p(\tau, \varphi)|_{X(w)} \not\equiv 0$ if and only if $w \geqslant \tau$ (in W/W_P)*
(3) *$\{p(\tau, \varphi) | w \geqslant \tau\}$ is a basis for $H^0(X(w), L)$.*

Remark 1.5. The basis $\{p(\tau, \varphi)\}$ is in fact constructed in [17] even over $\mathbf{Z}$. We shall denote the corresponding basis for $H^0(G_{\mathbf{Z}}|P_{\mathbf{Z}}, L_{\mathbf{Z}})$ by $\{P(\tau, \varphi)\}$ so that over any field k, we have $p(\tau, \varphi) = P(\tau, \varphi) \otimes 1$.

NOTATION 1.6. Let $\{Q(\tau, \varphi)\}$ be the basis of $V_{\mathbf{Z},\omega}$, the $\mathbf{Z}$-dual of $H^0(G_{\mathbf{Z}}/P_{\mathbf{Z}}, L_{\mathbf{Z}})$ dual to $\{P(\tau, \varphi)\}$. (Note that $V_{\mathbf{Z},\omega} \otimes \mathbf{Q}$ is the irreducible G-module (over $\mathbf{Q}$) with highest weight ω).

The group G_2. Among the two maximal parabolic subgroups $\{P_{\hat{\alpha}_1}, P_{\hat{\alpha}_2}\}$, $P_{\hat{\alpha}_1}$ is such that $m(\omega_1) = 2$ and hence, we have results for Schubert varieties in $G/P_{\hat{\alpha}_1}$ given by Theorem 1.4. We shall now recall (cf. [7]) results for $G/P_{\hat{\alpha}_2}$. Let us denote $\tau_0 = \mathrm{Id}$, $\tau_1 = s_2$, $\tau_2 = s_1 s_2$, $\tau_3 = s_2 s_1 s_2$, $\tau_4 = s_1 s_2 s_1 s_2$, $\tau_5 = s_2 s_1 s_2 s_1 s_2$ (note that $W^{P_{\hat{\alpha}_2}} = \{\tau_i, 0 \leqslant i \leqslant 5\}$).

We have (with notation as in [7])

THEOREM 1.7. *Let $P = P_{\hat{\alpha}_2}$. There exists a basis $\mathscr{B} = \{p(\tau_i), 0 \leqslant i \leqslant 5\} \cup \mathscr{B}'$, where $\mathscr{B}' = \{p(\tau, \varphi)$, $\tau = \tau_4$ or τ_3, $\varphi = \tau_2$ or τ_1, $r(\tau, \varphi)$, $q(\tau, \varphi)$, $(\tau, \varphi) = (\tau_4, \tau_3)$ or $(\tau_2, \tau_1)\}$ for $H^0(G/P, L)$, with similar properties as in Theorem* 1.4 (*for details refer to* [7]).

NOTATION 1.8. Denoting as above $V_{\mathbf{Z},\omega}$, the $\mathbf{Z}$-dual of $H^0(G_{\mathbf{Z}}/P_{\mathbf{Z}}, L_{\mathbf{Z}})$, we obtain a $\mathbf{Z}$-basis $\{Q(\tau_i), 0 \leqslant i \leqslant 5, Q(\tau, \varphi), \tau = \tau_4$ or τ_3, $\varphi = \tau_2$ or τ_1, $E(\tau, \varphi)$, $F(\tau, \varphi)$, $(\tau, \varphi) = (\tau_4, \tau_3)$ or $(\tau_2, \tau_1)\}$ (note that $V_{\mathbf{Z},\omega}$ is simply $\mathfrak{G}_{\mathbf{Z}}$ where $\mathfrak{G}$ is the Lie algebra of G and the above basis is simply the Chevalley basis of $\mathfrak{G}_{\mathbf{Z}}$ (cf. [7], § 4)).

§ 2. A conjecture towards completing standard monomial theory

Let A be a symmetrizable, generalized Cartan matrix. Let $\mathfrak{G}$ (resp. G) be the associated Kac–Moody Lie algebra (resp. Kac–Moody group). (For generalities on Kac–Moody Lie algebras and groups, one may refer to [5], [6], [24], [25]). Let W be the Weyl group. Let U be the universal enveloping algebra of $\mathfrak{G}$ and $U_{\mathbf{Z}}^{+}$, the $\mathbf{Z}$-subalgebra of U generated by $X_{\alpha}^{n}/n!$, $\alpha \in S$ (here S is the set of simple roots). Let λ be a dominant, integral weight and V_{λ} the integrable highest weight module (over $\mathbf{C}$) with highest weight λ. Let us fix a generator e for the highest weight space (note that e is unique up to scalars). For $\tau \in W$, let $e_{\tau} = \tau e$, $V_{\mathbf{Z},\tau} = U_{\mathbf{Z}}^{+} e_{\tau}$.

A conjectural basis for $V_{\mathbf{Z},\tau}$: $V_{\mathbf{Z},\tau}$ has a basis $\mathscr{B} = \{e_\tau, \tau \in W\} \cup \mathscr{B}'$, where $\mathscr{B}'$ is given as follows.

(1) *I, the indexing set for* $\mathscr{B}'$: I is given by

$$I = \left\{1 > \frac{P_{i_1}}{q_{i_1}} > \frac{P_{i_2}}{q_{i_2}} > \ldots > \frac{P_{i_t}}{q_{i_t}} > 0,\ \mu_{i_1} < \mu_{i_2} < \ldots < \mu_{i_t} < \mu_{i_{t+1}}\right\}$$

such that

(a) there exist elements $\mu_i \in W$, $0 \leqslant i \leqslant r+1$ with

$$\mu_0 = \mu_{i_1} < \mu_1 < \mu_2 < \ldots < \mu_{i_{t+1}} = \mu_{r+1},$$

where for $1 \leqslant l \leqslant t+1$, each $\mu_{i_l} = \mu_m$ for some m, $0 \leqslant m \leqslant r+1$; further, for $0 \leqslant i \leqslant r$, $X(\mu_i)$ is a Schubert divisor in $X(\mu_{i+1})$, say

$$\mu_i = s_{\beta_i}\mu_{i+1}, \qquad m_i = |(\mu_i(\lambda), \beta_i^*)|.$$

(b) There exist positive integers n_i, $0 \leqslant i \leqslant r$ such that

$$1 > \frac{n_0}{m_0} \geqslant \frac{n_1}{m_1} \geqslant \ldots \geqslant \frac{n_r}{m_r} > 0.$$

Further, in (b) $n_l/m_l < n_{l-1}/m_{l-1}$ if and only if $l \in \{i_1, \ldots, i_t\}$ and for such an l, $n_l/m_l = p_l/q_l$.

