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CHARACTERIZATION OF HILBERT CUBE MANIFOLDS: AN ALTERNATE PROOF*

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A characterization due to H. Toruńczyk states that locally compact absolute neighborhood retracts satisfying the disjoint n-cells property, for each $n = 1, 2, \ldots$, are Hilbert cube manifolds [To]. This represents just one of several types of manifolds that have been characterized in terms of their general position features. Two main ingredients went into Toruńczyk's original proof. The first of these was Edwards' result that the product of a locally compact ANR and the Hilbert cube is a Hilbert cube manifold (which made use of the work of Miller [Mi] showing that, for a compact ANR X, there is a cell-like map $f\colon M \to X \times [0, 1)$ from a Hilbert cube manifold M onto $X \times [0, 1)$). The second was West's Mapping Cylinder Theorem that can be restated: a cell-like map $f\colon M \to X$ from a Hilbert Cube manifold to an ANR, with the property that $\mathrm{CL}\{m\colon f^{-1}f(m) \neq m\}$ is a Z-set, is a near homeomorphism. The purpose of this exposition is to present yet a different approach that parallels the strategy used in the characterization of n-manifolds ($n \geqslant 5$) in [Ed_2]. The strategy has three essential ingredients: the existence of a resolution, a shrinking theorem for resolutions whose nondegenerate elements are contained in a "tame" closed subset, and a shrinking theorem for resolutions whose nondegenerate elements are contained in a countable union of "tame"closed subsets. This same strategy was employed successfully by Bestvina in his recent work [Be] that led to a characterization of manifolds modeled on k-dimensional Menger Universal Compacta, their characteristic general position property being the disjoint k-cells property. At that point, an analysis of these different proofs suggested that the same strategy ought to work in the infinite dimensional setting, for both Hilbert cube and Hilbert space manifolds. Furthermore, there was evidence suggesting proofs along these lines ought to be simpler. Exactly

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such a program for Hilbert space manifolds is carried out in [BBMW] and is carried out below for Hilbert cube manifolds.

The first step in all cases is to produce a "resolution". Our specific needs are met by Miller's result that, for a locally compact ANR X, there is a Hilbert cube manifold M and a cell-like map $f\colon M \to X \times [0, 1)$ (see [Ch]). Beyond that, we shall need the controlled version of Z-set unknotting and a standard form of. Bing Shrinking Theorem, both of these are stated below.

In Section 1 these last two results are combined reproducing a fairly standard form of a Bing Shrinking Theorem, both of these are stated below. onto an ANR with the property that CL $\{m\colon f^{-1}f(m) \neq m\}$ is a Z-set is a near homeomorphism. This is followed by the improvement that a resolution $g\colon M \to X$ is a near homeomorphism provided CL $\{x\colon g^{-1}(x) \neq \text{point}\}$ is a Z-set in X, the proof consisting of repeated uses of Z-set unknotting to produce an approximating resolution f that is a near homeomorphism as it satisfies the previous hypothesis.

In Section 2 the results of Section 1 are improved first by showing that a resolution $f\colon M \to X$ is a near homeomorphism provided the "nondegeneracy" set $\{m\colon f^{-1}f(m) \neq m\}$ is a countable union of Z-sets and, then, by extending this result to show that a resolution $g\colon M \to X$ is a near homeomorphism provided X satisfies the disjoint n-cells property for all n and the image of the "nondegeneracy" set, namely $\{x\colon g^{-1}(x) \neq \text{point}\}$, is a countable union of Z-sets. The proof of the first of these results involves only minor adjustments to the proof of the first result in Section 1 while the reduction of the second to the first is exactly the same as that in Section 1.

In Section 3 the Characterization Theorem is proved by using the results of Section 1 to reduce the general case to the case that is covered by results in Section 2.

Terminology and notation

Throughout, spaces are locally compact, separable, and metric. A *Z-set* in an ANR (absolute neighborhood retract) X is a closed subset A for which there are maps $\alpha\colon X \to X - A$ arbitrarily close to the identity. There are a multitude of different devices available for detecting that a closed subset is a Z-set. For our purposes, we shall need to know that a closed subset A of an ANR X is a Z-set provided each map α from the Hilbert cube into X can be approximated arbitrarily closely by maps into $X - A$. The Hilbert cube is denoted I^∞. A *near homeomorphism* is a map that can be approximated arbitrarily closely by homeomorphisms. A *proper map* is a map f for which $f^{-1}(K)$ is compact for each compact subset K. A *cell-like map* is a proper map f such that $f^{-1}(x)$ is a set having trivial shape for each point x.

