TOPICS IN ALGEBRA
BANACH CENTER PUBLICATIONS, VOLUME 26, PART 1
PWN POLISH SCIENTIFIC PUBLISHERS
WARSAW 1990

TILTING THEORY – AN INTRODUCTION

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Let A be a finite dimensional algebra (associative, with an identity) over an algebraically closed field. Following Happel and Ringel, we shall call a finitely generated A-module T_A a tilting module if it satisfies the following three conditions:

(T1) The projective dimension of T_A does not exceed one.

(T2) $\mathrm{Ext}_A^1(T, T) = 0$.

(T3) There exists a short exact sequence $0 \to A_A \to T'_A \to T''_A \to 0$ with T' and T'' direct sums of direct summands of T.

It was shown by Brenner–Butler and Happel–Ringel that there exists a close connection between the representation theories of the algebras A and $B = \mathrm{End}\, T_A$. This connection is known as tilting theory and these notes are meant as an introduction to this theory. Most of the basic results are proved here in detail and several examples are provided to illustrate the methods. Also, some of the more specialised aspects are surveyed, such as tilting-cotilting equivalence, and the theory of tilted and iterated tilted algebras.

Introduction

The present notes are a more or less faithful version of a series of five lectures given in the Banach semester on "Classical algebraic structures", session II: "Representations of finite dimensional algebras and related topics", held in Warsaw, April–May 1988. The aim of this series was to introduce the theory of tilting modules, as initiated in [27] and [43], and to present some of their applications in representation theory. We have tried to keep these notes as self-contained and introductory as possible, thus most of the basic results of

This paper is in final form and no version of it will be submitted for publication elsewhere.

tilting theory are proved here in detail. Also, several examples are provided. In order to limit the size of the notes, we have not given the space deserved to several developments (most notably Happel's results on derived categories [39]), and we have left out several others, such as the recent generalisations of the concept of tilting module (as in [58], see also [39], [30], or in [80]) and tilting objects in other categories (as in [35], [17], in [62], or in [33]). For the history of tilting theory, we refer the reader to [40], III.

The organisation of the notes follows closely that of the lectures, even though the notes are more complete. Section 1 is devoted to the introduction of tilting modules by means of torsion theories in mod A. Most of the material in this section is based on results of [16], [43], [21]. Section 2 contains the main theorems of tilting theory, as can be found in [27], [43], [21]. In Section 3, we give necessary and sufficient conditions for a torsion theory to be induced by a tilting module, as in [3], [70], [49], then we describe the torsion resolutions as introduced in [75]. In Section 4, we discuss those properties of an algebra which are preserved under the tilting process. Sections 5 and 6 are devoted respectively to the classes of tilted and iterated tilted algebras, introduced in [43] and [5], respectively.

Throughout these notes, k will denote a fixed algebraically closed field, and A a finite dimensional associative k-algebra with an identity. All our A-modules will be finitely generated right A-modules, and we shall denote their category by mod A. The corresponding (projectively) stable module category will be denoted by $\underline{\text{mod}}\,A$, and the standard duality on mod A by $D = \text{Hom}_k(-, k)$. For an additive category $\mathscr{C}$, we denote by ind $\mathscr{C}$ a full subcategory of $\mathscr{C}$ consisting of a complete set of representatives of the non-isomorphic indecomposable objects of $\mathscr{C}$. If, in particular, $\mathscr{C} = \text{mod}\,A$, we shall write ind $\mathscr{C}$ = ind A. For an A-module M, we denote by add(M) the full subcategory of mod A consisting of the direct sums of direct summands of M, and by Gen(M) (respectively, Cogen(M)) the full subcategory of mod A generated (respectively, cogenerated) by M. The projective (respectively, injective) dimension of an A-module M will be denoted by pd M (respectively, id M) and the global dimension of the algebra A by gl. dim A. For each vertex i of the ordinary quiver of A, we denote by e_i the corresponding primitive idempotent of A and by $S(i)$ the corresponding simple A-module. The projective cover (respectively, the injective envelope) of $S(i)$ will be denoted by $P(i)$ (respectively, $I(i)$). We shall use freely and without further reference properties of the Auslander–Reiten translations $\tau = D\text{Tr}$ and $\tau^{-1} = \text{Tr}D$, and of the Auslander–Reiten quiver Γ_A of A such as can be found in [15], [34] and [65].

1. Torsion theories and tilting modules

When the representation theory of an algebra A is difficult to study directly, it is sometimes convenient to replace A by another, simpler, algebra B and to reduce the problem from a problem on A to a problem on B. This occurs, for instance, in the classical Morita equivalence theorem. The latter is, however, of limited use in representation theory, since in this case, no change occurs on the level of the module categories. The main idea of tilting theory is, given an algebra A, to construct a module T_A, called a tilting module, such that, if $B = \operatorname{End} T_A$, then the categories mod A and mod B are reasonably close to each other (but generally not equivalent). Since this procedure can be seen as generalising Morita theory, we should expect to use the adjoint pair of functors $\operatorname{Hom}_A(T, -)$ and $-\otimes_B T$ to compare $\operatorname{mod} A$ and $\operatorname{mod} B$. Also, we should expect the full subcategory $\operatorname{Gen}(T_A)$ of $\operatorname{mod} A$ to be of interest, and, indeed, we shall derive sufficient conditions on T so that $\operatorname{Gen}(T_A)$ is the torsion class of a torsion theory in $\operatorname{mod} A$. We thus begin by recalling a few well-known facts about torsion theories in module categories.

1.1. Definition (Dickson [32]). A *torsion theory* in $\operatorname{mod} A$ is a pair $(\mathscr{T}, \mathscr{F})$ of classes of modules such that:

(i) $\operatorname{Hom}_A(M, N) = 0$ for all $M \in \mathscr{T}$ and $N \in \mathscr{F}$.
(ii) $\operatorname{Hom}_A(M, -)|_{\mathscr{F}} = 0$ implies $M \in \mathscr{T}$.
(iii) $\operatorname{Hom}_A(-, N)|_{\mathscr{T}} = 0$ implies $N \in \mathscr{F}$.

