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THE STATIONARY PHASE METHOD IN GEVREY CLASSES AND FOURIER INTEGRAL OPERATORS ON ULTRADISTRIBUTIONS

TODOR V. GRAMCHEV

Department of Differential Equations «Acad. G. Bonchev»
Institute of Mathematics, University of Sofia
Sofia, Bulgaria

1. Introduction

The subject of this paper is the asymptotic behaviour of some integrals for Gevrey symbols and Fourier integral operators in spaces of ultradistributions. Applications to hypoellipticity and propagation of singularities are also considered.

The classical Gevrey pseudodifferential operators were studied by L. Boutet de Monvel and P. Kree [1]. The basic ideas of analytic and Gevrey microlocal analysis were set out by L. Hörmander [11]. There the author posed the problem of developing the theory of analytic and Gevrey Fourier integral operators (F.I.O.'s). The analytic microlocal analysis was developed further by many authors, in particular by J. Sjöstrand [17], [18]. R. Lascar [15] announced results about F.I.O.'s in nonquasianalytic Denjoy–Carleman classes. Asymptotics in the Gevrey category were stated in [6].

Let $X \subset \mathbf{R}^n$ be an open set, $\sigma \geqslant 1$. We denote by $G^\sigma(X)$ the class of Gevrey functions of order σ, i.e. $f(x) \in G^\sigma(X)$ iff $f(x) \in C^\infty(X)$ and for every compact subset K of X ($K \Subset X$) there exists $C_K > 0$ such that

$$(1.1) \qquad |\partial_x^\alpha f(x)| \leqslant C_K^{|\alpha|+1}(\alpha!)^\sigma, \quad x \in K,$$

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{Z}_+^n, \ \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}, \ \alpha! = \alpha_1! \ldots \alpha_n!.$$

In particular, $G^1(X) = A(X)$ is the space of all real analytic functions in X. It is well known that $G_0^\sigma(X) \neq \{0\}$, $\sigma > 1$, while $G_0^1(X) = \{0\}$, where $G_0^\sigma(X) = G^\sigma(X) \cap C_0^\infty(X)$ (the G^σ functions with compact support) [11], [12], [20].

A sequence $f_k \in G^\sigma(X)$, $k = 1, 2, \ldots$, converges to $f \in G^\sigma(X)$ iff for any

$K \Subset X$ one can find a sequence $\varepsilon_k \to 0$ as $k \to \infty$, $\varepsilon_k > 0$, satisfying

(1.2) $$\sup_K |\partial_x^\alpha (f_k - f)(x)| \leqslant \varepsilon_k T^{|\alpha|} (\alpha!)^\sigma, \quad \alpha \in Z_+^n, \ T > 0 \text{ fixed}.$$

When $f_k \in G_0^\sigma(X)$ we have $f_k \to f$ as $k \to \infty$ in $G_0^\sigma(X)$ iff there exist $K \Subset X$ and a sequence $\varepsilon_k \to 0$, as $k \to \infty$, $\varepsilon_k > 0$, for which $\operatorname{supp} f_k \subset K$, $k = 1, 2, \ldots$, and (1.2) holds.

Let $\mathscr{E}'_\sigma(X) = [G^\sigma(X)]^*$, $\sigma \geqslant 1$, be the space of all σ-ultradistributions with compact support, and for $\sigma > 1$ put $\mathscr{D}'_\sigma(X) = [G_0^\sigma(X)]^*$ [1]. Evidently the Fourier transformation is well defined in $\mathscr{E}'_\sigma(X)$:

(1.3) $$\hat{u}(\xi) = \int e^{-ix\cdot\xi} u(x)\, dx := u(e^{-ix\cdot\xi}), \quad x \cdot \xi = x_1 \xi_1 + \ldots + x_n \xi_n.$$

Let $\sigma > 1$, $u \in \mathscr{D}'_\sigma(X)$ and $\varrho^0 = (x^0, \xi^0) \in T^*(X)\backslash 0$. The point ϱ^0 does not belong to the G^σ *wave front set* of u, $\mathrm{WF}_\sigma u$, iff one may choose a function $\varphi(x) \in G_0^\sigma(X)$, $\varphi(x^0) \neq 0$, an open cone $T \ni \xi^0$ in $R^n\backslash 0$ and a positive constant C with the property

(1.4)
$$|(\varphi u)\hat{}(\xi)| \leqslant C^{N+1} (N!)^\sigma \langle \xi \rangle^{-N}, \quad \xi \in T, \ N = 0, 1, \ldots, \ \langle \xi \rangle = (1+\xi^2)^{1/2}.$$

Remark 1.1. It is easy to deduce that the inequalities (1.4) are equivalent to

(1.5) $$|(\varphi u)\hat{}(\xi)| \leqslant C \exp(-|\xi|^{1/\sigma}/C), \quad \xi \in T, \ C > 0.$$

The G^σ symbols are considered in § 2. The asymptotics of certain integrals and the stationary phase method for G^σ symbols are the subject of § 3. The G^σ Fourier integral operators are investigated in § 4, and § 5 deals with some applications and examples.

