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CYCLIC HOMOLOGY OF GROUP RINGS*

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§ 1. Introduction

This note was inspired by two lectures on cyclic homology given by D. Burghelea during the Banach Center Topology Semester in May 1984. Among other things, they were devoted to the calculation of the cyclic homology groups for group rings kG, k being a field of characteristic zero.

As will be shown elsewhere, the cyclic homology seems to be the right tool when studying idempotents in group rings. In particular it generalizes the Stallings–Bass trace functions and hence provides a method of proving the Idempotent Conjecture for some new classes of torsion free groups.

In the present paper we offer a self-contained and purely algebraic calculation of the cyclic homology for group rings.

§ 2. Some homological algebra

In this section we merely fix the notation. We refer the reader to [1] for the details.

Let Λ be an associative ring. By a double complex C we mean a commuting array $\{C_{ij}\}_{i,j\geqslant 0}$ of left Λ-modules and Λ-homomorphisms in which all rows (C_{*j}, d_{hor}) and columns (C_{i*}, d_{ver}) are chain complexes. Maps between double complexes are defined in a natural way.

With a double complex C we associate its total complex $\operatorname{Tot} C$. It is the chain complex $(\operatorname{Tot} C)_n = \bigoplus_{i+j=n} C_{ij}$, $d_{n|C_{ij}} = d_{\text{hor}} + (-1)^i d_{\text{ver}}$. Obviously, any map between double complexes determines a chain map between their total complexes.

The total complex $\operatorname{Tot} C$ has two filtrations:

$$F^p_{\text{ver}} \operatorname{Tot} C_n = \bigoplus_{i\leqslant p} C_{i,n-i} \quad \text{and} \quad F^p_{\text{hor}} \operatorname{Tot} C_n = \bigoplus_{j\leqslant p} C_{n-j,j}, \qquad p = 0, 1, \ldots,$$

* This paper is in final form and no version of it will be submitted for publication elsewhere.

which give rise to two spectral sequences $\{E^r_{\mathrm{ver}}\}$, $\{E^r_{\mathrm{hor}}\}$, both converging to the homology of Tot C. They start from $E^0_{\mathrm{ver}} = E^0_{\mathrm{hor}} = C$. The table E^1_{ver} can be described as the vertical homology of C. Similarly, E^1_{hor} is the horizontal homology of C.

Let $\{E^r\}$ be one of those sequences. Any map $f\colon C \to D$ of double complexes induces maps $f^r\colon E^r(C) \to E^r(D)$, $r = 0, 1, \ldots$ and also a homomorphism $f_*\colon H_*(\mathrm{Tot}\, C) \to H_*(\mathrm{Tot}\, D)$. The only fact about spectral sequences we are going to use is

PROPOSITION 2.1 ([1], Ch. XV, Thm. 3.2). *If f^r is an isomorphism for some $r \geqslant 0$ then f_* is an isomorphism.* ∎

We now briefly review the homology theory of groups. Let G be a group and let k be a commutative ring with identity. In order to calculate the homology of G with coefficients in a right kG-module A we take any left kG-projective resolution $P \twoheadrightarrow k \to 0$ of the trivial kG-module k and set $H_n(G; A) = H_n(A \otimes_{kG} P_*)$. The result does not depend on the choice of P: given another resolution P' there is a chain map $f\colon P \to P'$, covering the identity on k. Any such a map identifies the homology defined either way.

One can always use the standard resolution $S \overset{\varepsilon}{\twoheadrightarrow} k \to 0$: $S_n = kG^{\otimes(n+1)} = k[G \times \ldots \times G]$, $d_n\colon S_n \to S_{n-1}$ is a k-linear map satisfying $d_n(x_0, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \ldots, \hat{x}_i, \ldots, x_n)$ for $x_i \in G$ and $\varepsilon\colon kG \to k$ is the augmentation.