The class $\mathscr{T}$ (respectively, $\mathscr{F}$) is called the *torsion class* (respectively, the *torsion-free class*) and its objects are called *torsion objects* (respectively, *torsion-free objects*). Clearly, $(\mathscr{T}, \mathscr{F})$ is a torsion theory in $\operatorname{mod} A$ if and only if $(D\mathscr{F}, D\mathscr{T})$ is a torsion theory in $\operatorname{mod} A^{\mathrm{op}}$. Any class $\mathscr{X}$ in $\operatorname{mod} A$ induces a torsion theory as follows:

$$\mathscr{F} = \{N_A \mid \operatorname{Hom}_A(-, N)|_{\mathscr{X}} = 0\}, \quad \mathscr{T} = \{M_A \mid \operatorname{Hom}_A(M, -)|_{\mathscr{F}} = 0\}.$$

Moreover, $\mathscr{T}$ is the smallest torsion class containing $\mathscr{X}$. Dually, $\mathscr{X}$ induces a torsion theory $(\mathscr{T}, \mathscr{F})$ such that $\mathscr{F}$ is the smallest torsion-free class containing $\mathscr{X}$. Another way to define a torsion theory is by means of an idempotent radical.

Definition. A *preradical* t is a subfunctor of the identity functor on $\operatorname{mod} A$, that is, it assigns to each module M a submodule tM such that each morphism $M \to N$ restricts to a morphism $tM \to tN$. A preradical t is said to be *idempotent* if $t^2 = t$ and is said to be a *radical* if $t(M/tM) = 0$ for every module M.

A torsion theory $(\mathscr{T}, \mathscr{F})$ induces an idempotent radical as follows: for every module M, tM is the *trace* of $\mathscr{T}$ in M, that is, it is the sum of all the submodules N of M such that $N \in \mathscr{T}$.

Torsion and torsion-free classes are characterised as follows:

Proposition. *Let $\mathscr{T}$ be a class of modules. The following conditions are equivalent*:

(i) *There exists a torsion theory* $(\mathscr{T}, \mathscr{F})$ *having* $\mathscr{T}$ *as torsion class.*
(ii) *There exists an idempotent radical t such that* $\mathscr{T} = \{M_A \mid tM = M\}$
(iii) $\mathscr{T}$ *is closed under images, direct sums and extensions.*

PROPOSITION. *Let* $\mathscr{F}$ *be a class of modules. The following conditions are equivalent*:

(i) *There exists a torsion theory* $(\mathscr{T}, \mathscr{F})$ *having* $\mathscr{F}$ *as torsion-free class.*
(ii) *There exists an idempotent radical t such that* $\mathscr{F} = \{N_A \mid tN = 0\}$.
(iii) $\mathscr{F}$ *is closed under submodules, direct products and extensions.*

For the proofs, we refer the reader to [32] or [71]

Thus, each of $\mathscr{T}$, $\mathscr{F}$ and t uniquely determines the others. Also, it follows directly from the above propositions that to each module M_A corresponds a short exact sequence $0 \to tM \to M \to M/tM \to 0$ with $tM \in \mathscr{T}$, $M/tM \in \mathscr{F}$ (called the *canonical sequence* for M) such that every short exact sequence $0 \to M' \to M \to M'' \to 0$ with $M' \in \mathscr{T}$, $M'' \in \mathscr{F}$ is isomorphic to the canonical sequence for M. An obvious consequence is that any simple module is either torsion or torsion-free.

DEFINITION. A torsion theory $(\mathscr{T}, \mathscr{F})$ is called *splitting* if, for every module M_A, the canonical sequence for M splits.

Clearly, a torsion theory $(\mathscr{T}, \mathscr{F})$ is splitting if and only if $\mathrm{Ext}_A^1(N, M) = 0$ for all $M \in \mathscr{T}$ and $N \in \mathscr{F}$, or, equivalently, if and only if every indecomposable A-module is either torsion or torsion-free.

1.2. Let now T_A be an A-module. We ask when $\mathrm{Gen}(T_A)$ is the torsion class of a torsion theory in mod A. For this purpose, we may clearly assume that T_A is *minimal*, that is, if $T = T' \oplus T''$, then $T' \notin \mathrm{Gen}(T'')$. Also, we shall let $B = \mathrm{End}\, T_A$ (so that T can also be considered as a left B-module).

LEMMA. *A module* M_A *belongs to* $\mathrm{Gen}(T_A)$ *if and only if the evaluation map* ε_M: $\mathrm{Hom}_A(T, M) \otimes_B T \to M$ *given by* $f \otimes t \to f(t)$ *is an epimorphism.*

Proof. We only need to prove the necessity. Suppose $M \in \mathrm{Gen}(T_A)$ and let $f_1, \ldots, f_d$ be a basis of the k-vector space $\mathrm{Hom}_A(T, M)$. Then there exists an epimorphism $[g_1, \ldots, g_m]$: $T^{(m)} \to M$. Since $g_j \in \mathrm{Hom}_A(T, M)$, we have $g_j = \sum_{i=1}^d \lambda_j^i f_i$ with $\lambda_j^i \in k$. Now any $x \in M$ can be written as $x = \sum_{j=1}^m g_j(t_j)$ with $t_j \in T$, but then

$$x = \sum_{j=1}^{m} \sum_{i=1}^{d} \lambda_j^i f_i(t_j) = \varepsilon_M\Big(\sum_{i,j} \lambda_j^i (f_i \otimes t_j)\Big).$$

Remarks. 1. While $\mathrm{Gen}(T_A)$ is clearly closed under images and direct sums, it is generally not closed under extensions (thus is not a torsion class). Let indeed A be any algebra having two non-isomorphic simple modules S_1, S_2 such that $\mathrm{Ext}_A^1(S_1, S_2) \neq 0$. Then $\mathrm{Gen}(S_1 \oplus S_2)$ is not closed under extensions.

