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MULTIPLICATIVE FUNCTION SYSTEMS AND THEIR APPLICATION IN DISCRETE INFORMATION PROCESSING

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In the recent decades, along with the study of various properties of the trigonometric system and of systems of orthogonal polynomials, there is an intensive research of the so-called multiplicative function systems, among which a special role is played by the Walsh, Crestenson–Levy and Price systems. A number of surveys concerning the properties of these systems have been published (see [1], [2], [12]). The multiplicative systems arise as character groups of certain abelian groups (see [11]), and also can be introduced by using products of Rademacher type functions (see [10], [5]). We will follow the latter approach.

2

Let $p_1, p_2, \ldots$ be a sequence of (not necessarily distinct) integers $\geqslant 2$. Put $m_0 = 1$, $m_n = p_n m_{n-1}$, $n = 1, 2, \ldots$, and for $x \in [0, \infty)$ and $k = 1, 2, \ldots$ let

$$x_k \equiv [x m_k] \pmod{p_k}, \qquad 0 \leqslant x_k \leqslant p_k - 1,$$

$$x_{-k} \equiv [x/m_{k-1}] \pmod{p_k}, \qquad 0 \leqslant x_{-k} \leqslant p_k - 1.$$

Then x can be written as

$$x = \sum_{k=1}^{\infty} x_{-k} m_{k-1} + \sum_{k=1}^{\infty} x_k/m_k = [x] + \{x\},$$

where the first sum, corresponding to the integer part of x, is always finite, since $x_{-k} = 0$ for $k \geqslant k_0 = k_0(x)$.

Note that if $x \in [0, 1]$, then all x_{-k} are zero, and if x is a positive integer, then all x_k are zero.

For $x \in [0, 1]$ the *functions of Rademacher type* are defined by

$$\chi_{m_{k-1}}(x) = \exp(2\pi i x_k/p_k), \quad k = 1, 2, \ldots,$$

and the *Price system* in Paley's numbering by

$$\chi_n(x) = \prod_{k=1}^{r(n)} \chi_{m_{k-1}}^{n-k}(x) = \exp\left(2\pi i \sum_{k=1}^{r(n)} x_k n_{-k}/p_k\right), \tag{1}$$

where

$$n = \sum_{k=1}^{r(n)} n_{-k} m_{k-1}.$$

For $p_k = p \geqslant 3$ for all $k = 1, 2, \ldots$ we obtain the *Crestenson–Levy system*, and for $p_k = 2$, $k = 1, 2, \ldots$, the *Walsh system* in Paley's numbering.

If $x \in [0, \infty)$ and $y \in [0, \infty)$, then the function

$$\chi(x, y) = \exp\left(2\pi i \sum_{k=1}^{\infty} \frac{x_k y_{-k} + x_{-k} y_k}{p_k}\right)$$

is called an *orthogonal kernel*. For integer $y = n$ we recover the Price system (1), i.e.

$$\chi(x, n) = \chi_n(x).$$

Analogously, $\chi(m, y) = \chi_m(y)$ for integer m. The functions χ have the following multiplicative properties:

(a) $$|\chi(x, y)| = 1, \quad \chi(x, y) = \chi(y, x).$$

(b) $$\chi(x, y) = \chi([x], \{y\})\chi(\{x\}, [y]).$$

(c) $$\chi(x, y)\chi(x, z) = \chi(x, y \oplus z), \quad \chi(x, y)\chi(z, y) = \chi(x \oplus z, y).$$

(d) $$\chi(x, y)\overline{\chi(x, z)} = \chi(x, y \ominus z).$$

Here

$$x \oplus y = \sum_{k=1}^{\infty} v_{-k} m_{k-1} + \sum_{k=1}^{\infty} v_k/m_k,$$

$$x \ominus y = \sum_{k=1}^{\infty} u_{-k} m_{k-1} + \sum_{k=1}^{\infty} u_k/m_k,$$

where

$$v_k \equiv (x_k + y_k) \pmod{p_k}, \quad u_k \equiv (x_k - y_k) \pmod{p_k},$$

$$0 \leqslant v_k \leqslant p_k - 1, \quad 0 \leqslant u_k \leqslant p_k - 1, \quad k = \pm 1, \pm 2, \ldots$$

If $f \in L(0, 1)$, then the numbers

$$c_n = c_n(f) = \int_0^1 f(x)\overline{\chi_n(x)}\,dx, \qquad n = 0, 1, \ldots,$$

are well defined; they are called the coefficients of the expansion of f with respect to the system $\Phi = \{\chi_n\}_{n=0}^{\infty}$. If $f \in L(0, \infty)$, then we can define the *multiplicative Fourier transform* (MFT) of f as

$$\hat{f}(v) = \int_0^{\infty} f(t)\overline{\chi(v, t)}\,dt, \qquad v \in [0, \infty). \tag{2}$$

The problems of convergence of the series

$$f(x) \sim \sum_{n=0}^{\infty} c_n \chi_n(x)$$

according to the properties of f have been studied by many authors (see the surveys [1], [2], [12]).