However, for the cyclic group C of order $n+1$ we can use

$$P\colon \qquad 0 \leftarrow kG \xleftarrow[1-X]{} kC \xleftarrow[N]{} kC \xleftarrow[1-X]{} kC \xleftarrow[N]{} \ldots$$

where X is the right multiplication by a fixed generator of C and $N = 1 + X + \ldots + X^n$.

LEMMA 2.2. *Assume the order of C is invertible in k. Then for any kC-module A the sequence* $\ldots \xleftarrow[N]{} A \xleftarrow[1-X]{} A \xleftarrow[N]{} A \xleftarrow[1-X]{} \ldots$ *is exact.*

Proof. Here A is a projective kC-module, hence $H_*(C; A) = 0$. ∎

From now on we will assume that k is a field of characteristic zero. Let Λ be an associative k-algebra with identity. We will use the abbreviation Λ^n for $\Lambda^{\otimes n}$, the n-fold tensor product of Λ over k.

Recall the Hochschild complex

$$\mathscr{H}\colon \qquad 0 \leftarrow \Lambda \xleftarrow[b_1]{} \Lambda^2 \xleftarrow[b_2]{} \Lambda^3 \xleftarrow[b_3]{} \Lambda^4 \leftarrow \ldots$$

where

$$b_n(a_0 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n + (-1)^n a_n a_0 \otimes \ldots \otimes a_1 \otimes \ldots \otimes a_n$$

For all $n \geqslant 0$ we have an action of the cyclic group $Z/(n+1)Z$ on Λ^{n+1}, generated by $T_n(a_0 \otimes \ldots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \ldots \otimes a_{n-1}$. The differentials b_n factor to give the chain complex:

$$0 \leftarrow \Lambda/(1-T_0)\Lambda \underset{\bar{b}_1}{\leftarrow} \Lambda^2/(1-T_1)\Lambda^2 \underset{\bar{b}_2}{\leftarrow} \ldots$$

The homology groups of this complex are called the cyclic homology groups of Λ:

$$HC_n(\Lambda) = \ker \bar{b}_n / \operatorname{im} \bar{b}_{n+1}, \qquad n = 0, 1, 2, \ldots$$

Following [2] we will identify $HC_*(\Lambda)$ with the homology of the total complex of some double complex $D(\Lambda)$. Consider a modified Hochschild complex:

$$\mathscr{H}': \qquad 0 \leftarrow \Lambda \overset{b'_1}{\leftarrow} \Lambda^2 \overset{b'_2}{\leftarrow} \Lambda^3 \overset{b'_3}{\leftarrow} \Lambda^4 \leftarrow \ldots,$$

$$b'_n(a_0 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n.$$

It is a contractible complex. Putting $\mathscr{H}$ and $\mathscr{H}'$ together, we obtain a double complex

$$D(\Lambda): \qquad \mathscr{H} \overset{1-T}{\leftarrow} \mathscr{H}' \overset{N}{\leftarrow} \mathscr{H} \overset{1-T}{\leftarrow} \mathscr{H}' \overset{N}{\leftarrow} \ldots$$

where $T_n\colon \Lambda^{n+1} \to \Lambda^{n+1}$ were defined earlier and $N_n = T_n^0 + \ldots + T_n^n$.

PROPOSITION 2.3. $HC_*(\Lambda) \approx H_*(\operatorname{Tot} D(\Lambda))$.

Proof. We apply the sequence $\{E^r_{\text{hor}}\}$ to $D(\Lambda)$. The zero degree column of E^1_{hor} coincides with the complex $(\Lambda^*/(1-T)\Lambda^*, \bar{b}_*)$ defining $HC_*(\Lambda)$ while the other columns are zero by Lemma 2.2. Thus the sequence collapses and $H_*(\operatorname{Tot} D(\Lambda)) \approx E^\infty \approx E^2 = HC_*(\Lambda)$. ∎

The formula $H_*(\operatorname{Tot} D(\Lambda))$ can be used as an alternative definition of cyclic homology. It proved to be better suited for algebras over arbitrary rings.