2. To the class $\mathrm{Gen}(T_A)$ is associated an idempotent preradical t defined as follows: for a module M, we let tM be the *trace* of T in M, that is, the sum of all the submodules of M which belong to $\mathrm{Gen}(T_A)$.

1.3. LEMMA [16]. *Let T_A be an A-module. Then*:

(i) $\mathrm{Gen}(T_A)$ *is a torsion class if and only if* $\mathrm{Ext}_A^1(T, -)|_{\mathrm{Gen}(T_A)} = 0$.

(ii) $\mathrm{Cogen}(T_A)$ *is a torsion-free class if and only if* $\mathrm{Ext}_A^1(-, T)|_{\mathrm{Cogen}(T_A)} = 0$.

Proof of (i). Suppose that $\mathrm{Gen}(T_A)$ is a torsion class, and let M be an indecomposable torsion module such that $\mathrm{Ext}_A^1(T, M) \neq 0$. Then there exists an indecomposable summand T_0 of T such that $\mathrm{Ext}_A^1(T_0, M) \neq 0$. Let us consider a non-split extension

$$0 \to M \xrightarrow{u} E \xrightarrow{v} T_0 \to 0.$$

Since M, $T_0 \in \mathrm{Gen}(T_A)$, we have $E \in \mathrm{Gen}(T_A)$ and thus there exists an epimorphism p: $T^{(m)} \to E$, for some m. Let us write $T^{(m)} = R \oplus T_0^{(m)}$; then the composition $f = vp$: $T^{(m)} \to T_0$ can be written as $f = [g, f_1, \ldots, f_m]$, with $g \in \mathrm{Hom}_A(R, T_0)$ and $f_i \in \mathrm{End}\, T_0$. Since f is an epimorphism,

$$T_0 = g(R) + \sum_{i=1}^{m} f_i(T_0).$$

Since v is not a retraction, no f_i is an isomorphism and consequently $f_i(T_0) \subseteq (\mathrm{rad\,End}\, T_0)(T_0)$ (because the indecomposability of T_0 implies that $\mathrm{End}\, T_0$ is local) for all i. So

$$T_0 = g(R) + (\mathrm{rad\,End}\, T_0)(T_0).$$

Applying Nakayama's lemma to the left $\mathrm{End}\, T_0$-module T_0 we get $T_0 = g(R)$, so that g is an epimorphism. This, however, contradicts the minimality of T.

In order to prove the sufficiency, we must show that $\mathrm{Gen}(T_A)$ is closed under extensions, so let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence, with M', $M'' \in \mathrm{Gen}(T_A)$. Since $\mathrm{Ext}_A^1(T, M') = 0$, we have a short exact sequence

$$0 \to \mathrm{Hom}_A(T, M') \to \mathrm{Hom}_A(T, M) \to \mathrm{Hom}_A(T, M'') \to 0.$$

Applying the right exact functor $- \otimes_B T$, and comparing with the original sequence, we obtain an exact commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}_A(T, M')\otimes_B T & \to & \mathrm{Hom}_A(T, M)\otimes_B T & \to & \mathrm{Hom}_A(T, M'')\otimes_B T & \to & 0 \\
\downarrow{\scriptstyle \varepsilon_{M'}} & & \downarrow{\scriptstyle \varepsilon_M} & & \downarrow{\scriptstyle \varepsilon_{M''}} & & \\
0 \longrightarrow M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0
\end{array}$$

By 1.2, $\varepsilon_{M'}$ and $\varepsilon_{M''}$ are surjective. By the Five Lemma, ε_M is surjective as well. Hence $M \in \mathrm{Gen}(T_A)$.

1.4. DEFINITION [16]. Let $\mathscr{C}$ be a subcategory of $\mathrm{mod}\, A$ which is closed under extensions. A non-zero module M_A in $\mathscr{C}$ is said to be Ext-*projective* (respectively, Ext-*injective*) in $\mathscr{C}$ if $\operatorname{Ext}_A^1(M, -)|_{\mathscr{C}} = 0$ (respectively, $\operatorname{Ext}_A^1(-, M)|_{\mathscr{C}} = 0$).

Thus, if M is Ext-projective in a torsion class $\mathscr{T}$, then, for every short exact sequence $0 \to N' \to N \to N'' \to 0$ with $N' \in \mathscr{T}$, the induced sequence

$$0 \to \operatorname{Hom}_A(M, N') \to \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(M, N'') \to 0$$

is exact. Also, if $\operatorname{Gen}(T_A)$ is a torsion class, it follows from 1.3 that T_A is Ext-projective in $\operatorname{Gen}(T_A)$. Dually, if $\operatorname{Cogen}(T_A)$ is a torsion-free class, then T_A is Ext-injective in $\operatorname{Cogen}(T_A)$. The following characterisation of Ext-projective and Ext-injective modules in torsion and torsion-free classes is due to Auslander and Smalø [16].

LEMMA. *Let* $(\mathscr{T}, \mathscr{F})$ *be a torsion theory in* $\operatorname{mod} A$, *and* t *denote its idempotent radical. Then*:

(i) *If* $M \in \mathscr{T}$ *is indecomposable, then* M *is* Ext-*projective in* $\mathscr{T}$ *if and only if* $\tau M \in \mathscr{F}$, *and* M *is* Ext-*injective in* $\mathscr{T}$ *if and only if* $M \simeq tI$ *for an injective* $I_A \notin \mathscr{F}$.

(ii) *If* $N \in \mathscr{F}$ *is indecomposable, then* N *is* Ext-*injective in* $\mathscr{F}$ *if and only if* $\tau^{-1} N \in \mathscr{T}$, *and* N *is* Ext-*projective in* $\mathscr{F}$ *if and only if* $N \simeq P/tP$ *for a projective* $P_A \notin \mathscr{T}$.