Z-SET UNKNOTTING. *Given a Z-embedding $F\colon A \times I \to M$ into a Hilbert* *cube manifold (i.e., a closed embedding onto a Z-set), there is an isotopy* $\{h_t\colon 0 \leqslant t \leqslant 1\}$ *of* M *such that* h_0 *is the identity,* $h_t \circ F|A \times \{0\} = F|A \times \{t\}$, *and, outside any prechosen neighborhood of the image of* F, *each* h_t *is the identity. Furthermore, if the "tracks"* $\{F(a \times I)\colon a \in A\}$ *refine an open cover* $\mathscr{W}$, *then we can require the "tracks" of the isotopy refine the cover also.*

BING SHRINKING THEOREM. *A proper map* $f\colon X \to Y$ *between locally compact spaces is a near homeomorphism provided, for open covers* $\mathscr{U}$ *of* Y *and* $\mathscr{V}$ *of* X, *there is a homeomorphism* h *of* X *such that* $f \circ h$ *is* $\mathscr{U}$*-close to* f *and such that the collection* $\{h(f^{-1}(y))\colon y \in Y\}$ *refines* $\mathscr{V}$.

There are a variety of sources for these results, perhaps a particularly appropriate one is Chapman's lecture notes [Ch].

Disjoint n-Cells Property. A space X satisfies this property provided each pair of maps α, $\beta\colon I^n \to X$ can be arbitrarily closely approximated by maps α^*, β^* such that $\alpha^*(I^n) \cap \beta^*(I^n) = \emptyset$. If an ANR X satisfies the disjoint n-cells property for all n, then a Baire category analysis shows that proper maps of locally compact spaces into X can be approximated by Z-embeddings [To].

1. Z-set shrinking

The setting in this section is a resolution $f\colon M \to X$ from a Hilbert cube manifold to an ANR with the property that its nondegeneracy set is contained in a Z-set. These fall into two classes depending on whether the nondegeneracy is measured in the domain M or the range X. In either case it follows easily that X satisfies the disjoint n-cells property for all n. We shall adopt the notation H_f for the set $\{x\colon f^{-1}(x) \neq \text{point}\}$ and N_f for the set $f^{-1}(H_f)$.

THEOREM 1.1. *Let* $f\colon M \to X$ *be a cell-like map from a Hilbert cube manifold to an* ANR.

(a) *If* $\mathrm{CL}(N_f)$ *is a Z-set, then* f *is a near homeomorphism.*

(b) *If* $\mathrm{CL}(H_f)$ *is a Z-set, then* f *is a near homeomorphism.*

Proof (a). In order to simplify the situation, we shall assume that M and X are compact and leave the details of the general case to the reader. In particular, the Bing Shrinking Theorem can be invoked once we have found, for each $\varepsilon > 0$, a homeomorphism $h\colon M \to M$ such that fh is ε-close to f and the diameter of each $h(f^{-1}(x))$ is less than ε. To this end we start by specifying a map G from the mapping cylinder of f, say $\mathscr{M}(f)$, to M. Describing the mapping cylinder as $M \times I \cup_f X$ where $f|_{M \times \{1\}} = f$, the map G is required to satisfy: (1) $G|_{M \times \{0\}}$ is the identity, and (2) $f(G(m, t))$ lies in the ε-neighborhood of $f(m)$ for $0 \leqslant t \leqslant 1$. Set $A = \mathrm{CL}(N_f)$ and

$\mathcal{M}(f; A) \subset \mathcal{M}(f)$ equal to the "naturally" occurring copy of the mapping cylinder of the restriction $f|_A$. Since $G|_{A\times\{0\}}$ is the identity and A is a Z-set, the restriction $G|_{\mathcal{M}(f,A)}$ can be approximated by a Z-embedding H such that $H|_{A\times\{0\}}$ is the identity. For a choice of t' sufficiently close to 1, $H|_{A\times\{t'\}}$ "shrinks" the nondgenerate point inverses of f to size less than ε. Using Z-set unknotting, the isotopy H: $A\times[0, t'] \to M$ extends, with control, to an isotopy of M whose t'-level serves as the shrinking homeomorphism h.