The simplest example of a k-algebra is k itself. It is easy to calculate from either definition that

$$HC_n(k) = \begin{cases} k & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

§ 3. An algebraic proof of Burghelea's Theorem

For a group G we denote by TG the set of its conjugacy classes. It decomposes into $T_0 G \cup T_\infty G$ where $T_0 G$ consists of classes of torsion elements (and 1) and $T_\infty G$ includes all other classes.

For any class $c \in TG$ choose, once for ever, a representative $z \in c \subseteq G$.

We agree to write G_c for the quotient group $C_G(z)/\langle z\rangle$ where $C_G(z) = \{y\in G|\ yz = zy\}$.

We are going to prove

THEOREM 3.1 (D. Burghelea). *Suppose k is a field of characteristic zero. There is an isomorphism*

$$\varrho\colon HC_*(kG) \xrightarrow{\simeq} \bigoplus_{c\in T_0 G} H_*(G_c)\otimes HC_*(k) \oplus \bigoplus_{c\in T_\infty G} H_*(G_c).$$

Remarks. $H_*(G_c)$ stands for the homology groups of G_c calculated with respect to the trivial kG_c-module of coefficients k. The tensor product is graded in the standard way. The isomorphism ϱ respects the gradings of both sides.

For simplicity, we write Λ for kG. By Prop. 2.3 we need to calculate $H_*(\operatorname{Tot} D(\Lambda))$. We perform that in five steps.

Step 1. Replace $D(\Lambda)$ by a new double complex $C(\Lambda)$, which is better suited for homology of groups.

First we construct $C(\Lambda)$. We use here the standard complex S introduced in section two, together with its modified (contractible) version S':

$$S'\colon \qquad 0 \leftarrow \Lambda \xleftarrow{d_1'} \Lambda^2 \xleftarrow{d_2'} \Lambda^3 \xleftarrow{d_3'} \Lambda^4 \leftarrow \ldots$$

$$d_n'(x_0, \ldots, x_n) = \sum_{i=0}^{n-1} (-1)^i (x_0, \ldots, \hat{x}_i, \ldots, x_n), \qquad x_i\in G.$$

Consider Λ as a right Λ-module with the action of G given by $\lambda x = x^{-1}\lambda x$ for $\lambda\in\Lambda$ and $x\in G$. We denote this module by $\tilde{\Lambda}$. We set

$$C(\Lambda)\colon \qquad \tilde{\Lambda}\otimes_\Lambda S \xleftarrow{1-\tau} \tilde{\Lambda}\otimes_\Lambda S' \xleftarrow{\nu} \tilde{\Lambda}\otimes_\Lambda S \xleftarrow{1-\tau} \ldots$$

with the maps $\tau_n\colon \tilde{\Lambda}\otimes_\Lambda \Lambda^{n+1} \to \tilde{\Lambda}\otimes_\Lambda \Lambda^{n+1}$ defined on generators as

$$\tau_n\big(y\otimes_\Lambda (x_0, \ldots, x_n)\big) = (-1)^n y\otimes_\Lambda (y^{-1}x_n, x_0, \ldots, x_{n-1})$$

for $y, x_i\in G$. They are well defined and satisfy $\tau_n^{n+1} = \mathrm{id}$. We also set $\nu_n = \tau_n^0 + \ldots + \tau_n^n$.