Proof of (i). Suppose that $\tau M \in \mathscr{F}$ and let $X \in \mathscr{T}$. By the Auslander–Reiten formula $\operatorname{Ext}_A^1(M, X) = D\overline{\operatorname{Hom}}_A(X, \tau M) \subseteq D\operatorname{Hom}_A(X, \tau M) = 0$. Thus, M is Ext-projective in $\mathscr{T}$. Conversely, suppose that $\tau M \notin \mathscr{F}$. Then, in the canonical sequence for τM:

$$0 \to t(\tau M) \xrightarrow{u} \tau M \xrightarrow{v} \tau M/t(\tau M) \to 0.$$

v is not a section, thus, if

$$0 \to \tau M \xrightarrow{f} E \xrightarrow{g} M \to 0$$

is the Auslander–Reiten sequence ending with M, there exists $h\colon E \to \tau M/t(\tau M)$ such that $hf = v$. Since v is surjective, so is h. We thus obtain an exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & t(\tau M) & \xrightarrow{f'} & \operatorname{Ker} h & \xrightarrow{g'} & M & \to & 0 \\
 & & {\scriptstyle u}\downarrow & & \downarrow & & \| & & \\
0 & \to & \tau M & \xrightarrow{f} & E & \xrightarrow{g} & M & \to & 0 \\
 & & {\scriptstyle v}\downarrow & & {\scriptstyle h}\downarrow & & & & \\
 & & \tau M/t(\tau M) & = & \tau M/t(\tau M) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Clearly, the first row does not split (for, if g' is a retraction, so is g). So $\mathrm{Ext}_A^1(M, t(\tau M)) \neq 0$ implying that M is not Ext-projective in $\mathscr{T}$. This proves the first assertion.

Now let $I_A \notin \mathscr{F}$ be injective and let $X \in \mathscr{T}$. Applying the functor $\mathrm{Hom}_A(X, -)$ to the canonical sequence for I yields

$$0 = \mathrm{Hom}_A(X, I/tI) \to \mathrm{Ext}_A^1(X, tI) \to \mathrm{Ext}_A^1(X, I) = 0.$$

Thus tI is Ext-injective in $\mathscr{T}$. Conversely, let M_A be Ext-injective in $\mathscr{T}$ and I denote its injective envelope. We claim that M is a direct summand of tI. Indeed, since $\mathrm{Hom}_A(-, I)|_{\mathscr{T}} \simeq \mathrm{Hom}_A(-, tI)|_{\mathscr{T}}$, we have an exact sequence $0 \to M \to tI \to tI/M \to 0$. Since tI is torsion, so is tI/M. But M is Ext-injective in $\mathscr{T}$, hence this sequence splits. The proof is complete.

1.5. Definition. A module T_A is called a *partial tilting module* if it satisfies the following two conditions:

(T1) $\mathrm{pd}\, T_A \leqslant 1$.
(T2) $\mathrm{Ext}_A^1(T, T) = 0$.

Lemma. *Let T_A be a partial tilting module. Then* $\mathrm{Gen}(T_A)$ *is a torsion class. Conversely, if* $\mathrm{Gen}(T_A)$ *is a torsion class, and T_A is a faithful module, then T_A is a partial tilting module.*

Proof. Let T_A be a partial tilting module, and let $M \in \mathrm{Gen}(T_A)$. Then there exists an epimorphism $T^{(m)} \to M$ for some m. By (T1), it induces an epimorphism $\mathrm{Ext}_A^1(T, T^{(m)}) \to \mathrm{Ext}_A^1(T, M)$. By (T2), we deduce that $\mathrm{Ext}_A^1(T, M) = 0$. Thus T is Ext-projective in $\mathrm{Gen}(T_A)$. By 1.3, $\mathrm{Gen}(T_A)$ is a torsion class.

Conversely, suppose that $\mathrm{Gen}(T_A)$ is a torsion class. Then, by 1.3, we have $\mathrm{Ext}_A^1(T, T) = 0$. On the other hand, if T_A is faithful, then there exists an epimorphism $T^{(m)} \to (DA)_A$ for some m. Therefore $DA \in \mathrm{Gen}(T_A)$. On the other hand, by 1.3, T is Ext-projective in $\mathrm{Gen}(T_A)$, therefore, by 1.4, τT is torsion-free. This implies that $\mathrm{Hom}_A(DA, \tau T) = 0$, that is, $\mathrm{pd}\, T_A \leqslant 1$ (see [65], 2.4, (1)).

Remarks. 1. If $\mathrm{Gen}(T_A)$ is a torsion class, then the corresponding torsion-free class is $\mathscr{F} = \{M_A \mid \mathrm{Hom}_A(T, M) = 0\}$. Note that, since T_A is Ext-projective in $\mathrm{Gen}(T_A)$, we always have $\tau T \in \mathscr{F}$.

2. If $\mathrm{Gen}(T_A)$ is a torsion class, but T_A is not faithful, it is generally not a partial tilting module. Indeed, if A is given by the quiver

$$\overset{1}{\circ} \xleftarrow{\beta} \overset{2}{\circ} \xleftarrow{\alpha} \overset{3}{\circ}$$

bound by $\alpha\beta = 0$, then $S(3)$ generates a torsion class, but $\mathrm{pd}\, S(3) = 2$.

1.6. Definition [43]. A partial tilting module is called a *tilting module* if it satisfies the additional property

(T3) There exists a short exact sequence $0 \to A_A \to T'_A \to T''_A \to 0$ with $T', T'' \in \mathrm{add}(T_A)$.