2. G^σ symbols and their properties

Let us recall [1], [6], [7], [17]

DEFINITION 2.1. Let $\Omega \subset R_x^n \times (R_\theta^d \backslash 0)$ be an open conic set (i.e. $(x, \theta) \in \Omega \Leftrightarrow (x, t\theta) \in \Omega$, $\forall t > 0$). The formal sum

(2.1) $$\sum_{k=0}^{\infty} p_{m-k}(x, \theta), \quad m \in R,$$

is called a *formal G^σ symbol* of order m in Ω, $\sigma \geqslant 1$, iff $p_{m-k} \in G^\sigma(\Omega)$, $\operatorname{ord}_\theta p_{m-k} = m-k$ (i.e. $p_{m-k}(x, t\theta) = t^{m-k} p_{m-k}(x, \theta)$, $t > 0$), $k = 0, 1, \ldots$, and for every compactly based conic set $\Omega^0 \Subset \Omega$ (i.e. $\Omega^0 \cap \{|\theta| = 1\}$ is compact in Ω) there exists $C_0 > 0$ satisfying

(2.2) $$|\partial_x^\alpha \partial_\theta^\beta p_{m-k}(x, \theta)| \leqslant C_0^{|\alpha|+|\beta|+k+1} (k!\,\alpha!\,\beta!)^\sigma |\theta|^{m-k-|\beta|},$$
$$(x, \theta) \in \Omega^0, \ k = 0, 1, \ldots, \ \alpha \in Z_+^n, \ \beta \in Z_+^d.$$

Let $FS^m_\sigma(\Omega)$ be the space of all formal G^σ symbols of order m in Ω. Evidently $FS^{m'}_{\sigma'}(\Omega) \subset FS^m_\sigma(\Omega)$, $m' \leqslant m$, $\sigma' \leqslant \sigma$.

DEFINITION 2.2. Let $\sum_{k=0}^\infty p_{m-k}(x, \theta) \in FS^m_\sigma(\Omega)$ and let $\Omega^0 \Subset \Omega$ be an open cone. We say that $p(x, \theta)$ is a *full G^σ realization* of $\sum_{k=0}^\infty p_{m-k}$ in Ω^0 if $p(x, \theta)$ is a G^σ function in x and there is a positive constant C such that

$$(2.3) \qquad \Big|\partial_x^\alpha \partial_\theta^\beta \Big(p - \sum_{k=0}^N p_{m-k}\Big)\Big| \leqslant C^{|\alpha|+|\beta|+N+1}(\alpha!\,\beta!\,N!)^\sigma |\theta|^{m-N-1},$$

$$(x, \theta) \in \Omega^0 \cap \{|\theta| \geqslant 1\},\ N = 0, 1, \ldots,\ \alpha \in \boldsymbol{Z}^n_+,\ \beta \in \boldsymbol{Z}^d_+.$$

The symbol $p(x, \theta)$ is called a *simple G^σ realization* of $\sum_{k=0}^\infty p_{m-k}$ in Ω^0 if for every $\beta \in \boldsymbol{Z}^d_+$ there exists $C_\beta > 0$ satisfying

$$(2.3) \qquad \Big|\partial_x^\alpha \partial_\theta^\beta \Big(p - \sum_{k=0}^N p_{m-k}\Big)(x, \theta)\Big| \leqslant C_\beta^{|\alpha|+N+1}(\alpha!\,N!)^\sigma |\theta|^{m-|\beta|-N-1},$$

$$(x, \theta) \in \Omega^0 \cap \{|\theta| \geqslant 1\},\ \alpha \in \boldsymbol{Z}^n_+,\ N = 0, 1, \ldots$$

Set

$$S_\sigma^{-\infty}(\Omega) = \{b(x, \theta):\ \forall \operatorname{cone} \Omega^0 \Subset \Omega,\ \exists C_0 > 0,$$
$$|\partial_x^\alpha b(x, \theta)| \leqslant C_0^{|\alpha|+1}(\alpha!)^\sigma \exp(-|\theta|^{1/\sigma}/C_0)\}.$$

This is an analogue of $S^{-\infty}$ in C^∞ theory [10].

THEOREM 2.1. *Let $\sum_{k=0}^\infty p_{m-k}(x, \theta) \in FS^m_\sigma(\Omega)$, $\Omega^0 \Subset \Omega$. Suppose $\sigma > 1$. Then there exists a full G^σ realization $p(x, \theta) \overset{\sigma}{\sim} \sum_{k=0}^\infty p_{m-k}(x, \theta)$ in Ω^0 and any other full G^σ realization $\tilde{p}(x, \theta) \overset{\sigma}{\sim} \sum_{k=0}^\infty p_{m-k}(x, \theta)$ satisfies*

$$(2.4) \qquad |\partial_x^\alpha \partial_\theta^\beta (p - \tilde{p})(x, \theta)| \leqslant C^{|\alpha|+|\beta|+N+1}(\alpha!\,\beta!\,N!)^\sigma \langle\theta\rangle^{m-|\beta|-N-1},$$

$$(x, \theta) \in \Omega^0,\ N = 0, 1, \ldots,\ \alpha \in \boldsymbol{Z}^n_+,\ \beta \in \boldsymbol{Z}^d_+,\ C > 0,\ C = C(p, \tilde{p}, \Omega^0).$$

In the case $\sigma = 1$ *one can find a simple G^1 (analytic) realization $p(x, \theta) \overset{1}{\sim} \sum_{k=0}^\infty p_{m-k}(x, \theta)$ in Ω^0 unique modulo the estimates*

$$(2.5) \qquad |\partial_x^\alpha \partial_\theta^\beta (p - \tilde{p})(x, \theta)| \leqslant C_\beta^{|\alpha|+N+1}\alpha!\,N!\,\langle\theta\rangle^{m-|\beta|-N-1},$$

$$(x, \theta) \in \Omega^0,\ \beta \in \boldsymbol{Z}^d_+,\ \alpha \in \boldsymbol{Z}^n_+,\ N = 0, 1, \ldots,\ C_\beta > 0.$$

Proof. Choose $h \in G_0^\sigma(\boldsymbol{R}^n)$ if $\sigma > 1$ ($h \in C_0^\infty(\boldsymbol{R}^n)$ when $\sigma = 1$) such that $\operatorname{supp} h \subset \{|\theta| \leqslant 2\}$, $h(\theta) = 1$ for $|\theta| \leqslant 1$.