(b). The strategy in this case is to approximate f by a resolution g satisfying the hypothesis of part (a). The "trick" is to use repeated applications of Z-set unknotting to "lift" back to M the property that the Z-embeddings of I^∞ into $X-\mathrm{CL}(H_f)$ are dense in the space of maps of I^∞ into X. Start by naming a countable collection of Z-embeddings α_i: $I^\infty \to M$, having pairwise disjoint images, that are dense in the space of maps of I^∞ into X. The composition $f\alpha_1$ is homotopic, by a "small homotopy", to a Z-embedding β: $I^\infty \to X-\mathrm{CL}(X_f)$. Using an approximate "lift" of the homotopy as a guide, Z-set unknotting produces a homeomorphism h_1 of M, fixed outside a small neighborhood of $f^{-1}f(\alpha_1(I^\infty))$, such that $h_1\alpha_1 = f^{-1}\beta$ and fh_1 is "close to" f. Set $f_1 = fh$ and observe that f_1 is one to one over the image of $\alpha_1(I^\infty)$ (this meaning that $f_1^{-1}f_1(m) = m$ for each $m \in \alpha_1(I^\infty)$). Recursively, a sequence $f_1, f_2, \dots$ is constructed so that $f_{i+1} = f_i h_{i+1}$ where h_{i+1} is a homeomorphism of M fixed outside a small neighborhood of $f_i^{-1}f_i(\alpha_{i+1}(I^\infty))$ missing $\bigcup\{\alpha_k(I^\infty)$: $1 \leqslant k \leqslant i\}$. The homeomorphism h_{i+1} is chosen so that $h_{i+1}\alpha_{i+1} = f_i^{-1}\beta$ where β: $I^\infty \to X-\mathrm{CL}(H_f)$ is a Z-embedding approximating $f_i\alpha_{i+1}$. Observe that $\{x$: $f_{i+1}^{-1}(x) \neq \text{point}\} = H_f$ and that $\bigcup\{f_{i+1}\alpha_k(I^\infty)$: $1 \leqslant k \leqslant i+1\} \subset X-\mathrm{CL}(H_f)$. Exercising sufficient control in specifying the "closeness" of f_{i+1} to f_i, the map $g = \lim\{f_i\}$ will be a cell-like map approximating f with $H_g \subset \mathrm{CL}(H_f)$ and with $g(\alpha_k(I^\infty)) \subset X-\mathrm{CL}(H_f)$. It follows easily that $\mathrm{CL}(N_g)$ is a Z-set.

2. σ-Z-set shrinking

The setting is a resolution f: $M \to X$ from a Hilbert cube manifold to an ANR with the property that the nondegeneracy set of f is a countable union of Z-sets. As in the last section, there are two measures that can be applied. The first is in the domain, namely, to the set $N_f = \{m$: $f^{-1}f(m) \neq m\}$; and the second is in the range, namely, to the set $H_f = \{x$: $f^{-1}(x) \neq \text{point}\}$. Both these sets are F_σ's as H_f is the union of the closed sets $H_i = \{x$: $\text{diameter}(f^{-1}(x)) \geqslant 1/i\}$, $i = 1, 2, \dots$, and N_f is the union of the $f^{-1}(H_i)$'s. In this setting, it is critically important to distinguish between measuring nondegeneracy in the domain versus the range. The existence of "cellular" null-sequence decompositions that are not "shrinkable" shows that merely assuming H_f is a σ-Z-set is not sufficient to assure that f is a near homeomorphism. While it follows easily from the assumption that N_f is a σ-Z-set that X satisfies the disjoint n-cells property, the assumption that H_f is a σ-Z-set does not insure that X satisfies the disjoint 2-cells property; an example can be found in [DW].

THEOREM 2.1. *Let f: $M \to X$ be cell-like map from a Hilbert cube manifold to an* ANR.

(a) *If N_f is a countable union of Z-sets, then f is a near homeomorphism.*

(b) *If H_f is a countable union of Z-sets and X satisfies the disjoint n-cells property for all n, then f is a near homeomorphism.*

Proof (a). For simplicity of exposition we assume that M and X are compact. Since f is a fine homotopy equivalence, there is a map G from the mapping cylinder of f, say $\mathcal{M}(f)$, to M. Using the description of the mapping cylinder given in the proof of Theorem 1.1, we require: (1) $G|_{M \times 0}$, is the identity, (2) $f(G(m, t))$ lies in the ε-neighborhood of $f(m)$ for $0 \leqslant t \leqslant 1$, and (3) the restriction of G to $\mathcal{M}(f) - M \times [0, t)$ is a Z-embedding into $M - N_f$ for $0 < t < 1$. First, let's consider the set $C_\varepsilon = \bigcup \{f^{-1}(x)$: diameter$(f^{-1}(x)) \geqslant \varepsilon\}$, the latter set being a compact Z-set that contains those elements that are in need of shrinking.