LEMMA 3.2. *There exists an isomorphism of double complexes*

$$\alpha\colon C(\Lambda) \xrightarrow{\simeq} D(\Lambda).$$

Proof. The map $\alpha_n\colon \tilde{\Lambda}\otimes_\Lambda \Lambda^{n+1} \to \Lambda^{n+1}$ is given on generators by

$$\alpha_n\big(\lambda\otimes_\Lambda (x_0, \ldots, x_n)\big) = x_n^{-1}\lambda x_0 \otimes x_0^{-1}x_1 \otimes \ldots \otimes x_{n-1}^{-1}x_n$$

and then extended by linearity. It is a k-isomorphism with the inverse given by

$$\alpha_n^{-1}(y_0\otimes \ldots \otimes y_n) = (y_1\ldots y_n y_0)\otimes_\Lambda (1, y_1, y_1 y_2, \ldots, y_1\ldots y_n) \qquad \text{for} \quad y_i\in G.$$

A simple calculation shows that α commutes with horizontal and vertical arrows of $D(\Lambda)$ and $C(\Lambda)$. ∎

COROLLARY 3.3. (i) *$C(\Lambda)$ is a double complex;*
(ii) *The odd degree columns of $C(\Lambda)$ are exact sequences.*

Step 2. Split $C(\Lambda)$ over the set TG.

Each conjugacy class $c \in TG$ is a right G-set via the action $y \cdot x = x^{-1}yx$ for $y \in c$, $x \in G$. Extending this action by linearity we obtain a right Λ-module $k[c]$. Clearly, $\tilde{\Lambda} \approx \bigoplus_{c \in TG} k[c]$ as right Λ-mudules. Consequently, each entry in $C(\Lambda)$ has a corresponding decomposition. It is easy to see that the arrows of $C(\Lambda)$ respect those decompositions. Thus $C(\Lambda) \approx \bigoplus_{c \in TG} C_c(\Lambda)$ as double complexes. Hence also $H_*(\operatorname{Tot} C(\Lambda)) \approx \bigoplus_{c \in TG} H_*(\operatorname{Tot} C_c(\Lambda))$.

What is left is to identify $H_*(\operatorname{Tot} C_c(\Lambda))$ with $H_*(G_c) \otimes HC_*(k)$ for $c \in T_0 G$ and with $H_*(G_c)$ for $c \in T_\infty G$. For that purpose let us choose a class $c \in TG$. Recall we have picked a representative $z \in c$. We write Λ_z for the group ring $k[C_G(z)]$ of its centralizer. It is a subring of Λ in the natural way.

Step 3. Simplify the double complex $C_c(\Lambda)$.

Recall we have the Shapiro isomorphisms

$$Sh_n\colon k[c] \otimes_\Lambda \Lambda^{n+1} \xrightarrow{\approx} (k \otimes_{\Lambda_z} \Lambda) \otimes_\Lambda \Lambda^{n+1} \xrightarrow{\approx} k \otimes_{\Lambda_z} \Lambda^{n+1}.$$

Applying Sh_n at proper positions of $C_c(\Lambda)$, we obtain an isomorphism $Sh\colon C_c(\Lambda) \xrightarrow{\approx} B(\Lambda)$ for

$$B(\Lambda)\colon \qquad k \otimes_{\Lambda_z} S(\Lambda) \xleftarrow{1-\tau} k \otimes_{\Lambda_z} S'(\Lambda) \xleftarrow{\nu} k \otimes_{\Lambda_z} S(\Lambda) \xleftarrow{1-\tau} \dots$$

where $\tau_n\colon k \otimes_{\Lambda_z} \Lambda^{n+1} \to k \otimes_{\Lambda_z} \Lambda^{n+1}$ is given by the formula

$$\tau_n(1 \otimes (x_0, \dots, x_n)) = 1 \otimes (-1)^n (z^{-1} x_n, x_0, \dots, x_{n-1}).$$

As before, $\nu_n = \tau_n^0 + \dots + \tau_n^n$.

It will be even more convenient to work with the double complex

$$B(\Lambda_z)\colon \qquad k \otimes_{\Lambda_z} S(\Lambda_z) \leftarrow k \otimes_{\Lambda_z} S'(\Lambda_z) \leftarrow k \otimes_{\Lambda_z} S(\Lambda_z) \leftarrow \dots$$

We have an obvious map $\beta\colon B(\Lambda_z) \to B(\Lambda)$ induced by the inclusion $\Lambda_z \hookrightarrow \Lambda$.