We will consider the problems of existence of the *inverse multiplicative Fourier transform* (IMFT)

$$f(x) \sim \int_0^{\infty} \hat{f}(v)\chi(v, x)\,dv, \qquad x \in [0, \infty), \tag{3}$$

the features of the discretization of the integrals (3) and (2), and some peculiarities of the numerical implementation of discrete multiplicative transformations.

3

Following Morgentaller [9], we introduce the notion of g-continuity. A function f defined on $[0, \infty)$ will be called *g-continuous* if

$$\sup_x |f(x \oplus h) - f(x)| \to 0 \quad \text{as } h \to 0+;$$

the quantity

$$\omega_g(1/m_n, f) = \sup_x \sup_{0 \leq h < 1/m_n} |f(x \oplus h) - f(x)|$$

is called the *modulus of g-continuity.*

In [7] it is shown that if $f \in L(0, \infty)$, then the MFT $\hat{f}$ defined by (2) is g-continuous on $[0, \infty)$. If, in addition, f is g-continuous on $[0, \infty)$ and $\hat{f} \in L(0, \infty)$, then the IMFT

$$f(x) = \int_0^{\infty} \hat{f}(v)\chi(v, x)\,dv \tag{4}$$

exists at each $x \in [0, \infty)$. Moreover, it is shown that (4) holds whenever the modulus of g-continuity $\omega_g(1/m_n, f)$ satisfies the Dini–Lipschitz condition

$$\lim_{n\to\infty} \omega_g(1/m_n, f) \ln m_n = 0$$

and the sequence $\{p_k\}_{k=1}^{\infty}$ is such that

$$r^{-1} \sum_{k=1}^{r} p_k \leqslant C \quad \text{for all } r = 1, 2, \ldots \tag{5}$$

Condition (5) may be satisfied by sequences $\{p_k\}$ with $\limsup_{k\to\infty} p_k = \infty$.

For $f \in L_p(0, \infty)$, $1 < p \leqslant 2$, the *direct multiplicative transform* is defined as

$$\hat{f}(y) = \lim_{a\to\infty} (p') \int_0^a f(x) \overline{\chi(x, y)}\, dx, \qquad 1/p + 1/p' = 1. \tag{6}$$

M. S. Bespalov [3] showed that (6) exists and satisfies

$$\|\hat{f}\|_{p'} \leqslant \|f\|_p, \qquad 1 < p \leqslant 2$$

(for $p = 2$ this is Parseval's inequality). The invertibility of the transformation (6) for $p = 2$ is shown in [1] (p. 84), and for $1 < p \leqslant 2$ by M. S. Bespalov in [4].

4

The transform (2) has an important property, applied in numerical information processing: it has a bounded support. It was shown by S. Yu. Zolotareva [14] and M. S. Bespalov [4] that if $f \in L_1(0, \infty)$ is g-continuous on $[0, \infty)$ and $\hat{f}(y) = 0$ for $y \geqslant m_r$, then f is a step function, constant on the intervals $\Delta_\nu(r) = [\nu/m_r, (\nu+1)/m_r)$, $\nu = 0, 1, \ldots$ The converse assertion is also true.

Comparing this with the Paley–Wiener theorem (see [15], p. 408) on the boundedness of the support of the Fourier transform, we may conclude that the step functions with discontinuities at $\{p_k\}$-adically rational points are analogues of entire functions of finite order, which ensure the boundedness of the support of the Fourier transform. Therefore such step functions may be taken to form the approximation machinery for functions $f \in L_1(0, \infty)$. A function g_r defined on $[0, \infty)$ is called $\{p_k\}$-*adically entire* of order $\leqslant r$ if it is constant on each of the intervals $\Delta_\nu(r) = [\nu/m_r, (\nu+1)/m_r)$, $\nu = 0, 1, \ldots$ Let $\mathfrak{N}_r$ denote the set of all $\{p_k\}$-adically entire functions of order r. We put

$$\mathscr{E}_{m_r}^{(p)} = \inf_{g_r \in \mathfrak{N}_r} \|f - g_r\|_{L_p(0,\infty)},$$

$$\omega_g^{(p)}(1/m_r, f) = \sup_{0 \leqslant h < 1/m_r} \|f(x \oplus h) - f(x)\|_{L_p(0,\infty)},$$

where

$$\|\varphi\|_{L_p(0,\infty)} = \Big(\int_0^\infty |\varphi(x)|^p\, dx\Big)^{1/p} \quad \text{for } 1 \leqslant p < \infty,$$

$$\|\varphi\|_{L_\infty(0,\infty)} = \sup_{x \in [0,\infty)} |\varphi(x)|.$$

For $f \in L_1(0, \infty) \cap L_p(0, \infty)$, $1 \leqslant p \leqslant \infty$, it is not difficult to prove the inequalities

$$\mathscr{E}_{m_r}^{(p)}(f) \leqslant \omega_g^{(p)}(1/m_r, f) \leqslant 2\mathscr{E}_{m_r}^{(p)}(f);$$

their analogues for the Walsh system were established by Watari [13], and for the Price system by A. V. Efimov [5].