The basic problem is that in shrinking these to small size we may "stretch" elements that were of size less than ε to a size greater than ε. There are two sources of difficulty. The first is that there is a sequence $f^{-1}(x_i)$, $i = 1, 2, \ldots$, that converges to a subset of $\mathcal{M}(f; C_\varepsilon) - C_\varepsilon$; in this case, the $f^{-1}(x_i)$'s form a null sequence (see restriction (3) in the choice of G). The second is that there is a sequence of $f^{-1}(x_i)$'s, not contained in C_ε, that converges to a subset of C_ε: in this case, the limit set may be a point but in general the best we can say is that it has diameter $\leqslant \varepsilon$.

As a first attempt, let's suppose that there are no sequences of the latter type; i.e., we are assuming that every sequence of $f^{-1}(x_i)$'s consisting of sets not contained in C_ε is a null sequence. We start by specifying a sequence of open sets W_i, $i = 1, 2, \ldots$, containing $G(\mathcal{M}(f; C_\varepsilon))$ and satisfying: (a) each $f^{-1}(x)$ contained in $W_1 - C_\varepsilon$ has diameter $\leqslant \varepsilon/3$; (b) $\mathrm{CL}(W_i) \subset W_{i+1}$: and (c) if $f^{-1}(x)$ meets W_{i+1}, then $f^{-1}(x) \subset W_i$. Next, we choose a partition $0 = t(1) < t(2) < \ldots < t(n) = 1$ such that the diameter of each $G(m \times [t(i), t(i+1)])$ is less than $\varepsilon/3$ and such that the sets $G(f^{-1}(x) \times \{t(n)\})$ have diameter $< \varepsilon$ for $f^{-1}(x) \subset C_\varepsilon$. A shrinking homeomorphism H is produced as a composition $H_n H_{n-1} \ldots H_1$ were the H_i's are determined successively as follows. Using Z-set unknotting, H_1 is chosen to extend a homeomorphism of the form GhG^{-1} where h is determined by a homeomorphism of $[t(1), 1]$ onto $[t(2), 1]$ that is the identity on $[t(2)-\delta, 1]$ for a "very small" value of δ. The choice of H_1 can easily be arranged so that it is the identity except on a "small" neighborhood of $G(C_\varepsilon \times [t(1), t(2)])$ contained in W_1 and, consequently, does not stretch any $f^{-1}(x)$ not contained in C_ε to size more than ε. Next, specify $i(2)$ so that $H_1(f^{-1}(x))$ has diameter less than $\varepsilon/3$ for each $f^{-1}(x)$ contained in $W_{i(2)} - C_\varepsilon$. Repeat the preceeding to produce H_2 using H_1 for G, $W_{i(2)}$ for W_1, $t(2)$ for $t(1)$, and $t(3)$ for $t(2)$. In this same manner, the H_i's for $i = 3, \dots n$ are constructed yielding the shrinking homeomorphism $H = H_n H_{n-1} \dots H_1$.

Ultimately we shall produce a shrinking homeomorphism defined essentially as above (for a different choice of ε) that handles the general situation. First let's consider the general situation when there may be sets $f^{-1}(x)$ near C_ε that are not "small" and analyse the potential for "stretching" by a homeomorphism defined as H was above. The "action" of H is essentially determined by its restriction to $\mathscr{M}(f; C_\varepsilon)$. The only point inverses $f^{-1}(x)$'s not in C_ε that are both near $\mathscr{M}(f; C_\varepsilon)$ and subject to possible "stretching" by H are near C_ε. More exactly any such point inverse $f^{-1}(x)$ is "near" a set $B \subset C_\varepsilon$ where B has roughly the same diameter as $f^{-1}(x)$. Then $G(f^{-1}(x))$ is near $G(\mathscr{M}(f; B))$; in fact, it is near some $G(B \times [t(i-1), t(i+1)])$. Since the $t(i)$'s were chosen so that $G(\{b\} \times [t(i-1), t(i+1)])$ is "small" for each $b \in B$, the size of $G(B \times \{t(i)\})$ essentially bounds the diameter of $G(f^{-1}(x))$.