Thus, trivial examples of tilting modules are provided by the Morita progenerators. Clearly, if T_A is a tilting module, then $\mathrm{Gen}(T_A)$ is a torsion class. The torsion theory induced by a tilting module T_A will be denoted by $(\mathscr{T}(T_A), \mathscr{F}(T_A))$ (thus, $\mathscr{T}(T_A) = \mathrm{Gen}(T_A)$).

LEMMA. *Let T_A be a tilting module. Then $\mathscr{T}(T_A) = \{M_A \mid \mathrm{Ext}_A^1(T, M) = 0\}$.*

Proof. Since T_A is Ext-projective in $\mathscr{T}(T_A)$, it follows that $\mathscr{T}(T_A) \subseteq \{M_A \mid \mathrm{Ext}_A^1(T, M) = 0\}$. Conversely, let M_A be a module such that $\mathrm{Ext}_A^1(T, M) = 0$. Applying the functor $\mathrm{Hom}_A(T, -)$ to the canonical sequence for M yields, by (T1), an epimorphism $\mathrm{Ext}_A^1(T, M) \to \mathrm{Ext}_A^1(T, M/tM)$. Thus $\mathrm{Ext}_A^1(T, M/tM) = 0$. Since $M/tM \in \mathscr{F}(T_A)$, we also have $\mathrm{Hom}_A(T, M/tM) = 0$. Therefore, applying the functor $\mathrm{Hom}_A(-, M/tM)$ to the short exact sequence of (T3) yields an exact sequence

$$0 \to \mathrm{Hom}_A(T'', M/tM) \to \mathrm{Hom}_A(T', M/tM) \to \mathrm{Hom}_A(A, M/tM) \to \mathrm{Ext}_A^1(T'', M/tM).$$

Hence $M/tM \cong \mathrm{Hom}_A(A, M/tM) = 0$ and $M = tM \in \mathscr{T}(T_A)$.

COROLLARY. *Let T_A be a tilting module. Then any injective A-module belongs to $\mathscr{T}(T_A)$. In particular, the* Ext*-injective modules in $\mathscr{T}(T_A)$ coincide with the injective A-modules.*

Remarks. 1. We shall see in 3.2 that, conversely, if $\mathscr{T} = \mathrm{Gen}(T_A)$ is a torsion class containing the injectives, then $\mathscr{T}$ is generated by a tilting module.

2. It follows from the Corollary that, if T_A is a tilting module, then $\mathscr{T}(T_A)$ is a hereditary torsion class only if T_A is a Morita progenerator.

3. Let P_A be an indecomposable projective-injective. Then P_A is a direct summand of any tilting module T_A. Indeed, since P is injective, there exists an epimorphism $T^{(m)} \to P$ which splits (because P is projective) and therefore $P \in \mathrm{add}(T)$. Thus, if A is self-injective, the tilting modules coincide with the Morita progenerators. The following is an example of an algebra which is not self-injective but such that the tilting modules coincide with the Morita progenerators: let A be the radical-square zero algebra given by the quiver

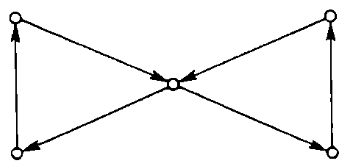

Fig. 1

of Fig. 1; it is easily seen that every indecomposable non-projective A-module has infinite projective dimension and so the stated property is satisfied. Actually, it is not hard to show that, for an algebra A, the tilting A-modules coincide with the Morita progenerators if and only if every indecomposable non-projective A-module has infinite projective dimension (see [40], III, 2.14(h)).

EXAMPLES. (i) *The APR-tilting modules*. The following construction, due to Auslander, Platzeck and Reiten [14], is a generalisation of the reflection functors of Bernstein, Gelfand and Ponomarev [20]. Let A be an algebra and let $S(i)_A$ be a simple projective non-injective. Then the module

$$T[i]_A = \tau^{-1}S(i)\oplus(\bigoplus_{j\neq i} P(j))$$

is a tilting module called the *APR-tilting module* associated to $S(i)$. Indeed, any neighbour of $S(i) = P(i)$ in Γ_A is an indecomposable projective $P(a)$ such that we have an arrow $a \to i$ in the ordinary quiver of A. Therefore we have a short exact sequence

$$0 \to P(i) \to \bigoplus_{a\to i} P(a) \to \tau^{-1}P(i) \to 0$$

which proves (T1) and (T3). Next, since $\operatorname{pd} T \leqslant 1$, we have $\operatorname{Ext}^1_A(T, T) \simeq D\operatorname{Hom}_A(T, \tau T) \simeq D\operatorname{Hom}_A(T, P(i)) = 0$, because $P(i)$ is simple. This shows (T2).

Moreover, $\operatorname{ind}\mathscr{F}(T_A) = \{P(i)\}$ and $\operatorname{ind}\mathscr{T}(T_A) = \operatorname{ind}A\backslash\{P(i)\}$ (in particular, $(\mathscr{T}(T_A), \mathscr{F}(T_A))$ is a splitting torsion theory). Indeed, $M\in\mathscr{T}(T_A)$ if and only if $0 = \operatorname{Ext}^1_A(T, M) = D\operatorname{Hom}_A(M, \tau T) = D\operatorname{Hom}_A(M, P(i))$, thus if and only if no indecomposable summand of M is isomorphic to $P(i)$. Since $P(i) = \tau T\in\mathscr{F}(T_A)$, this shows our claim.