(2) *The vectors in* $\mathscr{B}'$. To an element of I as in (1), we associate the vector $X_{-\beta_r}^{(n_r)} \ldots X_{-\beta_1}^{(n_1)} e_{\mu_0}$ (here for a real root β, $X_{-\beta}^{(n)}$ stands for $X^n_{-\beta}/n!$). Note that the above vector is a weight vector of weight

$$\left(1 - \frac{p_{i_1}}{q_{i_1}}\right)\mu_{i_1}(\lambda) + \left(\frac{p_{i_1}}{q_{i_1}} - \frac{p_{i_2}}{q_{i_2}}\right)\mu_{i_2}(\lambda) + \ldots + \frac{p_{i_t}}{q_{i_t}}\mu_{i_{t+1}}(\lambda).$$

§ 3. Singular loci of Schubert varieties

This section consists of two parts. In Part I, we decribe explicitly the singular loci of Schubert varieties for all groups of rank 2. In part II, we describe the singular loci of Schubert varieties in G/P, G classical and P belongs to a certain class of maximal parabolic subgroups. First, we recall results from [16], [9]. Let G be classical of rank n. In the sequel, for $1 \leqslant d \leqslant n$, we shall denote the maximal parabolic subgroup $P_{\hat{\alpha}_d}$ by just P_d. For $w \in W$, let

$$I(w) = \left\{ p(\lambda, \mu) \;\middle|\; \begin{array}{l} (1)\ (\lambda, \mu) \text{ is an admissible pair} \\ \quad \text{in } W/W_{P_j}, \text{ for some } j,\ 1 \leqslant j \leqslant n \\ (2)\ p(\lambda, \mu)|_{X(w)} \equiv 0 \end{array} \right\}.$$

THEOREM 3.1 (cf. [17]). *The ideal sheaf of $X(w)$ in G/B is generated by $I(w)$.*

Let $\tau \leqslant w$ and let J_τ be the (Jacobian) matrix $\|X_{-\beta}p(\lambda, \mu)\|$, evaluated at e_τ, where $\beta \in \tau(R^+)$ and $p(\lambda, \mu) \in I(w)$, then we have

THEOREM 3.2 (cf. [16]). $\operatorname{rank} J_\tau = \# R(w, \tau) = \{\beta \in \tau(R^+) |$ *there exists a* $p(\lambda, \mu) \in I(w)$ *with* $X_{-\beta} p(\lambda, \mu) = cp(\tau)$, $c \in k^*\}$ *(here* $p(\tau)$ *denotes the extremal weight vector* $p(\tau, \tau)$ *corresponding to the trivial admissible pair* (τ, τ)*).*

Let $T(w, \tau)$ be the Zariski tangent space to $X(w)$ at e_τ, then

THEOREM 3.3 (cf. [16]). *$T(w, \tau)$ is spanned by* $\{X_{-\beta}, \beta \in N(w, \tau)\}$ *where* $N(w, \tau) = \{\beta \in \tau(R^+) |$ *for every* $p(\lambda, \mu)$ *with* $X_{-\beta} p(\lambda, \mu) = cp(\tau)$, $c \in k^*$, $p(\lambda, \mu)|_{X(w)} \not\equiv 0\}$.

Using the above theorem and the explicit description of $p(\lambda, \mu)$ (cf. [8], [10]), an explicit description of $T(w, \tau)$ is given for classical groups in [9]. We take this occasion to state the results of [9] in a more compact form. We first recall (cf. [9]) few notation. Let G be classical of rank n. For $w \in W$, we shall denote by $X(w^{(d)})$ the projection of $X(w)$ under $G/B \to G/P_d$, $1 \leqslant d \leqslant n$. Given a d-tuple $(a_1, \ldots, a_d)$ of integers we shall denote by $(a_1, \ldots, a_d) \uparrow$ the d-tuple $(a_1, \ldots, a_d)$ with the entries arranged in ascending order. For $G = \mathrm{Sp}(2n)$ or $\mathrm{SO}(2n)$, we shall let $i' = 2n+1-i$, $1 \leqslant i \leqslant 2n$ and $|i| = \min\{i, i'\}$. For $G = \mathrm{SO}(2n+1)$, we shall let $i' = 2n+2-i$, $1 \leqslant i \leqslant 2n+1$, $i \neq n+1$ and $|i| = \min\{i, i'\}$. For $G = \mathrm{Sp}(2n)$ or $\mathrm{SO}(2n)$ (resp. $\mathrm{SO}(2n+1)$) we identify W as a subgroup of the symmetric group S_{2n} (resp. S_{2n+1}) as in [10]. For $1 \leqslant d \leqslant n$, we identify W^{P_d} with certain d-tuples of integers (as described in [10]). Let w, $\tau \in W$, $w \geqslant \tau$. Let $\beta = \tau(\alpha)$, $\alpha \in R^+$.

THEOREM 3.4. *Let G be classical. Then $T(w, \tau)$ is spanned by*

$$\{X_{-\beta} | w^{(d)} \geqslant \tau^{(\alpha,d)} \text{ (in } W^{P_d}), \ 1 \leqslant d \leqslant n\},$$

where $\tau^{(\alpha,d)} \in W^{P_d}$ is given as follows:

$$\tau^{(\alpha,d)} = (\tau s_\alpha)^{(d)}, \quad \text{if } \alpha = \varepsilon_j - \varepsilon_k, \ 1 \leqslant j < k \leqslant n$$

(here s_α denotes the reflection with respect to α). In the rest of the cases, $\tau^{(\alpha,d)}$ is given as follows.

The symplectic group $\mathrm{Sp}(2n)$: *Let* $\tau = (a_1 \ldots a_{2n})$.

