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ON THE ERROR ESTIMATION IN NUMERICAL METHODS

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To receive an error estimation of a given method for numerical solution of some problem usually additional assumptions are made about the solution of the problem. As a rule, these additional assumptions require the solution to possess derivatives of a certain order, belonging to definite functional spaces. These additional assumptions require from the solution of the problem more than the problem itself. For example in a numerical solution of the boundary problem for an equation of second order a bounded fourth derivative of the solution is generally required in order to obtain a good error estimate.

The purpose of this paper is to yield error estimations in numerical methods which are expressed by certain characteristics of solution and do not demand additional assumptions for them. In cases where some additional assumptions are set on the solution, the estimations already known, as well as several new ones follow from these estimates.

1

Let us consider those characteristics of the functions with which we shall make our estimates.

Let the function f be defined and bounded on the interval $[a, b]$. The local modulus of f of kth order in the point $x \in [a, b]$ is defined by

$$\omega_k(f, x; \delta) = \sup\{|\Delta_h^k f(t)|, t, t+kh \in [x-k\delta/2, x+k\delta/2] \cap [a, b]\}$$

where

$$\Delta_h^k f(t) = \sum_{m=0}^{k} (-1)^{k+m} \binom{k}{m} f(t+mh).$$

We define the following moduli:

$$\tau_k(f;\,\delta)_{L_p[a,b]} = \|\omega_k(f,\,x;\,\delta)\|_{L_p[a,b]}, \qquad 1 \leqslant p \leqslant \infty.$$

The case $p = 1$ is the most interesting. In this case we have:

$$\tau_k(f;\,\delta)_{L_1} = \int\limits_a^b \omega_k(f,\,x;\,\delta)\,dx.$$

The moduli $\tau_k(f;\,\delta)_{L_p}$ have many applications in different problems: for characterization of the best onesided approximation of functions in the metric L_p ([1], [2], [3]), for estimation of the error in approximation of functions by means of linear positive operators [10], [11] (in the metric L_p), for Hausdorff approximations of functions by piecewise monotone functions [7]. Here we shall give some other applications of the moduli $\tau_k(f;\,\delta)_{L_p}$.

For the history of the moduli $\tau_k(f;\,\delta)_{L_p}$ see [1], [2].

The most essential properties of $\tau_k(f;\,\delta)_{L_p}$ are the following ([1]–[4], [7]):

(i) $\tau_k(f;\,\delta)_{L_p} \leqslant \tau_k(f;\,\delta')_{L_p}$, $\delta \leqslant \delta'$;

(ii) $\omega_k(f;\,\delta)_{L_p} \leqslant \tau_k(f;\,\delta)_{L_p}$, $1 \leqslant p < \infty$;

(iii) $\omega_k(f;\,\delta)_{L_\infty} = \tau_k(f;\,\delta)_{L_\infty}$;

(iv) $\tau_k(f+g;\,\delta)_{L_p} \leqslant \tau_k(f;\,\delta)_{L_p} + \tau_k(g;\,\delta)_{L_p}$;

(v) $\tau_k(f;\,\delta)_{L_p} \leqslant \delta\tau_{k-1}\left(f';\,\dfrac{k}{k-1}\,\delta\right)_{L_p}$;

(vi) $\tau_1(f;\,\delta)_{L_p} \leqslant \delta\,\|f'\|_{L_p}$;

(vii) $\tau_1(f;\,\delta)_{L_1} \leqslant \delta \bigvee\limits_a^b f$ if $\bigvee\limits_a^b f < \infty$;

(viii) $\tau_k(f;\,\lambda\delta)_{L_p} \leqslant (2\lambda)^{k+1}\tau_k(f;\,\delta)_{L_p}$.

In the property (ii) $\omega_k(f;\,\delta)_{L_p}$ is the kth modulus of continuity of the function f in the metric L_p:

$$\omega_k(f;\,\delta)_{L_p} = \sup_{0<h\leqslant\delta} \left\{\int\limits_a^{b-kh} |\Delta_h^k f(t)|^p\,dt\right\}^{1/p}$$

and in (vii) $\bigvee\limits_a^b f$ denotes the variation of the function f in the interval $[a,\,b]$.