LEMMA 3.4. *$\beta_*\colon H_*(\operatorname{Tot} B(\Lambda_z)) \to H_*(\operatorname{Tot} B(\Lambda))$ is an isomorphism.*

Proof. We show that $\beta^1\colon E^1_{\text{ver}}(B(\Lambda_z)) \to E^1_{\text{ver}}(B(\Lambda))$ is an isomorphism. From Cor. 3.3(ii) we know that the odd degree columns of $B(\Lambda)$, $B(\Lambda_z)$ are exact. Thus $\beta^1\colon 0 \approx 0$ there.

In the even degree columns of E^1_{ver} we obtain $H_*(C_G(z); k)$ calculated from the resolution $S(\Lambda)$ and $S(\Lambda_z)$, respectively. But $\beta = \mathrm{id}_k \otimes_{\Lambda_z} \tilde{\beta}$ where $\tilde{\beta}\colon S(\Lambda_z) \to S(\Lambda)$ is a map of resolutions, covering id_k. Thus $\beta^1 = \tilde{\beta}_*$ is an isomorphism. Proposition 2.1 completes the argument. ∎

To simplify the notation we further assume that $\Lambda_z = \Lambda$, i.e. that z is central in G. Write $\bar{G}$ for $G/\langle z\rangle$ and $\bar{\Lambda}$ for $k\bar{G}$.

Step 4. Complete the case $c \in T_0 G$.

The natural homomorphism $\pi\colon \Lambda \to \bar{\Lambda}$ induces a map $\pi\colon B(\Lambda) \to B(\bar{\Lambda})$.

LEMMA 3.5. *$\pi_*\colon H_*(\operatorname{Tot} B(\Lambda)) \to H_*(\operatorname{Tot} B(\bar{\Lambda}))$ is an isomorphism.*

Proof. We use E^1_{ver} again. As before, the odd degree columns of E^1_{ver} are zero. In the even degree columns, π^1 coincides with the inflation homomorphism inf: $H_*(G;k) \to H_*(\bar{G};k)$. It is an isomorphism, as $\ker(G \to \bar{G})$ is a finite group and k is a field of characteristic zero. By Prop. 2.1 π_* is an isomorphism. ∎

The double complex $B(\bar{\Lambda})$ can be written as

$$k \otimes_{\bar{\Lambda}} \big(S(\bar{\Lambda}) \xleftarrow{1-\sigma} S'(\bar{\Lambda}) \xleftarrow{\nu} S(\bar{\Lambda}) \xleftarrow{1-\sigma} S'(\bar{\Lambda}) \xleftarrow{\nu} \dots\big)$$

where $\sigma_n(x_0, \dots, x_n) = (-1)^n (x_n, x_0, \dots, x_{n-1})$.

We compare it with

$$A_0\colon \qquad k \otimes_{\bar{\Lambda}} \big(S(\bar{\Lambda}) \leftarrow 0 \leftarrow S(\bar{\Lambda}) \leftarrow 0 \leftarrow S(\bar{\Lambda}) \leftarrow \dots\big).$$

LEMMA 3.6. *There exists a chain map $\tilde{\gamma}\colon S(\bar{\Lambda}) \to S(\bar{\Lambda})$ which makes the following diagram commutative*:

$$\begin{tikzcd}
S(\bar{\Lambda}) \arrow[d, "\tilde{\gamma}"'] & S'(\bar{\Lambda}) \arrow[l, "1-\sigma"'] \arrow[d] & S(\bar{\Lambda}) \arrow[l, "\nu"'] \arrow[d, "\tilde{\gamma}"'] & S'(\bar{\Lambda}) \arrow[l, "1-\sigma"'] \arrow[d] & \arrow[l] \\
S(\bar{\Lambda}) & 0 \arrow[l] & S(\bar{\Lambda}) \arrow[l] & 0 \arrow[l] & \arrow[l]
\end{tikzcd}$$