(1) $\alpha = 2\varepsilon_j$, $1 \leqslant d \leqslant n$. *Then* $\tau^{(\alpha,d)} = \tau s_\alpha^{(d)}$, $1 \leqslant d \leqslant n$.

(2) $\alpha = \varepsilon_j + \varepsilon_k$, $1 \leqslant j < k \leqslant n$. *Let* $s = \min\{|a_j|, |a_k|\}$, $r = \max\{|a_j|, |a_k|\}$. *Then*

$$\tau^{(\alpha,d)} = \begin{cases} (\tau s_\alpha)^{(d)} & \text{if } d < k, \\ (a_1, \ldots, \hat{a}_j, \ldots, \hat{a}_k, \ldots, a_d, s', r)\uparrow, & \text{if } k \leqslant d \leqslant n. \end{cases}$$

The orthogonal group $\mathrm{SO}(2n+1)$: *Let* $\tau = (a_1 \ldots a_{2n+1})$.

(1) $\alpha = \varepsilon_j + \varepsilon_k$, $1 \leqslant j < k \leqslant n$.

(a) *If $d < k$ or $d = n$, then* $\tau^{(\alpha,d)} = (\tau s_\alpha)^{(d)}$.

(b) *If* $k \leqslant d \leqslant n-1$, *then let* $s = \min\{|a_j|, |a_k|\}$, $r = \max\{|a_j|, |a_k|\}$. *Define s_i, $0 \leqslant i \leqslant c(d)$, as the integers*

$$r = s_0 < s_1 < s_2 < \ldots < s_c(d) \leqslant n$$

such that $s_i \notin \{|a_1|, \ldots, |a_d|\}$, $i \neq 0$.

If precisely one of $\{a_j, a_k\}$ *is* $> n$, *then*

$$\tau^{(\alpha,d)} = (a_1, \dots, \hat{a}_j, \dots, \hat{a}_k, \dots a_d, s', r)\uparrow$$

If a_j, a_k *are either both* $> n$ *or both* $\leqslant n$, *then*

$$\tau^{(\alpha,d)} = (a_1, \dots, \hat{a}_j, \dots, \hat{a}_k, \dots, a_d, s', s'_{c(d)})\uparrow$$

(2) $\alpha = \varepsilon_j,\ 1 \leqslant j \leqslant n$.
Define s_i, $0 \leqslant i \leqslant m(d)$, *as the integers*

$$s_0 = |a_j| < s_1 < s_2 < \dots < s_{m(d)} \leqslant n$$

such that $s_i \notin \{|a_1|, \dots, |a_d|\}$, $i \neq 0$. *Then*

$$\tau^{(\alpha,d)} = \begin{cases} (\tau s_\alpha)^{(d)}, & \text{if } d < j \text{ or } d = n, \\ (a_1, \dots, \hat{a}_j, \dots, a_d, s'_{m(d)})\uparrow, & \text{if } j \leqslant d \leqslant n-1. \end{cases}$$

The orthogonal group SO$(2n)$: *Let* $\tau = (a_1 \dots a_{2n})$ *and* $\alpha = \varepsilon_j + \varepsilon_k$, $1 \leqslant j < k \leqslant n$.

(a) *If* $d < k$ *or* $> n-2$, *then* $\tau^{(\alpha,d)} = (\tau s_\alpha)^{(d)}$.

(b) *If* $k \leqslant d \leqslant n-2$, *then let* $s = \min\{|a_j|, |a_k|\}$, $r = \max\{|a_j|, |a_k|\}$. *Define* s_i, $-l(d) \leqslant i \leqslant c(d)$, *as the integers* $s < s_{-l(d)} < s_{-l(d)+1} < \dots < s_{-1} < s_0 = r < s_1 < s_2 < \dots < s_{c(d)} \leqslant n$ *such that* $s_i \notin \{|a_1|, \dots, |a_d|\}$, $i \neq 0$.

If precisely one of $\{a_j, a_k\}$ *is* $> n$, *then*

$$\tau^{(\alpha,d)} = (a_1, \dots, \hat{a}_j, \dots, \hat{a}_k, \dots, a_d, s', r)\uparrow$$

If a_j, a_k *are either both* $> n$ *or both* $\leqslant n$, *then*

$$\tau^{(\alpha,d)} = \begin{cases} (a_1, \dots, \hat{a}_j, \dots, \hat{a}_k, \dots, a_d, s'_{c(d)-1}, s')\uparrow, & \text{if } (l(d), c(d)) \neq (0, 0), \\ (a_1, \dots, \hat{a}_j, \dots, \hat{a}_k, \dots, a_d, r', s')\uparrow, & \text{if } (l(d), c(d)) = (0, 0). \end{cases}$$

We have results for G_2 (similar to Theorems 3.1–3.3) which we recall below (cf. [7], [14]). Let G be of type G_2. For $P = P_{\hat{\alpha}_d}, j = 1, 2$, let us denote the basis vectors for $H^0(G/P, L)$ by $\{p^{(i)}(\tau, \varphi)\}$. To make it very precise, $p^{(i)}(\tau, \varphi)$ is just $p(\tau, \varphi)$, if either $j = 1$ or $j = 2$, $\tau \neq \tau_4, \tau_2$; for $j = 2$, $\tau = \tau_4, \tau_2$, we denote $q(\tau, \varphi)$ (resp. $r(\tau, \varphi)$) by $p^{(1)}(\tau, \varphi)$ (resp. $p^{(2)}(\tau, \varphi)$). For $w \in W$, let

$$I(w) = \{p^{(i)}(\lambda, \mu) \mid p^{(i)}(\lambda, \mu)|_{X(w)} \equiv 0\}.$$

THEOREM 3.5 (cf. [14]). *The ideal sheaf of* $X(w)$ *in* G/B *is generated by* $I(w)$.

Let J_τ be the Jacobian matrix $\|X_{-\beta} p^{(i)}(\lambda, \mu)\|$ evaluated at e_τ, where $\beta \in \tau(R^+)$ and $p^{(i)}(\lambda, \mu) \in I(w)$. Then we have

THEOREM 3.6 (cf. [14]). $\operatorname{rank} J_\tau = \# R(w, \tau)$, *where* $R(w, \tau) = \{\beta \in \tau(R^+) \mid$ *there exists a* $p^{(i)}(\lambda, \mu) \in I(w)$ *with* $X_{-\beta} p^{(i)}(\lambda, \mu) = cp(\tau),\ c \in k^*\}$.