At this point in the argument, we need to make use of the fact that the map f is a hereditary shape equivalence, namely, that G is defined on all of $M(f)$. (Thus far we have only used that the restriction of f to C_ε is a hereditary shape equivalence.) Using compactness, specify a $\delta > 0$ such that the diameter of $G(B \times \{t\})$ is less than $\varepsilon/3$ for each subset B having diameter less that δ and for each $0 \leqslant t \leqslant 1$. Finally, specify $p > 0$ so that if $f^{-1}(x)$ is not contained in C_p, then $f^{-1}(x)$ has diameter less than δ (measured in $M \times \{0\} \subset \mathscr{M}(f)$). Now the homeomorphism H described in the preceeding section with C_p in place of C_ε is the sought after shrinking homeomorphism.

(b) In exactly the same way as is done in the proof of part (b) of Theorem 1.1, the map f is approximated by a map that satisfies the hypothesis of part (a).

3. Characterization theorem

We are now in a position to present a proof of Toruńczyk's characterization of Hilbert cube manifolds. As a preliminary let's recall that, for a locally compact ANR X, the disjoint n-cells property for all n is a "minimal" way of stating that the subset of Z-embeddings of a compactum into X is dense in the space of mappings. The translation of this apparently weaker property to the strong embedding condition is accomplished using Baire category arguments. The particular version that we shall explicitly use is that, for such an ANR X, there is a countable set of Z-embeddings $\{s_i\colon I^\infty \to X\colon i$

$= 1, 2, \ldots\}$ having pairwise disjoint images such that every closed subset of $X - \bigcup s_i(I^\infty)$ is a Z-set.

THEOREM (Toruńczyk). *A locally compact* ANR X *satisfies the disjoint n-cells property for all n if and only if X is a Hilbert cube manifold.*

Proof. The strategy is to start with an arbitrary resolution $f\colon M \to X$ and then to improve the resolution using Theorem 1.1 until we obtain a resolution that Theorem 2.1 implies is a near homeomorphism. As we pointed out in the introduction, resolutions are not easy to obtain. Originally, Miller [Mi] produced a resolution of $X \times [0, 1)$ and West [We] further refined this to obtain a resolution of X (for any locally compact ANR X). For our purposes it suffices to know that $X \times [0, 1)$ has a resolution, for once we know that $X \times [0, 1)$ is a Hilbert cube manifold, it follows that $X \times [0, 1]$ is a Hilbert cube manifold and the projection to X is a resolution. In any case, we shall now proceed assuming that $f\colon M \to X$ is a resolution.

Our first goal is to approximate f by a resolution f_1 such that the restriction of f to $f^{-1}(s_1(I^\infty))$ is a homeomorphism. Let G_1 be the decomposition of M whose only nondegenerate elements are the sets $f^{-1}(x)$ for $x \in s_1(I^\infty)$. It follows, since f is a hereditary shape equivalence, that the decomposition map $h\colon M \to M/G_1$ is a resolution of the ANR M/G_1. Since s_1 is a Z-embedding, the map h satisfies part (b) of Theorem 1.1 and, hence, is a near homeomorphism. Set $f_1 = fh^{-1}g$ where g is a homeomorphism approximating h. Then f_1 is a cell-like map approximating f and is one to one over $s_1(I^\infty)$. In the same manner, f_1 can be approximated by f_2 such that f_2 is one to one over $s_1(I^\infty) \cup s_2(I^\infty)$ and agrees with f_1 except over a "small" neighborhood of $s_2(I^\infty)$ missing $s_1(I^\infty)$. In recursive fashion, f_i's are produced such that f_i approximates f_{i-1}, f_i is one to one over $\bigcup\{s_j(I^\infty)\colon 1 \leqslant j \leqslant i\}$, and f_i agrees with f_{i-1} except over a small neighborhood of $s_i(I^\infty)$ missing $\bigcup\{s_j(I^\infty)\colon 1 \leqslant j < i\}$ (the construction of the approximating homeomorphism in Theorem 1.1 allows for this extra control). If sufficient control is exercised in specifying the closeness of the f_i's, they will converge to a resolution $F\colon M \to X$ where F satisfies the hypothesis of part (b) of Theorem 2.1 as it is one to one over $\bigcup\{s_i(I^\infty)\colon 1 \leqslant i < \infty\}$.

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