(ii) Let A be given by the quiver of Fig. 2 bound by

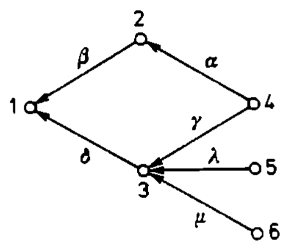

Fig. 2

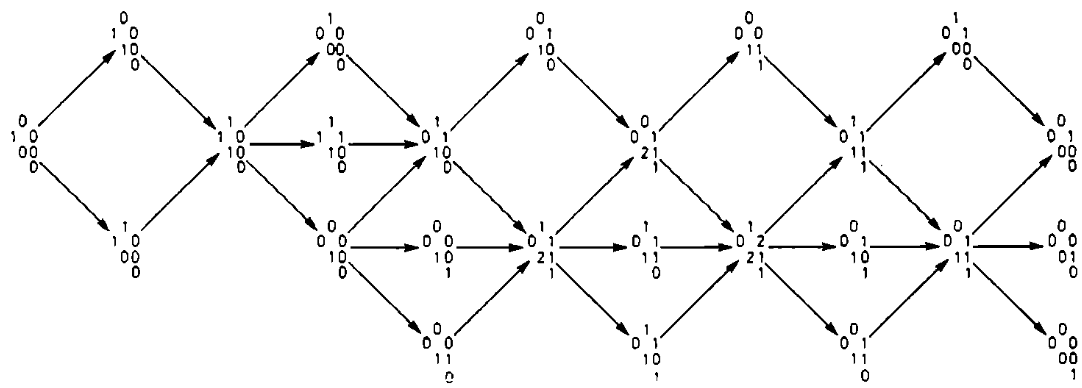

Fig. 3

$\alpha\beta = \gamma\delta$, $\lambda\delta = 0$, $\mu\delta = 0$. Then Γ_A is shown in Fig. 3, where modules are represented by their dimension vectors. We claim that

$$T_A = \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 0 & 0 \\ & & 0 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & 0 \\ & & 0 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 1 & 1 \\ & & 0 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 1 & 0 \\ & & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & 0 \\ & & 0 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 0 & 0 \\ & & 0 \end{smallmatrix}$$

is a tilting module. (Observe that $P(4) = \begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & 0 \\ & & 0 \end{smallmatrix}$ is projective-injective, thus is a summand of any tilting module.)

(T1) To show that $\operatorname{pd} T_A \leqslant 1$, we need only observe that we have short exact sequences

$$0 \to P(3) \to P(5) \oplus P(4) \to \begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 1 & 1 \\ & & 0 \end{smallmatrix} \to 0,$$

$$0 \to P(3) \to P(6) \oplus P(4) \to \begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 1 & 0 \\ & & 1 \end{smallmatrix} \to 0,$$

$$0 \to P(2) \to P(4) \to \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & 0 \\ & & 0 \end{smallmatrix} \to 0,$$

$$0 \to P(3) \to P(4) \to \begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 0 & 0 \\ & & 0 \end{smallmatrix} \to 0.$$

(T2) To prove that $\operatorname{Ext}_A^1(T, T) = D\operatorname{Hom}_A(T, \tau T) = 0$, we must show that $\operatorname{Hom}_A\left(P(1) \oplus P(4) \oplus \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & 0 \\ & & 0 \end{smallmatrix}, \tau T \right) = 0$: indeed, the only predecessors of τT which are in $\operatorname{add}(T)$ are $P(1)$, $P(4)$ and $\begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & 0 \\ & & 0 \end{smallmatrix}$. Since $(\tau T)e_1 = (\tau T)e_4 = 0$, we need only compute

$$\operatorname{Hom}_A\left(\begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & 0 \\ & & 0 \end{smallmatrix}, \tau T \right) = \operatorname{Hom}_A\left(\begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & 0 \\ & & 0 \end{smallmatrix}, \begin{smallmatrix} & 0 & \\ 0 & & 0 \\ & 1 & 1 \\ & & 1 \end{smallmatrix} \right) = 0.$$

(T3) follows from the last two short exact sequences of the proof of (T1) and the exact sequences

$$0 \to P(6) \to \begin{smallmatrix} & & 1 & \\ 0 & & & 1 \\ & 1 & & 0 \\ & & & 1 \end{smallmatrix} \to \begin{smallmatrix} & & 1 & \\ 0 & & & 1 \\ & 0 & & 0 \\ & & & 0 \end{smallmatrix} \to 0,$$

$$0 \to P(5) \to \begin{smallmatrix} & & 1 & \\ 0 & & & 1 \\ & 1 & & 1 \\ & & & 0 \end{smallmatrix} \to \begin{smallmatrix} & & 1 & \\ 0 & & & 1 \\ & 0 & & 0 \\ & & & 0 \end{smallmatrix} \to 0.$$

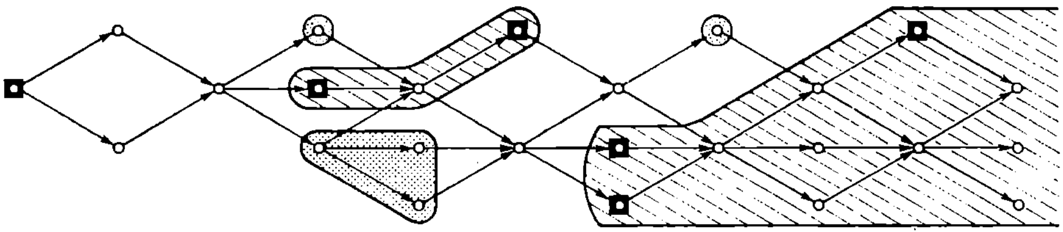

Fig. 4

In Fig. 4, the shaded regions indicate the subcategory $\mathscr{T}(T_A)$ and the dotted ones the subcategory $\mathscr{F}(T_A)$; the summands of T are denoted by small squares. A module M is in $\mathscr{T}(T_A)$ if and only if $\mathrm{Hom}_A(M, \tau T) = 0$: for instance, $\begin{smallmatrix} & & 1 & \\ 0 & & & 1 \\ & 1 & & 0 \\ & & & 0 \end{smallmatrix} \in \mathscr{T}(T_A)$ since

$$\mathrm{Hom}_A\left(\begin{smallmatrix} & & 1 & \\ 0 & & & 1 \\ & 1 & & 0 \\ & & & 0 \end{smallmatrix}, \tau T\right) = \mathrm{Hom}_A\left(\begin{smallmatrix} & & 1 & \\ 0 & & & 1 \\ & 1 & & 0 \\ & & & 0 \end{smallmatrix}, \begin{smallmatrix} & & 0 & \\ 0 & & & 0 \\ & 1 & & 1 \\ & & & 1 \end{smallmatrix}\right) = 0.$$

Similarly, N belongs to $\mathscr{F}(T_A)$ if and only if $\mathrm{Hom}_A(T, N) = 0$. The endomorphism algebra $B = \mathrm{End}\, T_A$ is given by the quiver of Fig. 5 bound by $\alpha\beta = \gamma\delta = \lambda\mu$, $\beta\nu = 0$, $\delta\nu = 0$, $\mu\nu = 0$.