THEOREM 3.7 (cf. [14]). $T(w, \tau)$, *the Zariski tangent space to* $X(w)$ *at* e_τ *is spanned by* $\{X_{-\beta}, \beta \in N(w, \tau)\}$, *where* $N(w, \tau) = \{\beta \in \tau(R^+) \mid$ *for every* $p^{(i)}(\lambda, \mu)$ *such that* $X_{-\beta} p^{(i)}(\lambda, \mu) = cp(\tau)$, $c \in k^*$, $p^{(i)}(\lambda, \mu)|_{X(w)} \not\equiv 0\}$.

Let G be classical. It can easily be seen that $X_{-\beta}\,p(\lambda, \mu) = cp(\tau)$, $c \in k^*$, if and only if $X_{-\beta}Q(\tau)$ ($Q(\tau) = Q(\tau, \tau)$ being as in Notation 1.6) when written as a $\mathbf{Z}$-linear combination of $Q(\theta, \delta)$'s, involves $Q(\lambda, \mu)$ with a coefficient that is non-zero in k. A similar remark applies to G_2 also. Hence in the following discussion, $T(w, \tau)$ will be determined by computing $X_{-\beta}Q(\tau)$.

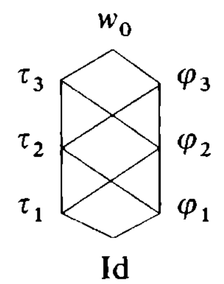
Part I. Singular loci of Schubert varieties for groups of rank 2

It is enough to discuss types C_2 and G_2 (since for A_2, it is easily seen that every Schubert variety is smooth). Let $P_{\hat{\alpha}_i} = P_i$, $i = 1, 2$ and L_i be the ample generator of $\operatorname{Pic}(G/P_i)$, $i = 1, 2$. For $w \in W$, let $S(w)$ denote the singular locus of $X(w)$. If $S(w) \neq \emptyset$, then it suffices to discuss the behaviour at e_τ with $\operatorname{codim} X(\tau)$ in $X(w) \geqslant 2$ in view of the fact that Schubert varieties are nonsingular in codimension 1 (cf. [2] or [17]).

Type C_2: We have (cf. [1])

$$R^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2\} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \quad \text{say.}$$

The following picture gives the configuration of Schubert varieties in G/B.



Let $\tau_1 = s_1$, $\tau_2 = s_2 s_1$, $\tau_3 = s_1 s_2 s_1$, $\varphi_1 = s_2$, $\varphi_2 = s_1 s_2$, $\varphi_3 = s_2 s_1 s_2$, $\tau_0 = \varphi_0 = \mathrm{Id}$ (here w_0 denotes the unique element of largest length in W and s_i denotes the reflection with respect to α_i, $i = 1, 2$.

With notation as in 1.6, we have (cf. [7]):

(a) The set $\{Q(\tau_i), 0 \leqslant i \leqslant 3\}$ is a $\mathbf{Z}$-basis for $V_{\mathbf{Z},\omega_1}$ (note that in this case, the trivial pairs (τ, τ) are the only admissible pairs in W^{P_1} and we have denoted $Q(\tau, \tau)$ by just $Q(\tau)$).

(b) The set $\{Q(\varphi_i), 0 \leqslant i \leqslant 3, Q(\varphi_2, \varphi_1)\}$ is a $\mathbf{Z}$-basis for $V_{\mathbf{Z},\omega_2}$ (note that in this case (φ_2, φ_1) is the only nontrivial admissible pair).

Let Q_i be the highest weight vector in $V_{\mathbf{Z},\omega_i}$ (cf. Notation 1.6). (Note that $Q_i = Q(\mathrm{Id})$.) We have (up to ± 1), with notation as in 1.6.

$$X_{-\alpha_1}Q_i = \begin{cases} Q(\tau_1), & i = 1, \\ 0, & i = 2, \end{cases} \qquad X_{-\alpha_2}Q_i = \begin{cases} 0, & i = 1, \\ Q(\varphi_1), & i = 2, \end{cases}$$

$$X_{-\alpha_3}Q_{\tau,i} = \begin{cases} Q(\tau_2), & i = 1, \\ Q(\varphi_2, \varphi_1), & i = 2, \end{cases} \qquad X_{-\alpha_4}Q_{\tau,i} = \begin{cases} Q(\tau_3), & i = 1, \\ Q(\varphi_2), & i = 2. \end{cases}$$

From this it follows (by considering the dim $T(w, \mathrm{Id})$ as given by Theorem 3.3, with $\tau = \mathrm{Id}$) that $X(\tau_3)$ is the only singular Schubert variety. We shall now compute dim $T(w, \tau)$, $w = \tau_3$ and $\tau = \tau_1$, φ_1 which will then enable us to determine $S(\tau_3)$.

Discussion at e_τ: We shall denote by $Q_{\tau,i}$ the extremal weight vector in $V_{\mathbf{Z},\omega_i}$ of weight $\tau(\omega_i)$, $i = 1, 2$. Further for $1 \leqslant j \leqslant 4$, we shall denote $\tau(\alpha_j)$ by β_j.

(1) $\tau = \varphi_1$. We have (up to ± 1)

$$X_{-\beta_1} Q_{\tau,i} = \begin{cases} Q(\tau_1), & i = 1, \\ 0, & i = 2, \end{cases} \qquad X_{-\beta_2} Q_{\tau,i} = \begin{cases} 0, & i = 1, \\ Q(\varphi_1), & i = 2, \end{cases}$$

$$X_{-\beta_3} Q_{\tau,i} = \begin{cases} Q(\tau_2), & i = 1, \\ Q(\varphi_2, \varphi_1), & i = 2, \end{cases} \qquad X_{-\beta_4} Q_{\tau,i} = \begin{cases} Q(\tau_3), & i = 1, \\ Q(\varphi_2), & i = 2. \end{cases}$$