Let R be a positive constant considerably larger than C_0 where C_0 is the constant in (2.2) for $\Omega^0 \Subset \Omega$. Put

$$(2.6) \qquad p(x, \theta) = \sum_{k=0}^\infty p_{m-k}(x, \theta)\left(1 - h\left(\frac{\theta}{(k+1)R}\right)\right).$$

The symbol $p(x, \theta)$ has the desired properties of full (or simple) G^σ realization. This is established by direct calculations using (1.1), (2.2) [1], [7],

[17]. The uniqueness modulo the estimates (2.4), (2.5) follows from Definition 2.2.

Remark 2.1. The relation $G_0^1(X) = \{0\}$ prevents us from constructing full analytic realizations.

We point out that (2.2), (2.3) imply

$$(2.7) \quad |\partial_x^\alpha \partial_\theta^\beta p(x,\theta)| \leqslant C_0^{|\alpha|+|\beta|+1}(\alpha!\,\beta!)^\sigma \langle\theta\rangle^{m-|\beta|},$$
$$(x,\theta)\in\Omega^0,\ C_0>0,\ \alpha\in Z_+^n,\ \beta\in Z_+^d,$$

if $p(x,\theta)$ is a full G^σ realization in Ω^0, and

$$(2.8) \quad |\partial_x^\alpha \partial_\theta^\beta p(x,\theta)| \leqslant [C_0(\beta)]^{|\alpha|+1}(\alpha!)^\sigma \langle\theta\rangle^{m-|\beta|},$$
$$(x,\theta)\in\Omega^0,\ \beta\in Z_+^d,\ C_0(\beta)>0,\ \alpha\in Z_+^n,$$

when $p(x,\theta)$ is a simple G^σ realization in Ω^0.

In fact, one can construct a full realization globally in Ω for $\sigma > 1$. More precisely, we have

THEOREM 2.2. *Let $\sum_{k=0}^{\infty} p_{m-k}(x,\theta)\in FS_\sigma^m(\Omega)$, $\sigma>1$. Then for every $\tau>\sigma$ there exists $p(x,\theta)$ belonging to G^τ in (x,θ) which is a full realization of $\sum_{k=0}^{\infty} p_{m-k}$ in an arbitrary cone $\Omega^0 \Subset \Omega$. The symbol $p(x,\theta)$ with this property is unique modulo $S_\tau^{-\infty}(\Omega)$.*

Proof. Define the following function:

$$(2.9) \qquad s(x,\theta) = \max_{\alpha,\beta,k}\left|\frac{\partial_x^\alpha \partial_\theta^\beta p_{m-k}(x,\theta)}{(\alpha!\,\beta!\,k!)^\sigma |\theta|^{m-k-|\beta|}}\right|^{\frac{1}{|\alpha|+|\beta|+k+1}}, \qquad (x,\theta)\in\Omega.$$

Choose $h(\theta)\in E_0^{1+\varepsilon}(R^n)$, $h(\theta)=1$ for $|\theta|\leqslant 1$, $\operatorname{supp} h\subset\{|\theta|\leqslant 2\}$, $0<\varepsilon<(\tau-\sigma)/4$. We can construct a function $R(x,\theta)\in G^{1+\varepsilon}(\Omega)$, $\operatorname{ord}_\theta R = 0$, satisfying

$$(2.10) \qquad R(x,\theta) \geqslant [4(n+d)]^{4(n+d)}\big(s(x,\theta)+2\big).$$

Then the desired symbol $p(x,\theta)$ is given by

$$p(x,\theta) = \sum_{k=0}^{\infty} p_{m-k}(x,\theta)\left(1-h\left(\frac{\theta}{(k+1)R(x,\theta)}\right)\right).$$

The check is based on the direct estimate of the derivatives of $p(x,\theta)$, $(x,\theta)\in\Omega$, using the inequalities

$$\left|(\partial_\theta^\alpha h)\left(\frac{\theta}{(k+1)R(x,\theta)}\right)\left(\frac{|\theta|}{(k+1)R(x,\theta)}\right)^l\right| \leqslant C^{|\alpha|}2^l(\alpha!)^{1+\varepsilon}, \qquad C>0,$$

and the rule for differentiation

$$\partial_x^k\left(\frac{1}{(f(x))^l}\right)=\sum_{j=0}^{k}(-1)^j\frac{(l+j)!}{l!\,j!}\frac{1}{(f(x))^{l+j}}\partial_y^k\big((f(y))^l-(f(x))^l\big)^j\big|_{y=x},$$
$$k=0,1,\ldots,\ l=0,1,\ldots,\ f(x)\in C^\infty,\ f(x)\neq 0.$$

Let

$$S_\sigma^m(\Omega)=\{p(x,\theta)\in G^\sigma(\Omega)\colon\ \forall\Omega^0\Subset\Omega,\ \forall\beta\in Z_+^d,\ \exists C_0(\beta)>0,$$
$$|\partial_x^\alpha\partial_\theta^\beta p(x,\theta)|\leqslant[C_0(\beta)]^{|\alpha|+1}(\alpha!)^\sigma\langle\theta\rangle^{m-|\beta|},\ \alpha\in Z_+^n\}.$$

One important question about the estimates above is whether they remain invariant under homogeneous G^σ diffeomorphisms.