2

The following interpolation theorem is very essential and shows the reasons because of which the moduli play a role in the error estimations in numerical methods:

THEOREM 1. *Let L_n be a linear operator which satisfies the conditions:*
(a) *if the bounded function f is p-integrable in $[a, b]$ then $L_n(f; x) \in L_p$;*
(b) $\|L_n(f; \cdot)\|_{L_p} \leqslant K \|f\|_{l^p_{\Sigma_n}}$, *where K is an absolute constant and*

$$\|f\|_{l^p_{\Sigma_n}} = \Big\{\sum_{i=1}^{n} |f(x_i)|^p \Delta_i\Big\}^{1/p}, \qquad \Delta_i = x_{i+1} - x_{i-1}, \qquad x_0 = a,\ x_{n+1} = b;$$

(c) *if $f \in W^r_p$ ($f \in W^r_p$ if $f^{(r-1)}$ is absolutely continuous and $f^{(r)} \in L_p$) then*

$$\|L_n(f; \cdot) - f\|_{L_p} \leqslant K_r d_n^s \|f^{(r)}\|_{L_p}, \qquad s \leqslant r,$$

where $d_n = \max\limits_{1 \leqslant i \leqslant n} \Delta_i$ and K_r is an absolute constant which depends eventually only on r.

Then for every p-integrable bounded function f in $[a, b]$ the following estimation holds for $d_n \leqslant 1$:

$$\|L_n(f; \cdot) - f\|_{L_p} \leqslant c\tau_r(f; d_n^{s/r})_{L_p}.$$

The constant c depends only on r, K, K_r.

Here and in what follows we shall separate the bounded function f from its class of equivalence in L_p and we assume that every function is given by its values in every point $x \in [a, b]$.

3

We shall give here estimations for approximation of bounded functions in the interval $[0, 1]$ by means of interpolating splines in respect to L_p metric.

We shall consider only parabolic and cubic spline interpolation on the uniform set of points.

Let us set:

$$x_i = ih, \qquad i = -1, 0, \ldots, n+2, \qquad h = 1/n.$$

For simplicity we shall consider only a periodical splines. We shall suppose that the function f is 1-periodic.

Let $s_2(f; x)$ be the parabolic spline with knots in the points $(x_i + x_{i+1})/2$, $i = 0, 1, \ldots, n-1$, which satisfy

$$s_2(f; x_i) = f(x_i), \qquad i = 0, 1, \ldots, n,$$
$$s_2^{(i)}(f; 0) = s_2^{(i)}(f; 1), \qquad i = 1, 2$$

(see [12]), and $s_3(f; x)$ be the cubic spline with knots in the points x_i,

$i = 0, 1, \ldots, n$, which satisfy

$$s_3(f; x_i) = f(x_i), \quad i = 0, 1, \ldots, n,$$
$$s_3^{(i)}(f; 0) = s_3^{(i)}(f; 1), \quad i = 1, 2.$$

Then the following theorems are valid:

THEOREM 2. *Let the* 1*-periodical function* f *be bounded and* p*-integrable in the interval* $[0, 1]$. *Then*

$$\|s_2(f; \cdot) - f\|_{L_p} \leqslant c\tau_3(f; h)_{L_p}$$

where c *is an absolute constant.*

THEOREM 3. *Let the* 1*-periodical function* f *be bounded and* p*-integrable in* $[0, 1]$. *Then*

$$\|s_3(f; \cdot) - f\|_{L_p} \leqslant c\tau_4(f; h)_{L_p},$$

where c *is an absolute constant.*

From Theorems 2 and 3 we can obtain many consequences using the properties of $\tau_k(f; \delta)_{L_p}$. For example:

COROLLARY 1. *Let the* 1*-periodical function* f *be bounded. If* $\bigvee_0^1 f < \infty$, *then*

$$\|s_2(f; \cdot) - f\|_L = O\Big(\bigvee_0^1 f \cdot n^{-1}\Big),$$

$$\|s_3(f; \cdot) - f\|_L = O\Big(\bigvee_0^1 f \cdot n^{-1}\Big).$$

If $\bigvee_0^1 f' < \infty$, *then*

$$\|s_2(f; \cdot) - f\|_L = O\Big(\bigvee_0^1 f' \cdot n^{-2}\Big),$$

$$\|s_3(f; \cdot) - f\|_L = O\Big(\bigvee_0^1 f' \cdot n^{-2}\Big).$$

If $\bigvee_0^1 f'' < \infty$, *then*

$$\|s_2(f; \cdot) - f\|_L = O\Big(\bigvee_0^1 f'' \cdot n^{-3}\Big),$$

$$\|s_3(f; \cdot) - f\|_L = O\Big(\bigvee_0^1 f'' \cdot n^{-3}\Big),$$

and if $\bigvee_0^1 f''' < \infty$, *then*

$$\|s_3(f; \cdot) - f\|_L = O\Big(\bigvee_0^1 f''' \cdot n^{-4}\Big).$$

COROLLARY 2. *If* $f''' \in L_p$, *respect.* $f^{(IV)} \in L_p$, *then*

$$\|s_2(f;\cdot)-f\|_{L_p} \leqslant ch^3\|f'''\|_{L_p},$$

$$\|s_3(f;\cdot)-f\|_{L_p} \leqslant ch^4\|f^{(IV)}\|_{L_p}.$$

This property is obtained by other way in [12].

4. Application to the quadrature formulas

Now we shall consider the application of the moduli $\tau_k(f;\delta)_{L_p}$ to the estimation of the error of the quadrature formulas. An estimation of the error of the composite quadrature formulas of Newton–Cotes type is given in [4]. Let us mention here only tree special cases:

(a) *The rectangular rule*:

$$(1)\qquad \left|\int_0^1 f(x)\,dx-\frac{1}{n}\sum_{i=1}^{n} f\big((2i-1)/2n\big)\right| \leqslant c'\tau_2(f;n^{-1})_L.$$

(b) *The trapezoidal rule*:

$$(2)\qquad \left|\int_0^1 f(x)\,dx-\frac{1}{n}\left(f(0)+2\sum_{i=1}^{n-1} f\left(\frac{i}{n}\right)+f(1)\right)\right| \leqslant c''\tau_2(f;n^{-1})_L.$$

(c) *The Simpson rule*:

$$(3)\qquad \left|\int_0^1 f(x)\,dx-\frac{1}{6n}\left\{f(0)+2\sum_{i=1}^{n-1} f\left(\frac{i}{n}\right)+4\sum_{i=1}^{n-1} f\left(\frac{2i-1}{2n}\right)+f(1)\right\}\right| \leqslant c'''\tau_4(f;n^{-1})_L.$$

Let us underline that the estimations are obtained without some additional restrictions on the function f — we use only that f is bounded. If f has some properties we obtain something about the order of the error. Usualy the rectangular rule and trapezoidal rule have the error $O(n^{-2})$ if f'' is bounded, and the Simpson rule has the error $O(n^{-4})$ if $f^{(IV)}$ is bounded. From the estimations (1), (2), (3) and the properties of the moduli $\tau_k(f;\delta)_{L_p}$ it follows that the rectangular rule and trapezoidal rule have the error $O(n^{-2})$ if $\bigvee_0^1 f' < \infty$ and the Simpson rule has the error $O(n^{-4})$ if $\bigvee_0^1 f''' < \infty$.

In the general case, estimations for error of the composite quadrature formulas are given by K. Ivanov in [5].

Let us have the composite quadrature formula

$$\int_0^1 f(x)\,dx = \sum_{i=0}^{m} A_i f(x_i) + R(f)$$

which is construct in the following way: we divide the interval $[0,1]$ in n equal parts $\Delta_i = [(i-1)/n, i/n]$, $i = 1, 2, \ldots, n$, and in each part we apply the quadrature formula $L(f, \Delta_i)$, which is exact for the algebraical polynomials of kth degree. Then we have $R(f) = O\big(\tau_{k+1}(f; n^{-1})_L\big)$.