A. As above, considering dim $T(w, \tau)$, $w = \tau_3$, we obtain that $X(\tau_3)$ is smooth at e_τ.

(2) $\tau = \tau_1$. We have (up to ± 1)

$$X_{-\beta_1} Q_{\tau,i} = \begin{cases} Q(\mathrm{Id}), & i = 1, \\ 0, & i = 2, \end{cases} \qquad X_{-\beta_2} Q_{\tau i} = \begin{cases} 0, & i = 1, \\ Q(\varphi_2), & i = 2, \end{cases}$$

$$X_{-\beta_3} Q_{\tau,i} = \begin{cases} Q(\tau_3), & i = 1, \\ Q(\varphi_2, \varphi_1), & i = 2, \end{cases} \qquad X_{-\beta_4} Q_{\tau,i} = \begin{cases} Q(\tau_2), & i = 1, \\ Q(\varphi_1), & i = 2. \end{cases}$$

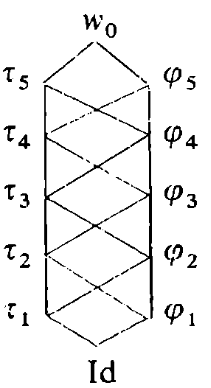
B. By considerations as above we obtain that $X(\tau_3)$ is singular at e_τ. Hence we obtain

THEOREM 3.8. *With notation as above, $X(\tau_3)$ is the only singular Schubert variety in G/B and $S(\tau_3) = X(\tau_1)$.*

Type G_2: We have (cf. [1])

$$\begin{aligned} R^+ &= \{\alpha_1, \alpha_2, \alpha_1, +\alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\} \\ &= \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}, \quad \text{say.} \end{aligned}$$

The following diagram gives the configuration of Schubert varieties in G/B.



Let $\varphi_1 = s_1$, $\varphi_2 = s_2 s_1$, $\varphi_3 = s_1 s_2 s_1$, $\varphi_4 = s_2 s_1 s_2 s_1$, $\varphi_5 = s_1 s_2 s_1 s_2 s_1$, $\tau_1 = s_2, \tau_2 = s_1 s_2, \tau_3 = s_2 s_1 s_2, \tau_4 = s_1 s_2 s_1 s_2, \tau_5 = s_2 s_1 s_2 s_1 s_2, \tau_0 = \varphi_0 = \mathrm{Id}$.

With notation as in 1.8, we have

(a) $\{Q(\varphi_i),\ 0 \leqslant i \leqslant 5,\ Q(\varphi_3, \varphi_2)\}$ is a $\mathbf{Z}$-basis for $V_{\mathbf{Z},\omega_1}$,

(b) $\{Q(\tau_i),\ 0 \leqslant i \leqslant 5,\ Q(\tau, \varphi),\ \tau = \tau_4$ or τ_3, $\varphi = \tau_2$ or τ_1, $E(\tau, \varphi)$, $F(\tau, \varphi)$, $(\tau, \varphi) = (\tau_4, \tau_3)$ or $(\tau_2, \tau_1)\}$ is a $\mathbf{Z}$-basis for $V_{\mathbf{Z},\omega_2}$. As remarked earlier, $V_{\mathbf{Z},\omega_2} = \mathfrak{G}_{\mathbf{Z}}$ and the above basis is simply the Chevalley basis. To be very precise (cf. [7]), the extremal weight vectors $Q(\tau_i)$, $0 \leqslant i \leqslant 5$ are $X_{\pm\beta}$, β being a long root. The nonextremal weight vectors are given by

$$X_{2\alpha_1+\alpha_2} = E(\tau_2, \tau_1), \qquad X_{\alpha_1+\alpha_2} = F(\tau_2, \tau_1),$$

$$X_{-(2\alpha_1+\alpha_2)} = F(\tau_4, \tau_3), \qquad X_{-(\alpha_1+\alpha_2)} = E(\tau_4, \tau_3),$$

$$X_{\alpha_1} = Q(\tau_3, \tau_1), \qquad X_{-\alpha_1} = Q(\tau_4, \tau_2), \qquad H_{\alpha_1} = Q(\tau_4, \tau_1), \qquad H_{\alpha_2} = Q(\tau_3, \tau_2).$$

As before, we shall denote by Q_i the highest weight vector in $V_{\mathbf{Z},\omega_i}$, $i = 1, 2$. (note that $Q_i = Q(\mathrm{Id})$). We have (up to ± 1)

$$X_{-\alpha_1} Q_i = \begin{cases} Q(\varphi_1), & i = 1, \\ 0, & i = 2, \end{cases} \qquad X_{-\alpha_2} Q_i = \begin{cases} 0, & i = 1, \\ Q(\tau_1), & i = 2, \end{cases}$$

$$X_{-\alpha_3} Q_i = \begin{cases} Q(\varphi_2), & i = 1, \\ E(\tau_2, \tau_1), & i = 2, \end{cases} \qquad X_{-\alpha_4} Q_i = \begin{cases} Q(\varphi_3, \varphi_2), & i = 1, \\ F(\tau_2, \tau_1), & i = 2, \end{cases}$$

$$X_{-\alpha_5} Q_i = \begin{cases} Q(\varphi_3), & i = 1, \\ Q(\tau_2), & i = 2, \end{cases} \qquad X_{-\alpha_6} Q_i = \begin{cases} Q(\varphi_4), & i = 1, \\ Q(\tau_4, \tau_1) + 2Q(\tau_3, \tau_2), & i = 2. \end{cases}$$

Hence by consideration of $\dim T(w, \mathrm{Id})$, we find that the singular Schubert varieties are given by $X(w)$, $w = \varphi_i$, $3 \leqslant i \leqslant 5$, and $w = \tau_j$, $j = 4, 5$. To determine $S(w)$, we compute as above $X_{-\beta} Q(\tau)$, $\beta \in \tau(R^+)$, $\tau \leqslant w$. We shall denote by $Q_{\tau,i}$ the extremal weight vector in $V_{\mathbf{Z},\omega_i}$ of weight $\tau(\omega_i)$, $i = 1, 2$. Further, for $1 \leqslant j \leqslant 6$, we shall denote $\tau(\alpha_j)$ by β_j.