THEOREM 2.3. *Let* $\varkappa(x,\theta)=(z,\eta)=(z(x,\theta),\eta(x,\theta))\in R_z^n\times(R_\eta^d\backslash 0)$, $z,\eta\in G^\sigma(\Omega)$, $\mathrm{ord}_\theta z=0$, $\mathrm{ord}_\theta\eta=1$ *and* $\varkappa$ *is a diffeomorphism between* Ω *and* $\tilde\Omega=\varkappa(\Omega)\subset R_z^n\times(R_\eta^d\backslash 0)$. *Then* $\varkappa$ *induces the isomorphisms*

$$\text{(2.11)}\qquad \begin{aligned}&\varkappa^*\colon\ FS_\sigma^m(\Omega)\to FS_\sigma^m(\tilde\Omega),\\ &\varkappa^*\colon\ S_\sigma^m(\Omega)/S_\sigma^{-\infty}(\Omega)\to S_\sigma^m(\tilde\Omega)/S_\sigma^{-\infty}(\tilde\Omega),\end{aligned}$$

and if $\tilde p(z,\eta)$ *is a full (simple)* G^σ *realization of* $\sum_{k=0}^\infty\tilde p_{m-k}(z,\eta)$ *then* $\tilde p(\varkappa(x,\theta))$ *is also a full (simple)* G^σ *realization of* $\sum_{k=0}^\infty\tilde p_{m-k}(\varkappa(x,\theta))$.

Proof. Let $\sum_{k=0}^\infty\tilde p_{m-k}(z,\eta)\in FS_\sigma^m(\tilde\Omega)$ and put $p_{m-k}(x,\theta)=\tilde p_{m-k}(\varkappa(x,\theta))$. We have

$$\partial_{x_j}p_{m-k}(x,\theta)=L_j(z,\eta,\partial_z,\partial_\eta)\tilde p_{m-k}(z,\eta)|_{(z,\eta)=\varkappa(x,\theta)},\quad j=1,\ldots,n,$$
$$\partial_{\theta_s}p_{m-k}(x,\theta)=T_s(z,\eta,\partial_z,\partial_\eta)\tilde p_{m-k}(z,\eta)|_{(z,\eta)=\varkappa(x,\theta)},\quad s=1,\ldots,d,$$

where

$$\text{(2.12)}\qquad \begin{aligned}L_j&=\sum_{\nu=1}^n\frac{\partial z_\nu}{\partial x_j}(\varkappa^{-1}(z,\eta))\,\partial_{z_\nu}+\sum_{\mu=1}^d\frac{\partial\eta_\mu}{\partial x_j}(\varkappa^{-1}(z,\eta))\,\partial_{\eta_\mu},\quad j=1,\ldots,n,\\ T_s&=\sum_{\nu=1}^n\frac{\partial z_\nu}{\partial\theta_s}(\varkappa^{-1}(z,\eta))\,\partial_{z_\nu}+\sum_{\mu=1}^d\frac{\partial\eta_\mu}{\partial\theta_s}(\varkappa^{-1}(z,\eta))\,\partial_{\eta_\mu},\quad s=1,\ldots,d.\end{aligned}$$

Thus

$$\text{(2.13)}\qquad \partial_x^\alpha\partial_\theta^\beta p_{m-k}(x,\theta)=L^\alpha T^\beta\tilde p_{m-k}(z,\eta)|_{(z,\eta)=\varkappa(x,\theta)},$$
$$L^\alpha=L_1^{\alpha_1}\ldots L_n^{\alpha_n},\qquad T^\beta=T_1^{\beta_1}\ldots T_d^{\beta_d}.$$

LEMMA 3.1. *Let* $f(t)\in G^\sigma(\Delta)$, *and let* $\mathscr{A}$ *be a family of* G^σ *functions on* $\Delta\subset R$ *such that for some positive constants* C_0,C_1,C_2,

$$\text{(2.14)}\qquad \begin{aligned}&|d_t^j f(t)|\leqslant C_0C_1^j(j!)^\sigma,\quad t\in\Delta,\ j=0,1,\ldots\\ &|d_t^j a(t)|\leqslant C_2^{j+1}(j!)^\sigma,\quad t\in\Delta,\ j=0,1,\ldots,\ a(t)\in\mathscr{A}.\end{aligned}$$

Then there exists a constant $C_3 > 0$ depending only on C_1, C_2 so that if $Q_\nu(t, \partial_t) = a_\nu(t) d_t$, $a_\nu(t) \in \mathscr{A}$, $\nu = 1, \ldots, N$, $N \in \mathbf{Z}_+$, we have

$$|Q_1 \ldots Q_N f(t)| \leqslant C_0 C_3^N (N!)^\sigma, \qquad t \in \Delta. \tag{2.15}$$

Proof of the lemma. One obtains inductively

$$Q_1 \ldots Q_N f(t) = \sum\nolimits^N a_1(t) d_t^{s_1} a_2(t) \ldots d_t^{s_{N-1}} a_N(t) d_t^{s_N} f(t)$$

where $\sum^N$ denotes the sum over $s_1, \ldots, s_N$ with $s_1 + \ldots + s_N = N$, $\sum_{\nu=1}^j s_\nu \leqslant j$, $j = 1, \ldots, N$.