5

The discretization of the MFT (2) is understood to be the approximate calculation of the integral (2) by the rectangular formula with finite sum, i.e. passing from the MFT to the sum

$$\tilde{f}(y) = \sum_{k=0}^{N-1} f(t_k)\overline{\chi(t_k, y)}\,\Delta t_k.$$

It is proved (see [8]) that if, for $f \in L_1(0, \infty)$, the MFT $\hat{f}$ has a bounded support, i.e. $\hat{f}(y) = 0$ for $y \geqslant m_r$, then for equally spaced discretization knots ($\Delta t_k = 1/m_r$) and for $N = m_n m_r$ we have

$$\tilde{f}(y) = \begin{cases} m_n^{-1} \sum_{k=0}^{N-1} f(k/m_r)\overline{\chi(k/m_r, y)} & \text{for } 0 \leqslant y < m_r, \\ 0 & \text{for } m_r \leqslant y < \infty, \end{cases} \tag{7}$$

and the relation between $\tilde{f}(y)$ and the original $\hat{f}(y)$ is

$$\tilde{f}(y) = \begin{cases} m_r^{-1} \int_0^{1/m_n} \hat{f}(y \ominus v)\, dv & \text{for } 0 \leqslant y < m_r, \\ 0 & \text{for } m_r \leqslant y < \infty. \end{cases} \tag{8}$$

Since, by (8), $\tilde{f}$ is constant on the intervals $[l/m_n, (l+1)/m_n)$, $l = 0, 1, \ldots$, we obtain from (7) the discrete multiplicative transformation (DMT)

$$\tilde{f}(l/m_n) = m_n^{-1} \sum_{k=0}^{N-1} f(k/m_r)\overline{\chi(k/m_r, l/m_n)}, \qquad l = 0, 1, \ldots, N-1, \tag{9}$$

whose inverse is

$$(10)\qquad f(k/m_r) = m_r^{-1} \sum_{l=0}^{N-1} \tilde{f}(l/m_n)\chi(k/m_r, l/m_n), \qquad k = 0, 1, \ldots, N-1.$$

To compute the transforms (9) and (10), fast algorithms are applied (see [6]).

6

If $p_k = p$ for $k = 1, 2, \ldots$, then using the relation between the orthogonal Crestenson–Levy kernels and the Crestenson–Levy functions:

$$\chi(v/p^k, m/p^r) = \chi(v, m/p^{k+r}) = \chi_v(m/p^{k+r}),$$

one sees that the discrete multiplicative transformation of a column vector $X = (x_0, x_1, \ldots, x_{p^m-1})^T$ to a column vector $Y = (y_0, y_1, \ldots, y_{p^m-1})^T$ may be written in the form

$$y_l = p_m^{-1} \sum_{k=0}^{p^m-1} x_k \overline{\chi_k(l/p^m)}, \qquad l = 0, 1, \ldots, p^m-1,$$

or in matrix form

$$Y = p_m^{-1} WX, \qquad W = (w_{l,k})_{l,k=0}^{p^m-1},$$

where $w_{l,k} = \overline{\chi_k(l/p^m)}$, $l, k = 0, 1, \ldots, p^m-1$. The matrix W may be factored into a product of m weakly filled $p^m \times p^m$-matrices W_v, $v = 1, \ldots, m$. In contrast to the discrete Fourier transform, the matrices W_v may be identical. For instance, there exists a matrix B such that $W = CB^m$, where C is the matrix of a p-adic permutation of the column vector $B^m X$. Moreover, B has block structure: the entries in every pth row are the same, possibly in different columns. For instance, for $p = 3$ any 3 rows can be obtained from the matrix

$$B_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \tilde{q} & \tilde{q}^2 \\ 1 & \tilde{q}^2 & \tilde{q} \end{bmatrix}, \qquad \tilde{q} = \exp(2\pi i/3),$$

by adding $3^m - 3$ zero columns. Such a structure of B permits the vector BZ to be calculated by parallel computations on p^{m-1} processors of the same kind.

Note, moreover, that it is more reasonable to calculate not in the complex basis $(1, i)$, but in the basis $(1, q, \ldots, q^{p-2})$, where $q = \exp(2\pi i/p)$. This permits the vector $B^m X$ to be computed using addition only (without multiplication).

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