(1) $\tau = \varphi_1$. We have (up to ± 1)

$$X_{-\beta_1} Q_{\tau,i} = \begin{cases} Q(\mathrm{Id}), & i = 1, \\ 0, & i = 2, \end{cases} \qquad X_{-\beta_2} Q_{\tau,i} = \begin{cases} 0, & i = 1, \\ Q(\tau_2), & i = 2, \end{cases}$$

$$X_{-\beta_3} Q_{\tau,i} = \begin{cases} Q(\varphi_3), & i = 1, \\ F(\tau_2, \tau_1), & i = 2, \end{cases} \qquad X_{-\beta_4} Q_{\tau,i} = \begin{cases} Q(\varphi_3, \varphi_2), & i = 1, \\ E(\tau_2, \tau_1), & i = 2, \end{cases}$$

$$X_{-\beta_5} Q_{\tau,i} = \begin{cases} Q(\varphi_2), & i = 1, \\ Q(\tau_1), & i = 2, \end{cases} \qquad X_{-\beta_6} Q_{\tau,i} = \begin{cases} Q(\varphi_5), & i = 1, \\ Q(\tau_4, \tau_1) + 2Q(\tau_3, \tau_2), & i = 2, \end{cases}$$

(A) From this we obtain (by considering $\dim T(w, \tau)$) that e_{φ_1} is singular on $X(\varphi_i)$, $i = 3, 4, 5$ and $X(\tau_4)$.

(2) $\tau = \tau_1$:

$$X_{-\beta_1} Q_{\tau,i} = \begin{cases} Q(\varphi_2), & i = 1, \\ 0, & i = 2, \end{cases} \qquad X_{-\beta_2} Q_{\tau,i} = \begin{cases} 0, & i = 1, \\ Q(\mathrm{Id}), & i = 2, \end{cases}$$

$$X_{-\beta_3} Q_{\tau,i} = \begin{cases} Q(\varphi_1), & i = 1, \\ E(\tau_2, \tau_1), & i = 2, \end{cases} \qquad X_{-\beta_4} Q_{\tau,i} = \begin{cases} Q(\varphi_3, \varphi_2), & i = 1, \\ Q(\tau_3, \tau_1), & i = 2, \end{cases}$$

$$X_{-\beta_5} Q_{\tau,i} = \begin{cases} Q(\varphi_4), & i = 1, \\ Q(\tau_3), & i = 2, \end{cases} \qquad X_{-\beta_6} Q_{\tau,i} = \begin{cases} Q(\varphi_3), & i = 1, \\ Q(\tau_4, \tau_2), & i = 2. \end{cases}$$

(B) From this we obtain that e_{τ_1} is singular on $X(\varphi_i)$, $X(\tau_i)$, $i = 4, 5$.

(3) $\tau = \varphi_2$:

$$X_{-\beta_1} Q_{\tau,i} = \begin{cases} Q(\mathrm{Id}), & i = 1, \\ 0, & i = 2, \end{cases} \qquad X_{-\beta_2} Q_{\tau,i} = \begin{cases} 0, & i = 1, \\ Q(\tau_3), & i = 2, \end{cases}$$

$$X_{-\beta_3} Q_{\tau,i} = \begin{cases} Q(\varphi_4), & i = 1, \\ Q(\tau_3, \tau_1), & i = 2, \end{cases} \qquad X_{-\beta_4} Q_{\tau,i} = \begin{cases} Q(\varphi_3, \varphi_2), & i = 1, \\ E(\tau_2, \tau_1), & i = 2, \end{cases}$$

$$X_{-\beta_5} Q_{\tau,i} = \begin{cases} Q(\varphi_1), & i = 1, \\ Q(\mathrm{Id}), & i = 2, \end{cases} \qquad X_{-\beta_6} Q_{\tau,i} = \begin{cases} Q(\varphi_5), & i = 1, \\ Q(\tau_3), & i = 2, \end{cases}$$

(C) From this we obtain that e_{φ_2} is singular on $X(\varphi_i)$, $i = 4, 5$.

(4) $\tau = \tau_2$:

$$X_{-\beta_1} Q_{\tau,i} = \begin{cases} Q(\varphi_3), & i = 1, \\ 0, & i = 2, \end{cases} \qquad X_{-\beta_2} Q_{\tau,i} = \begin{cases} 0, & i = 1, \\ Q(\mathrm{Id}), & i = 2, \end{cases}$$

$$X_{-\beta_3} Q_{\tau,i} = \begin{cases} Q(\mathrm{Id}), & i = 1, \\ F(\tau_2, \tau_1), & i = 2, \end{cases} \qquad X_{-\beta_4} Q_{\tau,i} = \begin{cases} Q(\varphi_3, \varphi_2), & i = 1, \\ Q(\tau_4, \tau_2), & i = 2, \end{cases}$$

$$X_{-\beta_5} Q_{\tau,i} = \begin{cases} Q(\varphi_5), & i = 1, \\ Q(\tau_4), & i = 2, \end{cases} \qquad X_{-\beta_6} Q_{\tau,i} = \begin{cases} Q(\varphi_2), & i = 1, \\ Q(\tau_3, \tau_2), & i = 2. \end{cases}$$

(D) From this it follows that e_{τ_2} is singular on $X(\varphi_5)$ and $X(\tau_4)$.

(5) $\tau = \varphi_3$:

$$X_{-\beta_1} Q_{\tau,i} = \begin{cases} Q(\varphi_1), & i = 1, \\ 0, & i = 2, \end{cases} \qquad X_{-\beta_2} Q_{\tau,i} = \begin{cases} 0, & i = 1, \\ Q(\tau_4), & i = 2, \end{cases}$$

$$X_{-\beta_3} Q_{\tau,i} = \begin{cases} Q(\varphi_5), & i = 1, \\ Q(\tau_4, \tau_2), & i = 2, \end{cases} \qquad X_{-\beta_4} Q_{\tau,i} = \begin{cases} Q(\varphi_3, \varphi_2), & i = 1, \\ F(\tau_2, \tau_1), & i = 2, \end{cases}$$

$$X_{-\beta_5} Q_{\tau,i} = \begin{cases} Q(\mathrm{Id}), & i = 1, \\ Q(\mathrm{Id}), & i = 2, \end{cases} \qquad X_{-\beta_6} Q_{\tau,i} = \begin{cases} Q(\varphi_4), & i = 1, \\ Q(\tau_3, \tau_2), & i = 2. \end{cases}$$

(E) From this it follows that e_{φ_3} is singular on $X(\varphi_5)$.