Put $M_N = \sum^N (s_1! \ldots s_N!)^\sigma$. Then

$$M_{N+1} = M_N + \sum\nolimits^N (s_1! \ldots s_N!)\big((s_1+1)^\sigma + \ldots + (s_N+1)^\sigma\big).$$

One establishes by induction the inequality

$$M_N \leqslant 2^{\sigma N}(N!)^\sigma, \qquad N = 0, 1, \ldots$$

Taking $C_3 = \max(2^\sigma C_2^2, 2^\sigma C_1 C_2)$ we obtain (2.15).

Let $\Omega^0 \Subset \Omega$ be an open cone and let $\tilde{\Omega}^0 = \varkappa(\Omega^0)$. Take $\tilde{C}_0$ to be a positive constant for which the corresponding estimates of $\tilde{p}_{m-k}(z, \eta)$ are valid in $\tilde{\Omega}^0$. The definition of T_j, L_s shows that $L^\alpha T^\beta \tilde{p}_{m-k}(z, \eta)$ consists of $(n+d)^{|\alpha|+|\beta|}$ terms of the type (2.15). One proves with similar arguments that

$$|L^\alpha T^\beta \tilde{p}_{m-k}(z, \eta)| \leqslant \tilde{C}_0^{k+1} \tilde{C}_3^{|\alpha|+|\beta|} (\alpha! \beta! k!)^\sigma, \qquad (z, \eta) \in \tilde{\Omega}^0 \cap \{|\eta| = 1\},$$

and the fact that $\varkappa$ is a homogeneous G^σ diffeomorphism implies $\sum_{k=0}^\infty p_{m-k}(x, \theta) \in FS_\sigma^m(\Omega)$.

The other assertions of Theorem 2.3 are deduced in the same way, using the considerations of Lemma 3.1.

3. The stationary phase method in the class of G^σ symbols

Let $\Omega = X \times Y \times \mathbf{R}^+$, where X, Y are open domains in $\mathbf{R}^n$, $\mathbf{R}^l$ respectively. As in the C^∞ case we will study the asymptotics for $\lambda \to +\infty$ of the integral

$$I(y, \lambda) = \int e^{i\lambda\Phi(x,y)} g(x, y, \lambda)\, dx. \tag{3.1}$$

Here $\Phi(x, y)$ is a real-valued real-analytic function in $\bar{X} \times \bar{Y}$ and $g(x, y, \lambda) \in S_\sigma^m(\Omega)$, $\sigma > 1$. We require that

(3.2) For every $K_1 \Subset Y$ there exists $K_2 \Subset X$ such that $g(x, y, \lambda) = 0$ when $x \notin K_2$, $y \in K_1$, $\lambda > 0$.

PROPOSITION 3.1. *Let the condition* (3.2) *be valid and let $\Phi_x \neq 0$ for $(x, y) \in \operatorname{supp} g$. Then $I(y, \lambda) \in S_\sigma^{-\infty}(Y \times \mathbf{R}^+)$.*

Proof. Choose and fix an arbitrary $K_1 \Subset Y$ and let $K_2 \Subset X$ be a set satisfying (3.2). Using a partition of unity with G_0^σ functions we may assume without loss of generality that $\Phi_{x_1}(x, y) \neq 0$, $(x, y) \in K_2 \times K_1$. This allows us to make the following change of variables: $x \to z$, $z_1 = \Phi(x, y)$, $z_j = x_j$, $j = 2, \ldots, n$. Put

$$\tilde{g}(z, y, \lambda) = g\big(x(z, y), y, \lambda\big) |\det x_z'(z, y)|.$$

Thus we get $I(y, \lambda) = \int e^{i\lambda z_1} \tilde{g}(z, y, \lambda)\, dz$ and integration by parts gives

$$\begin{aligned} |\partial_y^\beta I(y, \lambda)| &\leqslant \Big|(-i\lambda)^{N+[m]+1} \int e^{i\lambda z_1} \partial_{z_1}^{N+[m]+1} \tilde{g}(z, y, \lambda)\, dz\Big| \\ &\leqslant C^{|\beta|+N+1} (\beta!\, N!)^\sigma \lambda^{-N}, \qquad N = 0, 1, \ldots, C = C(m, K_1, K_2) > 0, \end{aligned}$$

or equivalently

$$|\partial_y^\beta I(y, \lambda)| \leqslant C_1^{|\beta|+1} (\beta!)^\sigma \exp(-\lambda^{1/\sigma}/C_1)$$

which proves the proposition.

Now we suppose that for each $y \in Y$ there is a unique critical point $x(y)$ of Φ with respect to x, i.e. $\Phi_x\big(x(y), y\big) = 0$, and in addition this critical point is nondegenerate:

$$\det Q(y) \neq 0, \quad y \in Y, \quad Q(y) = d_x^2 \Phi\big(x(y), y\big). \tag{3.3}$$

Next we require that $g(x, y, \lambda)$ be a full (simple) G^σ realization of $\sum_{k=0}^\infty g_{m-k}(x, y) \lambda^{m-k} \in FS_\sigma^m(\Omega)$, $\sigma > 1$.