In [5] there are given also estimations for the composite quadrature formulas containing derivatives of the function f.

5

Let us consider now the Cauchy problem:

$$\text{(4)} \qquad \begin{gathered} u' = f(x, u), \qquad x_0 \leqslant x \leqslant x_0 + X, \\ u(x_0) = u_0. \end{gathered}$$

If we solve (4) by means of Runge–Kutta methods it is possible to obtain an estimation of the error by means of $\tau_k(f; \delta)_{L_p}$.

To show how we can apply $\tau_k(f; \delta)$ we shall consider the Runge–Kutta method with the local error h^3. It is well known that all formulas with a local error h^3 are of the form [9]:

$$\text{(5)} \qquad y_{i+1} = y_i + p_1 k_1 + p_2 k_2, \qquad y_0 = u_0,$$

where

$$\text{(6)} \qquad \begin{gathered} k_1 = hf(x_i, y_i), \qquad k_2 = hf(x_i + ah, y_i + \beta k_1), \\ p_1 + p_2 = 1, \qquad p_1 a = p_2 \beta = 1/2. \end{gathered}$$

THEOREM 4. *Let the function $f(x, u)$ satisfy $|f_u'| < L$ and u be the solution of* (4), *and let $\{y_i\}_{i=0}^{n}$, $y_0 = u(x_0) = u_0$, $x_n = x_0 + nh = x_0 + X$, are obtained from* (5), (6). *Then we have*:

$$\max_{0 \leqslant i \leqslant n} |y_i - u_i| = O\big(\tau_2(u'; h)_L + h\tau(u'; h)_L\big).$$

COROLLARY. *If* $\bigvee\limits_{x_0}^{x_0+X} u'' < \infty$, *then*

$$\max_{0 \leqslant i \leqslant n} |y_i - u_i| = O(h^2).$$

Let us solve now the problem (4) by means of the finite difference method:

$$\text{(7)} \qquad L_j(\nu) \equiv \sum_{i=0}^{k} a_{-i}\nu_{j-i} - h\sum_{i=0}^{k} b_{-i} f(x_{j-i}, \nu_{j-i}) = 0, \qquad a_0 \neq 0.$$

If we know $\nu_0, \nu_1, \ldots, \nu_{k-1}$ from (7) we can obtain $\nu_k, \nu_{k+1}, \ldots$ We shall assume that

$$\nu_0 = u_0 = u(x_0), \quad \nu_1 = u(x_1), \quad \ldots, \quad \nu_{k-1} = u(x_{k-1}). \tag{8}$$

THEOREM 5. *Let $L_j(P) = 0$ if P is an algebraical polynomial of m-th degree and let $|\partial f/\partial u| \leqslant L$. Let the roots of the difference equation $\sum_{i=0}^{k} a_{-i}\nu_{j-i} = 0$ be in absolute value $\leqslant 1$ (if there are multiple roots we assume that they are strictly < 1). Then if u is the solution of (4) and $\{\nu_i\}_{i=0}^{n}$ are obtained from (7), (8), then*

$$\max_{0\leqslant i\leqslant n} |u_i - \nu_i| = O\big(\tau_m(y'; h)_L\big).$$

6

Let us consider the boundary problem for an ordinary linear differential equation of a second order in the interval $[0, 1]$:

$$\begin{gathered}\big(k(x)\,u'(x)\big)' - g(x)u(x) = -f(x), \quad 0 < x < 1,\\ u(0) = \alpha, \quad u(1) = \beta,\end{gathered} \tag{9}$$

where $0 < c_1 \leqslant k(x) \leqslant c_2$, $g(x) \geqslant 0$.

In [13] there are obtained estimations of the error by means of $\tau_2(u'; h)_L$, $\tau_2(k; h)_L$, $\tau_2(g; h)_L$, $\tau_2(f; h)_L$, if we solve numerically the equation (9) by means of the homogeneous conservative scheme.