(6) $\tau = \tau_3$:

$$X_{-\beta_1} Q_{\tau,i} = \begin{cases} Q(\varphi_4), & i=1, \\ 0, & i=2, \end{cases} \qquad X_{-\beta_2} Q_{\tau,i} = \begin{cases} 0, & i=1, \\ Q(\tau_1), & i=2, \end{cases}$$

$$X_{-\beta_3} Q_{\tau,i} = \begin{cases} Q(\mathrm{Id}), & i=1, \\ Q(\tau_3, \tau_1), & i=2, \end{cases} \qquad X_{-\beta_4} Q_{\tau,i} = \begin{cases} Q(\varphi_3, \varphi_2), & i=1, \\ E(\tau_4, \tau_3), & i=2, \end{cases}$$

$$X_{-\beta_5} Q_{\tau,i} = \begin{cases} Q(\varphi_5), & i=1, \\ Q(\tau_5), & i=2, \end{cases} \qquad X_{-\beta_6} Q_{\tau,i} = \begin{cases} Q(\varphi_1), & i=1, \\ Q(\tau_3, \tau_2), & i=2. \end{cases}$$

(F) Hence we see that e_{τ_3} is smooth on all $X(w)$, $w \geqslant \tau_3$.

Now using (A) through (F) (and the fact that Schubert varieties are nonsingular in codimension one), we obtain

THEOREM 3.9. *Let G be of type G_2. The singular Schubert varieties in G/B are given by $X(w)$, $w = \varphi_i$, $i = 3, 4, 5$ and $w = \tau_j$, $j = 4, 5$. Further $S(\varphi_i) = X(\varphi_{i-2})$, $i = 3, 4, 5$ and $S(\tau_4)$ (resp. $S(\tau_5)$) is $X(\tau_2)$ (resp. $X(\tau_1)$).*

Part II. Singular loci of Schubert varieties in G/P

In this part, we shall describe singular loci of Schubert varieties in G/P, G classical and P is a maximal parabolic subgroup given as follows:

A_n: P is any maximal parabolic subgroup.

B_n, C_n: $P = P_i$, $i = 1$ or n.

D_n: $P = P_i$, $i = 1, n-1$ or n.

The proof of the results in this part will apear in [21].

A_n: Let $P = P_d$. Then recall (cf. [10], for example) that W^P, the set of minimal representatives in W/W_P is given by

$$W^P = \{(a_1, \ldots, a_d) \mid 1 \leqslant a_1 < \ldots < a_d \leqslant n+1\}.$$

For $\boldsymbol{a} = (a_1, \ldots, a_d)$, let us denote the associated Schubert variety in G/P by $X(\boldsymbol{a})$ and its singular locus by $S(\boldsymbol{a})$; further, let us denote by $\lambda_{\boldsymbol{a}}$ the partition

$$\lambda_{\boldsymbol{a}}\colon\ a_d - d \geqslant a_{d-1} - (d-1) \geqslant \ldots \geqslant a_1 - 1.$$

THEOREM 3.10. *$S(\boldsymbol{a}) = \bigcup_{\boldsymbol{b}} X(\boldsymbol{b})$ where $\boldsymbol{b} \leqslant \boldsymbol{a}$ and $\lambda_{\boldsymbol{a}}/\lambda_{\boldsymbol{b}}$ is a hook (refer to [22] for the definition of a hook; also recall that $\boldsymbol{b} = (b_1, \ldots, b_d) \leqslant \boldsymbol{a} = (a_1, \ldots, a_d) \Leftrightarrow b_t \leqslant a_t$, $1 \leqslant t \leqslant d$). In particular, $X(\boldsymbol{a})$ is smooth if and only if $\boldsymbol{a}$ is a rectangle.*

The above Theorem (and also the Theorems below) is proved by computing $m_\tau(w)$, the multiplicity of $X(w)$ at e_τ (cf. [20]).

C_n: For $P = P_1$, G/P is a projective space and every Schubert variety is smooth. For $P = P_n$, we have (cf. [10])

$$W^P = \left\{ (a_1, \ldots, a_n) \,\middle|\, \begin{array}{l} (1)\ 1 \leqslant a_1 < \ldots < a_n \leqslant 2n, \\ (2)\ \text{for } 1 \leqslant i \leqslant n, \text{ precisely one of} \\ \quad \{i, 2n+1-i\} \text{ belongs to } \{a_1, \ldots a_n\} \end{array} \right\}.$$

For $\boldsymbol{a} = (a_1, \ldots, a_n)$, denoting as above $\lambda_{\boldsymbol{a}}$: $a_n - n \geqslant a_{n-1} - (n-1) \geqslant \ldots \ldots \geqslant a_1 - 1$, we have $\lambda_{\boldsymbol{a}}$ is self-dual. With notation as above, we have

THEOREM 3.11. *$S(\boldsymbol{a}) = \bigcup_{\boldsymbol{b}} X(\boldsymbol{b})$, where $\boldsymbol{b} \leqslant \boldsymbol{a}$ and $\lambda_{\boldsymbol{a}}/\lambda_{\boldsymbol{b}}$ is a sum of two hooks dual to each other or a self-dual hook (different from a box). In particular, $X(\boldsymbol{a})$ is smooth if and only if $\lambda_{\boldsymbol{a}}$ is a square.*

B_n, $P = P_n$ and D_n, $P = P_{n-1}, P_n$: Let G_1 be of type B_{n-1} and G_2 of type D_n. Let W_i, $i = 1, 2$ be the corresponding Weyl groups. We have (cf. [10])

$$W_1^{P_{n-1}} = \left\{ (a_1, \ldots, a_{n-1}) \,\middle|\, \begin{array}{l} (1)\ 1 \leqslant a_1 < \ldots < a_n \leqslant n_{n-1} \leqslant 2n-1, \\ (2)\ \text{for } 1 \leqslant i \leqslant n-1, \text{ precisely one of} \\ \quad \{i, i'\} \in (a_1, \ldots, a_{n-1}) \text{ where } i' = 2n-i \end{array} \right\},$$

$$W_2^{P_n} = \left\{ (a_1, \ldots, a_n) \,\middle|\, \begin{array}{l} (1)\ 1 \leqslant a_1 < \ldots < a_n \leqslant 2n, \\ (2)\ \text{for } 1 \leqslant i \leqslant n, \text{ precisely one of} \\ \quad \{i, i'\} \in (a_1, \ldots, a_n) \text{ where } i = 2n+1-i \end{array} \right\}.$$