THEOREM 3.1. *Under the above conditions we have*

$$I(y, \lambda) = e^{i\lambda \Phi(x(y), y)} q(y, \lambda) \tag{3.4}$$

where $q(y, \lambda)$ *is a full (simple)* $G^{2\sigma-1}$ *realization in* $Y \times \mathbf{R}^+$ *of*

$$\sum_{k=0}^\infty q_{-n/2+m-k}(y) \lambda^{-n/2+m-k} \in FS_{2\sigma-1}^{m-n/2}(Y \times \mathbf{R}^+). \tag{3.5}$$

If in addition $\sum_{k=0}^\infty g_{m-k}(x, y) \lambda^{m-k}$ *is a formal analytic* (G^1) *symbol near every point* $\big(x(y), y\big)$, $y \in Y$, *then no loss of Gevrey regularity occurs,*

$$\sum_{k=0}^\infty q_{-n/2+m-k}(y) \lambda^{-n/2+m-k} \in FS_1^{m-n/2}(Y \times \mathbf{R}^+),$$

and $q(y, \lambda)$ *is a full (simple)* G^σ *realization of this last formal symbol in* $Y \times \mathbf{R}^+$.

Proof. As in the C^∞ case (using the Morse lemma for analytic functions) one has to consider, after an appropriate change of variables $x \to z$, the integral ($x = z$ again)

$$q(y, \lambda) = \int e^{i\lambda(Q(y)x, x)/2} g(x, y, \lambda)\, dx. \tag{3.6}$$

According to the well known formula,

$$(3.7)\qquad q_{-n/2+m-k}(y) = \frac{e^{i(\pi/4)\operatorname{sign} Q(y)}}{\sqrt{|\det Q(y)|}} \sum_{j+s=k} \frac{(2i)^{-s}}{s!} (Q^{-1}(y)\,\partial_x, \partial_x)^s g_{m-j}(0, y)$$

and the expression above yields the loss of $G^{\sigma-1}$ regularity. The estimate of the remainder (after the Fourier transformation of g and $e^{i\lambda(Q(y)x,x)/2}$ in (3.6)) follows from the following assertion: if $\mathscr{A}$ is a family consisting of G_0^σ functions supported in a fixed compact subset K and for some $C > 0$

$$\sup|\partial_x^\alpha r(x)| \leqslant C^{|\alpha|+1}(\alpha!)^\sigma, \qquad \alpha \in Z_+^n,\ r(x) \in \mathscr{A},$$

then there is a positive constant C_1 depending only on C and K such that

$$|\hat{r}(\xi)| \leqslant C_1^{N+1}(N!)^\sigma \langle\xi\rangle^{-N}, \qquad N = 0, 1, \ldots,\ r(x) \in \mathscr{A}.$$

This assertion is deduced using integration by parts in $\hat{r}(\xi)$.

4. G^σ Fourier integral operators

Let $\sigma > 1$, $\Omega = X \times Y \times \Gamma$, where X, Y are open domains in $\boldsymbol{R}_x^n$, $\boldsymbol{R}_y^l$ respectively and Γ is an open cone in $\boldsymbol{R}_\theta^d \backslash 0$.

We suppose that $\Phi(x, y, \theta)$ is an analytic phase function in Ω, i.e. $\Phi(x, y, \theta) \in A(\bar{\Omega})$, $\operatorname{ord}_\theta \Phi = 1$, $\operatorname{Im}\Phi \geqslant 0$, $\Phi_\theta = 0 \Rightarrow \Phi_x \neq 0$, $\Phi_y \neq 0$.

Consider the F.I.O.

$$(4.1)\qquad Iu(x) = \int e^{i\Phi(x,y,\theta)} p(x, y, \theta) u(y)\, dy\, d\theta, \qquad u(y) \in G_0^\sigma(Y),$$

where $\operatorname{supp}_\theta p \subset \Gamma^0 \Subset \Gamma$, $|\partial_x^\alpha \partial_\theta^\beta \partial_y^\gamma p| \leqslant C^{|\alpha|+|\beta|+|\gamma|+1}(\alpha!\,\beta!\,\gamma!)^\sigma \langle\theta\rangle^{m-|\beta|}$, $C > 0$, $\forall \alpha, \beta, \gamma$.

The theorem below was stated by R. Lascar [15] for distributions.

THEOREM 4.1. *The operator I has the following properties*:

(i) *I: $G_0^\sigma(Y) \to G^\sigma(X)$ continuously and extends naturally to a continuous linear operator I: $\mathscr{E}'_\sigma(Y) \to \mathscr{D}'(X)$.*

(ii) *For any ultradistribution $u \in \mathscr{E}'_\sigma(Y)$*

$$\operatorname{WF}_\sigma(Iu) \subset \{(x, \xi) \in T^*X\backslash 0 \colon \exists (y, \theta), (x, y, \theta) \in \operatorname{supp} p,$$
$$\Phi_\theta(x, y, \theta) = 0, (y, -\Phi_y(x, y, \theta)) \in WF_\sigma u, \xi = \Phi_x(x, y, \theta)\}.$$

Sketch of the proof. Let $K_1 \Subset Y$, $K_2 \Subset X$ be arbitrary fixed compact sets and let $u(y) \in G_0^\sigma(K_1)$ satisfy for some $C_1 > 0$

$$(4.2)\qquad \sup|\partial_y^\beta u(y)| \leqslant C_1^{|\beta|+1}(\beta!)^\sigma, \qquad \forall \beta \in Z_+^l.$$

One can write $p(x, y, \theta) = p_1(x, y, \theta) + p_2(x, y, \theta)$ so that $\Phi_\theta \neq 0$ on $\operatorname{supp} p_1$, $\Phi_x \neq 0$, $\Phi_y \neq 0$ on $\operatorname{supp} p_2$.