7

Let us consider the heat conduction equation:

$$\begin{gathered}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x, t), \quad 0 < x < 1,\ 0 < t \leqslant T,\\ u(x, 0) = u_0(x), \quad 0 < x < 1,\\ u(0, t) = u_1(t), \quad u(1, t) = u_2(t), \quad 0 \leqslant t \leqslant T.\end{gathered} \tag{10}$$

Let us solve the problem (10) by means of difference scheme (see [8], p. 278):

$$\begin{gathered}y_t = \Lambda\big(\sigma\hat{y} + (1-\sigma)y\big) + \varphi, \quad (x, t) \in \omega_{h\tau},\\ y(0, t) = u_1(t), \quad y(1, t) = u_2(t), \quad t \in \omega_\tau,\\ y(x, 0) = u_0(x), \quad x \in \overline{\omega}_h,\end{gathered} \tag{11}$$

where

$$\overline{\omega}_h = \{x_i = ih,\ i = 0, 1, \ldots, n\}, \qquad \omega_\tau = \{t_j = j\tau,\ j = 0, 1, \ldots, j_0\},$$
$$\overline{\omega}_{h\tau} = \overline{\omega}_h \times \omega_\tau = \{(ih, j\tau),\ i = 0, 1, \ldots, n;\ j = 0, 1, \ldots, j_0\},$$
$$\omega_{h\tau} = \{(ih, j\tau),\ i = 1, 2, \ldots, n-1;\ j = 1, 2, \ldots, j_0\}.$$

Here y and φ are network functions defined on the net $\overline{\omega}_{h\tau}$,

$$\Lambda y_i = y_{\bar{x}x,i} = (y_{i-1} - 2y_i + y_{i+1})/h^2, \qquad 0 \leqslant \sigma \leqslant 1, \qquad y_j^i = y, \qquad y_i^{j+1} = \hat{y},$$

$$y_t = (\hat{y} - y)/\tau, \quad \dot{u} = \frac{\partial u}{\partial t}, \quad u' = \frac{\partial u}{\partial x}, \quad \bar{u} = u(x_i, t_{j+1/2}), \quad t_{j+1/2} = t_j + \frac{\tau}{2}.$$

Let us set:

$$z = y - u. \tag{12}$$

From (11) and (12) it follows:

$$\begin{aligned} z_t &= \Lambda\big(\sigma\hat{z} + (1-\sigma)z\big) + \psi, \qquad (x, t) \in \omega_{h\tau}, \\ z(0, t) &= z(1, t) = 0, \qquad t \in \omega_\tau, \\ z(x, 0) &= 0, \qquad x \in \overline{\omega}_h, \end{aligned} \tag{13}$$

where $\psi = \Lambda\big(\sigma\hat{u} + (1-\sigma)u\big) - u_t + \varphi$.

From the above we obtain:

$$\begin{aligned} \psi &= \Lambda\big(\sigma\hat{u} + (1-\sigma)u\big) - \big(\sigma\hat{u}'' + (1-\sigma)u''\big) + \big(\sigma\hat{u}'' + (1-\sigma)u''\big) - u_t + \varphi \\ &= \big[\Lambda\big(\sigma\hat{u} + (1-\sigma)u\big) - \big(\sigma\hat{u}'' + (1-\sigma)u''\big)\big] + [\sigma\hat{\dot{u}} + (1-\sigma)\dot{u} - u_t] - \\ &\qquad - [\sigma\hat{f} + (1-\sigma)f - \varphi]. \end{aligned} \tag{14}$$

We shall use the following

Lemma 1. *Let us set $x_{i+1} = x_i + h$, $x_{i-1} = x_i - h$. Then*

$$\left| y'_{i+1} - \frac{y_{i-1} - 4y_i + 3y_{i+1}}{2h} \right| \leqslant c\omega_2(y', x_i; h),$$

$$\left| y'_{i-1} - \frac{-3y_{i-1} + 4y_i - y_{i+1}}{2h} \right| \leqslant c\omega_2(y', x_i; h).$$