Let θ: $W_1^{P_{n-1}} \to W_2^{P_n}$ be the map $\theta(a_1, \ldots, a_{n-1}) = (a_1, \ldots, a_n)$, where $a_n = n$ or $(n+1)$ according as $\# \{i | a_i > n\}$ is even or odd (under the map θ, the i' appearing in $(a_1, \ldots, a_{n-1})$ (resp. $\theta(a_1, \ldots, a_{n-1})$) should be understood as $2n-i$ (resp. $2n+1-i$)). It turns out that θ is a bijection and that for $\boldsymbol{b} \leqslant \boldsymbol{a}$, $\boldsymbol{a}, \boldsymbol{b} \in W_1^{P_{n-1}}$, the multiplicity of $X(\boldsymbol{a})$ at $\boldsymbol{b}$ and that of $X(\theta(\boldsymbol{a}))$ at $\theta(\boldsymbol{b})$ are equal. A similar remark applies to the map

$$\delta\colon W_2^{P_n} \to W_2^{P_{n-1}}, \qquad \delta(a_1, \ldots, a_n) = (b_1, \ldots, b_{n-1}),$$

where $(b_1, \ldots, b_{n-1})$ is obtained from $(a_1, \ldots, a_{n-1})$ by replacing n by n' $(= n+1)$ (resp. n' by n), if n (resp. n') $\in \{a_1, \ldots, a_{n-1}\}$. Hence we shall state the results just for the case B_n, $P = P_n$. For $\boldsymbol{a} = (a_1, \ldots, a_n) \in W^{P_n}$ denoting by $\lambda_{\boldsymbol{a}}$ the partition $a_n - n \geqslant a_{n-1} - (n-1) \geqslant \ldots \geqslant a_1 - 1$, we have $\lambda_{\boldsymbol{a}}$ is self-dual. With notation as above, we have

THEOREM 3.12. *$S(\boldsymbol{a}) = \bigcup_{\boldsymbol{b}} X(\boldsymbol{b})$, where $\boldsymbol{b} \leqslant \boldsymbol{a}$ and $\lambda_{\boldsymbol{a}}/\lambda_{\boldsymbol{b}}$ is either*

(a) *sum of two hooks dual to each other which are either disjoint or connected at one box or*

(b) *a double hook which is self-dual.*

B_n, $P = P_1$: We have

$$W^{P_1} = \left\{ i \,\middle|\, \begin{array}{l} (1)\ 1 \leqslant i \leqslant 2n+1, \\ (2)\ i \neq n+1 \end{array} \right\}.$$

Further

$$X_{-(e_1-e_j)} Q(\mathrm{Id}) = Q(j),$$
$$X_{-(e_1+e_j)} Q(\mathrm{Id}) = Q(2n+2-j),$$
$$X_{-e_1} Q(\mathrm{Id}) = Q(n+2, n),$$

(note that in this case $(\tau, \varphi) = (n+2, n)$ is the only nontrivial admissible pair). From this we see, by considering $\dim T(w, \mathrm{Id})$, that $X(w) = X(j)$ is smooth for $j = 1, 2, \ldots, n, 2n+1$ and $X(w)$, $w = 2n+1-j$, $1 \leqslant j \leqslant n-1$, is singular.

Proceeding as in Part I, if $X(\tau) = X(j)$, $j \leqslant n-1$, then

$$X_{-\tau(e_1-e_i)} Q(\tau)) = \begin{cases} Q(i) & i > j, \\ Q(i-1), & i \leqslant j, \end{cases}$$

$$X_{-\tau(e_1+e_i)} Q(\tau) = \begin{cases} Q(2n+2-i), & i > j, \\ Q(2n+3-i), & i \leqslant j, \end{cases}$$

$$X_{-\tau(e_1)} Q(\tau) = Q(n+2, n).$$

From this we find that for $X(w) = X(2n+1-j)$, $\dim T(w, \tau) = 2n-j > 2n-j-1 = \dim X(w)$ and for $X(2n+2-j)$, $\dim T(w, \tau) = 2n-j = \dim X(w)$. Hence we obtain

THEOREM 3.13. *Let $X(w) = X(i)$. Then $X(w)$ is smooth for $i = 1, 2, \ldots, n, 2n+1$. For $n+2 \leqslant i \leqslant 2n$, $S(w) = X(2n+1-i)$.*

D_n, $P = P_1$: We have

$$W^{P_1} = \{i,\ 1 \leqslant i \leqslant 2n\}.$$

Further

$$X_{-(e_1-e_i)} Q(\mathrm{Id}) = Q(i), \qquad X_{-(e_1+e_i)} Q(\mathrm{Id}) = Q(2n+1-i).$$

From this, we find as above that $X(w) = X(j)$ is smooth for $j = 1, 2, \ldots, n, n+1, 2n$ and $X(w)$, $w = 2n-j$, $j \leqslant n-2$ is singular. Proceeding as above, if $X(\tau) = X(j)$, then in W^P

$$\tau s_{e_1-e_i} = \begin{cases} i, & i > j, \\ i-1. & i \leqslant j, \end{cases}$$

$$\tau s_{e_1+e_i} = \begin{cases} 2n+1-i, & i > j, \\ 2n+2-i. & i \leqslant j. \end{cases}$$

From this we see that for $X(w) = X(2n-j)$, $\dim T(w, \tau) = 2n-j-1 > 2n-j-2 = \dim X(w)$ and for $X(w) = X(2n+1-j)$, $\dim T(w, \tau) = 2n-j-1 = \dim X(w)$.

Hence we obtain

THEOREM 3.14. *Let $X(w) = X(i)$. Then $X(w)$ is smooth for $i = 1, 2, \ldots, n, n+1, 2n$ and is singular for $i = n+2, \ldots, 2n-1$. For $n+2 \leqslant i \leqslant 2n-1$, $S(w) = X(2n-i)$.*

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