Let

$$I_s u(x) = \int e^{i\Phi(x,y,\theta)} p_s(x, y, \theta) u(y)\, dy\, d\theta, \qquad s = 1, 2. \tag{4.3}$$

Evidently $|\operatorname{Re}\Phi_\theta| + |\operatorname{Im}\Phi_\theta| \neq 0$ on $\operatorname{supp} p_1$.

As in the proof of Proposition 3.1 one may assume $\operatorname{Re}\Phi_{\theta_1} \neq 0$ on $\operatorname{supp} p_1(x, y, \theta)$, and making the homogeneous change of variables $\eta_1 = \Phi(x, y, \theta)$, $\eta_j = \theta_j$, $j = 2, \ldots, n$, we obtain (with $\tilde{p} = p_1 |\det \theta'_\eta|$)

$$I_1 u(x) = \int e^{i(\eta_1 + i\Phi_0(x,y,\eta))} \tilde{p}(x, y, \eta) u(y)\, dy\, d\theta, \qquad \Phi_0 \text{ real}. \tag{4.4}$$

Integration by parts with respect to η_1 leads to

$$|\partial_x^\alpha I_1 u(x)| \leqslant C_2^{|\alpha|+1} (\alpha!)^\sigma, \qquad x \in K_2,\ C_2 = C_2(C_1, K_1, K_2) > 0.$$

Concerning $I_2 u(x)$ we have

$$\begin{aligned} \partial_x^\alpha I_2 u(x) &= \int \partial_x^\alpha (e^{i\Phi} p_2) u(y)\, dy\, d\theta \\ &= \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \sum_{j=0}^{|\beta|} \int e^{i\Phi(x,y,\theta)} c_j^\beta(x, y, \theta)\, \partial_x^{\alpha-\beta} p_2(x, y, \theta) u(y)\, dy\, d\theta. \end{aligned} \tag{4.5}$$

Here $\partial_x^\beta(e^{i\Phi}) = \sum_{j=0}^{|\beta|} e^{i\Phi} c_j^\beta(x, y, \theta)$, $\operatorname{ord}_\theta c_j^\beta = j$, $j = 0, 1, \ldots, |\beta|$, $\beta \in Z_+^n$, and there exists $C_3 > 0$ satisfying in particular (Φ analytic)

$$|\partial_y^\gamma c_j^\beta(x, y, \theta)| \leqslant C_3^{|\gamma|+|\beta|+1} (|\beta|-j)!\, \gamma!\, |\theta|^j, \qquad \gamma \in Z_+^l,\ \beta \in Z_+^n, \tag{4.6}$$
$$j = 0, 1, \ldots, |\beta|,\ (x, y, \theta) \in \operatorname{supp} p_2,\ (x, y) \in K_2 \times K_1.$$

The estimates above imply the existence of a positive constant C_4 satisfying

$$\left|\partial_y^\gamma \left(c_j^\beta(x, y, \theta)\, \partial_x^{\alpha-\beta} p_2(x, y, \theta)\right)\right| \leqslant C_4^{|\gamma|+|\alpha|+1} \left(\gamma!\,(|\alpha|-j)!\right)^\sigma \langle\theta\rangle^{m+j}, \tag{4.7}$$
$$\gamma \in Z_+^l,\ \alpha \in Z_+^n,\ \beta \in Z_+^n,\ \beta \leqslant \alpha,\ j = 0, 1, \ldots, |\beta|,\ (x, y) \in K_2 \times K_1.$$

In view of (4.7) and the arguments in the proof of Theorem 2.3, if ${}^tL = \sum_{j=1}^{l} i^{-1} k_j(x, y, \theta)\, \partial_{y_j}$, $k_j = |\theta|\, \bar{\Phi}_{y_j}/|\Phi_y|^2$, then $|\theta|^{-1} L(e^{i\Phi}) = e^{i\Phi}$ and for some $C_5 > 0$ (assuming $p_2 = 0$ for $|\theta| \leqslant 1$)

$$|L^\nu(c_j^\beta \cdot \partial_x^{\alpha-\beta} p_2 \cdot u)(x, y, \theta)| \leqslant C_5^{\nu+|\alpha|+1} \left((|\alpha|-j)!\, \nu!\right)^\sigma \langle\theta\rangle^{m+j}, \tag{4.8}$$
$$\nu = 0, 1, \ldots,\ x \in K_2,\ y \in K_1,\ \alpha, \beta \in Z_+^n,\ \beta \leqslant \alpha,\ j = 0, 1, \ldots, |\beta|.$$

Thus when $x \in K_2$ we get

$$\begin{aligned} |\partial_x^\alpha I_2 u(x)| &= \left| \int e^{i\Phi} \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \sum_{j=0}^{|\beta|} \frac{L^{j+m+d+1}}{|\theta|^{j+m+d+1}} (c_j^\beta \cdot \partial_x^{\alpha-\beta} p_2 \cdot u)\, dy\, d\theta \right| \\ &\leqslant \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \sum_{j=0}^{|\beta|} C_6^{|\alpha|+m+d+1} (\alpha!)^\sigma \int_{K_1} dy \int_{|\theta| \geqslant 1} |\theta|^{-d-1}\, d\theta \leqslant C_7^{|\alpha|+1} (\alpha!)^\sigma. \end{aligned} \tag{4.9}$$

The fact that C_7 depends only on C_1, K_1, K_2, p means that I: $G_0^\sigma(Y)$

$\to G^\sigma(X)$ is continuous. This proves (i) because the formal adjoint ${}^tI$ of I acts continuously from $G_0^\sigma(X)$ to $G^\sigma(Y)$.