Proof. Let Σ_{n+1} be the symmetric group permuting the coordinates $(x_0, \dots, x_n)$ in $\bar{\Lambda}^{n+1}$. We define $\tilde{\gamma}_n\colon \bar{\Lambda}^{n+1} \to \bar{\Lambda}^{n+1}$ by the formula

$$\tilde{\gamma}_n(x_0, \dots, x_n) = \frac{1}{(n+1)!} \sum_{g \in \Sigma_{n+1}} \operatorname{sign}(g)\, g(x_0, \dots, x_n).$$

It is easy to check that $\tilde{\gamma}$ is a chain endomorphism of $S(\bar{\Lambda})$ which covers id_k and such that $\tilde{\gamma}(1-\sigma) = 0$. ∎

Consider the map $\gamma = \mathrm{id}_k \otimes_{\bar{\Lambda}} \tilde{\gamma}\colon B(\bar{\Lambda}) \to A_0$.

LEMMA 3.7. *$\gamma_*\colon H_*(\operatorname{Tot} B(\bar{\Lambda})) \to H_*(\operatorname{Tot} A_0)$ is an isomorphism.*

Proof. Use again E^1_{ver}: $\tilde{\gamma}$ covers id_k, so γ^1 is an isomorphism; Prop. 2.1 again does the job. ∎

The homology of $\operatorname{Tot} A_0$ can easily be calculated:

$$H_n(\operatorname{Tot} A_0) = H_n(\bar{G}) \oplus H_{n-2}(\bar{G}) \oplus \dots = \big(H_*(\bar{G}) \otimes HC_*(k)\big)_n.$$

Returning to the general notation we conclude:

– if $c \in T_0 G$ then $H_*(\operatorname{Tot} C_c(\Lambda)) \approx H_*(G_c) \otimes HC_*(k)$.

Step 5. Complete the case $c \in T_\infty G$.

Assume the symplifying assumption: $\Lambda = \Lambda_z$ is valid again. We start with π: $B(\Lambda) \to B(\bar{\Lambda})$ as before, although π_* is not an isomorphism this time. Instead of A_0 we use

$$A_\infty: \qquad k \otimes_{\bar{\Lambda}} (S(\bar{\Lambda}) \leftarrow 0 \leftarrow 0 \leftarrow 0 \leftarrow 0 \leftarrow \ldots).$$

Let $\tilde{\delta}$ be the map

$$\begin{tikzcd} S(\bar{\Lambda}) \arrow[d, "\bar{\gamma}"] & S'(\bar{\Lambda}) \arrow[l, "1-\sigma"'] \arrow[d] & S(\bar{\Lambda}) \arrow[l, "\nu"'] \arrow[d] & S'(\bar{\Lambda}) \arrow[l, "1-\sigma"'] \arrow[d] & \arrow[l] \\ S(\Lambda) & 0 \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & \arrow[l] \end{tikzcd}$$

and set $\delta = \mathrm{id}_k \otimes_{\bar{\Lambda}} \tilde{\delta}$.

We intend to prove that $(\delta\pi)_*$: $H_*(\operatorname{Tot} B(\Lambda)) \to H_*(\operatorname{Tot} A_\infty)$ is an isomorphism. For that we need two lemmata.

LEMMA 3.8. *Let* $\sigma = \sigma_n$: $\Lambda^{n+1} \to \Lambda^{n+1}$ *be given by* $\sigma(x_0, \ldots, x_n) = (-1)^n(z^{-1}x_n, x_0, \ldots, x_{n-1})$. *If the order of* z *is infinite then* $1-\sigma$ *is injective.*