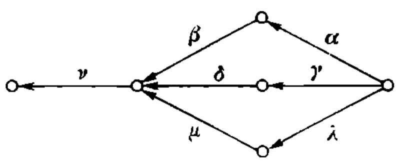

Fig. 5

1.7. The following lemma, due to Bongartz [21], justifies the name of partial tilting modules.

LEMMA. *Let T_A be a partial tilting module. Then there exists a module X_A such that $T \oplus X$ is a tilting module.*

Proof. Let $e_1, \ldots, e_d$ be a basis of the k-vector space $\mathrm{Ext}_A^1(T, A)$, and consider the exact sequence

$$\text{(e)} \qquad 0 \to A \to X \to T^{(d)} \to 0$$

defined as the amalgamated sum ("pushout") along the codiagonal map $A^{(d)} \to A$ of the exact sequence $\bigoplus_{i=1}^{d} e_i$. Since $\mathrm{pd}\, T_A \leqslant 1$, it follows from [25], 8.1, Corollaire 2, that $\mathrm{pd}\, X_A \leqslant 1$. Applying the functor $\mathrm{Hom}_A(T, -)$ to (e) gives

$$\ldots \to \mathrm{Hom}_A(T, T^{(d)}) \to \mathrm{Ext}_A^1(T, A) \to \mathrm{Ext}_A^1(T, X) \to 0.$$

By construction, the first morphism is surjective. Hence $\mathrm{Ext}_A^1(T, X) = 0$. Moreover, applying respectively $\mathrm{Hom}_A(-, T)$ and $\mathrm{Hom}_A(-, X)$ to (e) yields

$$0 = \mathrm{Ext}_A^1(T^{(d)}, T) \to \mathrm{Ext}_A^1(X, T) \to \mathrm{Ext}_A^1(A, T) = 0,$$

$$0 = \mathrm{Ext}_A^1(T^{(d)}, X) \to \mathrm{Ext}_A^1(X, X) \to \mathrm{Ext}_A^1(A, X) = 0.$$

It follows that $\mathrm{Ext}_A^1(X, T) = 0$, $\mathrm{Ext}_A^1(X, X) = 0$ and consequently we have $\mathrm{Ext}_A^1(T \oplus X, T \oplus X) = 0$. Since (e) is the sequence of (T3), $T \oplus X$ is indeed a tilting module.

1.8. In the remainder of this section, we shall assume that T_A is a tilting module.

LEMMA. *For every $M \in \mathscr{T}(T_A)$, there exists a short exact sequence*

$$0 \to K \to T_0 \xrightarrow{f_0} M \to 0$$

with $T_0 \in \mathrm{add}(T)$ and $K \in \mathscr{T}(T_A)$.

Proof. Let $f_{01}, \ldots, f_{0d}$ be a basis of the k-vector space $\mathrm{Hom}_A(T, M)$ and consider the short exact sequence

$$0 \to K \to T_0 \xrightarrow{f_0} M \to 0$$

where $T_0 = T^{(d)}, f_0 = [f_{01}, \ldots, f_{0d}]\colon T_0 \to M$ and $K = \mathrm{Ker} f_0$. Applying the functor $\mathrm{Hom}_A(T, -)$ yields

$$\ldots \to \mathrm{Hom}_A(T, T_0) \xrightarrow{\mathrm{Hom}_A(T, f_0)} \mathrm{Hom}_A(T, M) \to \mathrm{Ext}_A^1(T, K) \to 0.$$

By construction, $\mathrm{Hom}_A(T, f_0)$ is surjective. Hence $\mathrm{Ext}_A^1(T, K) = 0$ and so $K \in \mathscr{T}(T_A)$. This completes the proof.

COROLLARY. *Let X_A be an* Ext-*projective module in $\mathscr{T}(T_A)$. Then* $X \in \mathrm{add}(T)$. *Thus, if T_A is minimal* (*in the sense of* 1.2), *then it is the direct sum of all non-isomorphic* Ext-*projective indecomposable modules in $\mathscr{T}(T_A)$.*

Proof. By the lemma, there exists a short exact sequence $0 \to K \to T_0 \to X \to 0$ with $T_0 \in \mathrm{add}(T)$ and $K \in \mathscr{T}(T)$. Since X is Ext-projective in $\mathscr{T}(T)$, it splits and so $X \in \mathrm{add}(T)$. The second statement follows from the fact that T_A itself is Ext-projective in $\mathscr{T}(T_A)$, and from the minimality of T_A.