Proof. From

$$y'_{i+1} - \frac{y_{i-1} - 4y_i + 3y_{i+1}}{2h}$$
$$= y'_{i+1} - 2y'_{i+1/2} + y'_i + 2y'_{i+1/2} - y'_i - \frac{y_{i-1} - 4y_i + 3y_{i+1}}{2h}$$

we obtain:

$$(15)\quad \left|y'_{i+1}-\frac{y_{i-1}-4y_i+3y_{i+1}}{2h}\right| \leqslant \omega_2(y', x_i; h)+2\left|y'_{i+1/2}-\frac{y_{i+1}-y_i}{h}\right|+\left|y'_i-\frac{y_{i+1}-y_{i-1}}{2h}\right|.$$

We have

$$(16)\quad \begin{gathered}\left|y'_{i+1/2}-\frac{y_{i+1}-y_i}{h}\right|\leqslant \tfrac{1}{2}\omega_2(y', x_{i+1/2}; h/2)\leqslant \tfrac{1}{2}\omega_2(y', x_i; h),\\ \left|y'_i-\frac{y_{i+1}-y_{i-1}}{2h}\right|\leqslant \omega_2(y', x_i; h).\end{gathered}$$

From (15) and (16) the first inequality in the lemma follows. The proof of the second inequality is analogously.

In [13] the inequality:

$$(17)\quad \left|y''_i-\frac{y_{i-1}-2y_i+y_{i+1}}{h^2}\right|\leqslant \tfrac{1}{2}\omega_2(y'', x_i; h).$$

is proved.

We have

$$\begin{aligned}\sigma\dot{\hat{u}}+(1-\sigma)\dot{u}-u_t &= \sigma\left[\dot{\hat{u}}-\frac{u-4\bar{u}+3\hat{u}}{\tau}\right]+\sigma\frac{u-4\bar{u}+3\hat{u}}{\tau}+\\ &+(1-\sigma)\left[\dot{u}-\frac{-3u+4\bar{u}-\hat{u}}{\tau}\right]+(1-\sigma)\frac{-3u+4\bar{u}-\hat{u}}{\tau}-\frac{\hat{u}-u}{\tau}\\ &=\sigma\left[\dot{\hat{u}}-\frac{3\hat{u}-4\bar{u}+u}{\tau}\right]+(1-\sigma)\left[\dot{u}-\frac{-3u+4\bar{u}-\hat{u}}{\tau}\right]-\\ &\quad-\frac{4[(\sigma-\frac{1}{2})(\hat{u}-2\bar{u}+u)]}{\tau}.\end{aligned}$$

From here and Lemma 1 it follows that

$$(18)\quad |\sigma\dot{\hat{u}}+(1-\sigma)\dot{u}-u_t| \leqslant c_1\omega_2^t(\dot{u}, (x_i, t_j); \tau)+\frac{4|\sigma-\frac{1}{2}|\,\omega_2^t(\dot{u}, (x_i, t_{j+1/2}); \tau/2)}{\tau}.$$

Let us set

$$(19)\quad \varphi=\sigma\hat{f}+(1-\sigma)f.$$

From (14), (15), (17), (18) we obtain:

$$|\psi_i^j| \leqslant c_1 \omega_2^t\big(\dot{u}, (x_i, t_j); \tau\big) + c_2 \omega_2^x\big(u'', (x_i, t_{j+1}); h\big) + c_3 \omega_2^x\big(u'', (x_i, t_j); h\big) + 4|\sigma - \tfrac{1}{2}| \frac{\omega_2^t\big(u, (x_i, t_j); \tau\big)}{\tau} \tag{20}$$

$$\leqslant c_1 \omega_2^t\big(\dot{u}, (x_i, t_j); \tau\big) + c_2 \omega_2^x\big(u'', (x_i, t_{j+1}); h\big) + c_3 \omega_2^x\big(u'', (x_i, t_j); h\big) + c|\sigma - \tfrac{1}{2}| \omega^t\big(\dot{u}, (x_i, t_j); \tau\big).$$

Here the upper index show the variable, in respect of which the local modulus is taken.