Proof. Assume that $\sigma(a) = a$ for some $a \in \Lambda^{n+1} \setminus \{0\}$. Then $a = \sigma^{n+1}(a) = z^{-1}a$, i.e. $(1-z)a = 0$. Thus $1-z$ is a zero divisor in the group ring $k[G \times \ldots \times G]$. Let $H = \langle z \rangle \subseteq G \times \ldots \times G$. Considering ag instead of a, if necessary, we can assume that the support of a intersects H nontrivially. Let $\bar{a}$ be the linear projection of a onto $kH \subseteq k[G \times \ldots \times G]$. Then $\bar{a} \neq 0$ and so $1-z$ is a zero divisor of kH. But this is impossible, as $H \approx Z$ is orderable. ∎

LEMMA 3.9. *The sequence*

$$\bar{S}: \qquad 0 \leftarrow \Lambda/(1-\sigma_0)\Lambda \xleftarrow{\bar{d}_1} \Lambda^2/(1-\sigma_1)\Lambda^2 \xleftarrow{\bar{d}_2} \ldots$$

is a left $\bar{\Lambda}$*-projective resolution of the trivial* $\bar{\Lambda}$*-module* k.

Proof. It is an easy exercise to show that $\Lambda^{n+1}/(1-\sigma^{n+1})\Lambda^{n+1} = \Lambda^{n+1}/(1-z^{-1})\Lambda^{n+1}$ is a free $\bar{\Lambda}$-module. But the natural epimorphism

$$\Lambda^{n+1}/(1-\sigma^{n+1})\Lambda^{n+1} \twoheadrightarrow \Lambda^{n+1}/(1-\sigma)\Lambda^{n+1}$$

splits, e.g. by $\dfrac{1}{n+1}(1+\sigma+\ldots+\sigma^n)$. Thus all terms of $\bar{S}$ are projective $\bar{\Lambda}$-modules.

To prove the exactness of $\bar{S} \twoheadrightarrow k$ consider the diagram of chain complexes:

$$\begin{tikzcd}
S' \arrow[r, two heads] \arrow[d, tail, "1-\sigma"'] & 0 \arrow[d] \\
S \arrow[r, two heads] \arrow[d, two heads, "\text{proj.}"'] & k \arrow[d, "="] \\
\bar{S} \arrow[r, two heads] & k
\end{tikzcd}$$

The previous lemma shows that all columns are exact. Also the first two rows are exact. Thus so is the third one. ∎

LEMMA 3.10. $(\delta\pi)_*$: $H_*(\mathrm{Tot}\, B(\Lambda)) \to H_*(\mathrm{Tot}\, A_\infty)$ is an isomorphism.

Proof. We compare the E^2_{hor}-terms for $B(\Lambda)$ and A_∞. First notice that the E^1_{hor}-tables vanish outside the zero degree column.

The zero degree column of $E^1_{\mathrm{hor}}(B(\Lambda))$ is equal to

$$0 \leftarrow k \otimes_\Lambda \Lambda/(1-\tau_0)k \otimes_\Lambda \Lambda \leftarrow k \otimes_\Lambda \Lambda^2/(1-\tau_1)k \otimes_\Lambda \Lambda^2 \leftarrow \ldots$$

We identify it with $k \otimes_{\bar{\Lambda}} \bar{S}$ where $\bar{S}$ is the resolution from Lemma 3.9. Thus $E^2_{\mathrm{hor}}(B(\Lambda))$ has $H_*(\bar{G};k)$ in its zero degree column and zeros elsewhere. The same is clearly true for $E^2_{\mathrm{hor}}(A_\infty)$.

Now, $(\delta\pi)^1$ is induced by a chain map of $\bar{\Lambda}$-projective resolutions $\bar{\delta\pi}$: $\bar{S} \to S(\bar{\Lambda})$. Thus $(\delta\pi)^2$ is an isomorphism and so is $(\delta\pi)_*$. ∎

Obviously, $H_*(\mathrm{Tot}\, A_\infty) = H_*(\bar{G})$. Returning to the general notation, we conclude:

$$- \text{ if } c \in T_\infty G \text{ then } H_*(\mathrm{Tot}\, C_c(\Lambda)) \approx H_*(G_c)$$

what completes the proof of Burghelea's Theorem.

References

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