The inclusion (ii) is established with the methods in (i) (integration by parts, applying the estimates of the type (4.8), (4.9)) and taking into account Theorem 2 from [6] (see also [7], Chap. 3).

5. Some applications and examples

Let now $X = Y$, $\Gamma \subset \mathbf{R}^n \backslash 0$, $\Phi(x, y, \theta) = (x-y)\cdot\theta$ and let $p(x, y, \theta)$ be a full G^σ realization of $\sum_{k=0}^{\infty} p_{m-k}(x, y, \theta) \in FS_\sigma^m(\Omega)$, $\sigma > 1$. Consider the p.d.o.

$$p(x, D)u(x) = \int e^{i(x-y)\theta} p(x, y, \theta) u(y)\, dy\, d\theta. \tag{5.1}$$

If we do not impose further restrictions on $\sum_{k=0}^{\infty} p_{m-k}(x, y, \theta)$ near the diagonal $\{x = y\}$ the usual operations for p.d.o.'s like expressing the full symbol $\sigma_p(x, \xi)$, transposition and composition of two p.d.o.'s will lead to the loss of some sort of $\sigma - 1$ regularity.

So naturally we require that in a neighbourhood of $\{x = y\}$, $\sum_{k=0}^{\infty} p_{m-k}$ be a formal analytic symbol with respect to (x, y), i.e.

$$|\partial_x^\alpha \partial_y^\beta \partial_\theta^\gamma p_{m-k}(x, y, \theta)| \leqslant C^{|\alpha|+|\beta|+|\gamma|+k+1} \alpha!\, \beta!\, (k!\, \gamma!)^\sigma |\theta|^{m-k-|\gamma|}, \qquad |x-y| < \varepsilon. \tag{5.2}$$

(If $\Gamma = \mathbf{R}^n \backslash 0$ one can assume $\sum_{k=0}^{\infty} p_{m-k} \in FS_1^m$ with respect to θ.) Then all usual operations preserve the class of the corresponding p.d.o.'s.

Let $\sum_{k=0}^{\infty} p_{1-k}(x, \xi) \in FS_1^1(X \times \mathbf{R}^n \backslash 0)$. Suppose $p_1(x, \xi)$ is real-valued, $d_{x,\xi} p_1 \neq 0$ on $p_1 = 0$ and $N_p = p_1^{-1}(0) \cap (T^* X \backslash 0)$ contains no radial points. If $\sigma > 1$ and $p(x, \xi)$ is a full G^σ realization of $\sum_{k=0}^{\infty} p_{1-k}(x, \xi)$, then Egorov's theorem is valid in the G^σ category. More precisely, for every $\varrho^0 \in N_p$ such that $d_\xi p_1(\varrho^0) \neq 0$ there exists a G^σ F.I.O.

$$Ev(x) = \int e^{iS(x,\eta')} a(x, \eta') \hat{v}(\eta')\, đ\eta', \qquad \eta' = (\eta_2, \ldots, \eta_n), \tag{5.3}$$

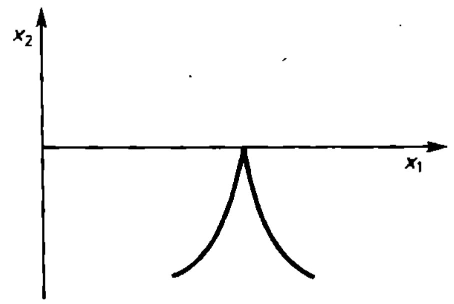
transforming microlocally, near ϱ^0, $p(x, D)$ into D_{y_1}, i.e.

$$p_1 \circ E - E \circ D_{y_1} = R, \qquad Rw \in G^\sigma, \qquad w \in \mathscr{E}'_\sigma. \tag{5.4}$$

In the case $d_\xi p_1(\varrho^0) = 0$, $d_x p_1(\varrho^0) \neq 0$, one can construct a G^σ F.I.O. E_1 of the type (5.3) for which microlocally near ϱ^0

$$p_1 \circ E_1 - E_1 \circ (y_1 |\eta|) = R_1. \tag{5.5}$$

In particular, the relations (5.4), (5.5) imply that $\mathrm{WF}_\sigma u \backslash \mathrm{WF}_\sigma(pu)$ is invariant under the Hamiltonian flow of H_{p_1}, which generalizes Theorem 7.3 of [11]. For example, the Tricomi operator $x_2 D_{x_1}^2 + D_{x_2}^2$ has the properties $x_2 \xi_1^2 + \xi_2^2 = p_1(x_2, \xi)$, $p_1(0, \xi_1, 0) = 0$, $d_{x_1} p_1(0, \xi_1, 0) \neq 0$, N_{p_1} contains no radial points.



Let now $d_\xi p_m(x, \xi) \neq 0$ and the famous (P) condition hold near $\varrho^0 \in N_p$. Then $p(x, D)$ is G^σ hypoelliptic in a conic neighbourhood $\Gamma \ni \varrho^0$, i.e.

$$\mathrm{WF}_\sigma u \cap \Gamma = \mathrm{WF}_\sigma(pu) \cap \Gamma, \quad u \in \mathscr{E}'_\sigma(X). \tag{5.6}$$

The equality above can be proved by constructing a microlocal parametrix following the ideas of F. Treves and using formal asymptotic solutions in the spaces of formal analytic symbols, for formal analytic p.d.o.'s [8].

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