LEMMA 2. *Let the function f be defined and bounded in the interval $[a, b]$, $x_i = a + ih$, $h = (b-a)/n$, $i = 0, 1, \ldots, n$. Then*

$$\Big(\sum_{i=1}^{n} h\omega_k^2(f, x_i; h)\Big)^{1/2} \leqslant c\tau_k(f; h)_{L_2}.$$

Proof.

$$\Big(\sum_{i=1}^{n} h\omega_k^2(f, x_i; h)\Big)^{1/2} = \Big(\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \omega_k^2(f, x_i; h)\,dx\Big)^{1/2}$$

$$\leqslant \Big(\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \omega_k^2(f, x; 3h)\,dx\Big)^{1/2} = \tau_k(f; 3h)_{L_2} \leqslant c\tau_k(f; h)_{L_2},$$

what proves the lemma.

From (20) and $ab \leqslant (a^2+b^2)/2$ it follows that

$$|\psi_i^j|^2 \leqslant c_4 \big[\omega_2^t\big(\dot{u}, (x_i, t_j); \tau\big)\big]^2 + c_5 \big[\omega_2^x\big(u'', (x_i, t_{j+1}); h\big)\big]^2 + c_6 \big[\omega_2^x\big(u'', (x_i, t_j); h\big)\big]^2 + c_7 \big[|\sigma - \tfrac{1}{2}| \omega^t\big(\dot{u}, (x_i, t_j); \tau\big)\big]^2. \tag{21}$$

Let us denote

$$\max_{0 \leqslant j \leqslant j_0} \tau_k\big(u'', (\cdot, t_j); h\big)_{L_p} = \tau_k(u''; h)_{L_p},$$

$$\max_{0 \leqslant i \leqslant n} \tau_k\big(\dot{u}, (x_i, \cdot); \tau\big)_{L_p} = \tau_k(\dot{u}; \tau)_{L_p}.$$

The estimation from [8], p. 294

$$\max_{i,j} |z_i^j| \leqslant \frac{1}{2\sqrt{2\varepsilon}} \Big(\sum_{j=0}^{j_0} \tau \sum_{i=1}^{n-1} h |\psi_i^j|^2\Big)^{1/2},$$

$$\sigma \geqslant \sigma_\varepsilon = \frac{1}{2} - \frac{1-\varepsilon}{4} \cdot \frac{h^2}{\tau}, \quad 0 < \varepsilon \leqslant 1,$$

Lemma 2 and (21) give us

THEOREM 6. *Let u be the solution of* (10) *and let y be the solution of* (11). *If* $\sigma \geqslant \sigma_\varepsilon = \frac{1}{2} - \frac{1-\varepsilon}{4} \cdot \frac{h^2}{\tau}$, $0 < \varepsilon \leqslant 1$, *then*

$$\max_{i,j} |u_i^j - y_i^j| \leqslant c_8 \tau_2^t(\dot{u};\ \tau)_{L_2} + c_9 \tau_2^x(u'';h)_{L_2} + c_{10} |\sigma - \tfrac{1}{2}| \tau^t(\dot{u};\tau)_{L_2}.$$

Let us write

$$L_m^k = \left\{ u(x,t) \colon \left\| \frac{\partial^k u}{\partial x^k} \right\|_{L_2} < \infty, \left\| \frac{\partial^m u}{\partial t^m} \right\|_{L_2} < \infty \right\}.$$

From Theorem 6 and the properties of $\tau_k(f;\ \delta)_{L_p}$ we have:

COROLLARY 1. *If* $u \in L_2^3$ *then*

$$\max_{i,j} |u_i^j - y_i^j| = O(h+\tau).$$

COROLLARY 2. *If* $u \in L_2^4$ *then*

$$\max_{i,j} |u_i^j - y_i^j| = O(h^2+\tau). \tag{22}$$

COROLLARY 3. *If* $\sigma = 1/2$ *and* $u \in L_3^4$ *then*

$$\max_{i,j} |u_i^j - y_i^j| = O(h^2+\tau^2). \tag{23}$$

In [8] the estimations (22), (23) are obtained by the assumptions $u \in C_2^4$, $u \in C_3^4$, $\sigma = 1/2$ (C_m^k denotes the class of functions $u(x,t)$ for which $\frac{\partial^k u}{\partial x^k}$ is continuous and $\frac{\partial^m u}{\partial t^m}$ is continuous).

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