1.9. Lemma. *A module M is torsion if and only if the evaluation map ε_M: $\mathrm{Hom}_A(T, M)\otimes_B T\to M$ is bijective.*

Proof. We only need to show the necessity, so let $M\in\mathscr{T}(T_A)$. By 1.8, there exists a short exact sequence $0\to K_0\to T_0\to M\to 0$ with $T_0\in\mathrm{add}(T)$ and $K_0\in\mathscr{T}(T)$. Applying 1.8 to K_0, we find another short exact sequence $0\to K_1\to T_1\to K_0\to 0$ with $T_1\in\mathrm{add}(T)$ and $K_1\in\mathscr{T}(T)$. Since K_0 and K_1 are torsion, we have short exact sequences

$$0\to \mathrm{Hom}_A(T, K_0)\to \mathrm{Hom}_A(T, T_0)\to \mathrm{Hom}_A(T, M)\to 0,$$

$$0\to \mathrm{Hom}_A(T, K_1)\to \mathrm{Hom}_A(T, T_1)\to \mathrm{Hom}_A(T, K_0)\to 0.$$

Thus the sequence $\mathrm{Hom}_A(T, T_1)\to \mathrm{Hom}_A(T, T_0)\to \mathrm{Hom}_A(T, M)\to 0$ is exact. Applying the right exact functor $-\otimes_B T$ and comparing with the original sequence, we obtain an exact commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}_A(T, T_1)\otimes_B T & \to & \mathrm{Hom}_A(T, T_0)\otimes_B T & \to & \mathrm{Hom}_A(T, M)\otimes_B T & \to & 0\\
\downarrow{\scriptstyle \varepsilon_{T_1}} & & \downarrow{\scriptstyle \varepsilon_{T_0}} & & \downarrow{\scriptstyle \varepsilon_M} & & \\
T_1 & \to & T_0 & \to & M & \to & 0
\end{array}$$

Since ε_T is clearly bijective, so are ε_{T_0} and ε_{T_1}. Therefore ε_M is bijective.

Corollary. *The torsion submodule of a module M in the torsion theory $(\mathscr{T}(T_A), \mathscr{F}(T_A))$ is given by $tM = \mathrm{Hom}_A(T, M)\otimes_B T$.*

Proof. Since $tM\in\mathscr{T}(T_A)$, we have

$$tM \xrightarrow{\sim} \mathrm{Hom}_A(T, tM)\otimes_B T\xrightarrow{\sim} \mathrm{Hom}_A(T, M)\otimes_B T.$$

2. The main theorems

In this section, we return to our basic problem: given an algebra A, and a tilting module T_A with $B = \mathrm{End}\, T_A$, we wish to compare $\mathrm{mod}\, A$ and $\mathrm{mod}\, B$.

2.1. Theorem (Brenner–Butler). *Let A be an algebra, let T_A be a tilting module and $B = \mathrm{End}\, T_A$. Then:*

(a) *${}_BT$ is a tilting module and $A \xrightarrow{\sim} \mathrm{End}({}_BT)$, canonically.*

(b) *The functors $\mathrm{Hom}_A(T, -)$ and $-\otimes_B T$ induce mutually inverse equivalences between the full subcategories*

$$\mathscr{T}(T_A) = \{M_A \mid \mathrm{Ext}^1_A(T, M) = 0\} \quad \textit{and} \quad \mathscr{Y}(T_A) = \{N_B \mid \mathrm{Tor}^B_1(N, T) = 0\},$$

while the functors $\mathrm{Ext}^1_A(T, -)$ and $\mathrm{Tor}^B_1(-, T)$ induce mutually inverse equivalences between the full subcategories

$$\mathscr{F}(T_A) = \{M_A \mid \mathrm{Hom}_A(T, M) = 0\} \quad \textit{and} \quad \mathscr{X}(T_A) = \{N_B \mid N\otimes_B T = 0\}.$$

(c) *We have*

$$\mathrm{Tor}_1^B(-, T)\mathrm{Hom}_A(T, -) = 0 = (- \otimes_B T)\mathrm{Ext}_A^1(T, -),$$
$$\mathrm{Hom}_A(T, -)\mathrm{Tor}_1^B(-, T) = 0 = \mathrm{Ext}_A^1(T, -)(- \otimes_B T).$$

Proof. (i) For any module M_A, we have $\mathrm{Tor}_1^B(\mathrm{Hom}_A(T, M), T) = 0$ (thus, $\mathrm{Hom}_A(T, M) \in \mathcal{Y}(T_A)$).

Indeed, since $\mathrm{Hom}_A(T, M) \simeq \mathrm{Hom}_A(T, tM)$, we may assume that $M \in \mathcal{T}(T)$. By 1.8, there exists a short exact sequence $0 \to K \to T_0 \to M \to 0$ with $T_0 \in \mathrm{add}(T)$ and $K \in \mathcal{T}(T)$. Applying the functor $\mathrm{Hom}_A(T, -)$ yields a short exact sequence

$$0 \to \mathrm{Hom}_A(T, K) \to \mathrm{Hom}_A(T, T_0) \to \mathrm{Hom}_A(T, M) \to 0.$$

Applying now the right exact functor $- \otimes_B T$ and comparing with the original sequence yields an exact commutative diagram

$$\begin{array}{ccccccccc} 0 \to \mathrm{Tor}_1^B(\mathrm{Hom}_A(T, M), T) & \to & \mathrm{Hom}_A(T, K)\otimes_B T & \to & \mathrm{Hom}_A(T, T_0)\otimes_B T & \to & \mathrm{Hom}_A(T, M)\otimes_B T & \to & 0 \\ & & \downarrow{\scriptstyle \varepsilon_K} & & \downarrow{\scriptstyle \varepsilon_{T_0}} & & \downarrow{\scriptstyle \varepsilon_M} & & \\ 0 & \to & K & \to & T_0 & \to & M & \to & 0 \end{array}$$

since the projectivity of $\mathrm{Hom}_A(T, T_0)$ implies that $\mathrm{Tor}_1^B(\mathrm{Hom}_A(T, T_0), T) = 0$. By 1.9, the vertical maps are isomorphisms. Hence the result.

(ii) $\mathrm{pd}\,_B T \leqslant 1$.

Let $0 \to A_A \to T'_A \to T''_A \to 0$ be the short exact sequence of (T3). Applying $\mathrm{Hom}_A(-, {}_BT_A)$ yields

$$0 \to \mathrm{Hom}_A(T'', {}_BT_A) \to \mathrm{Hom}_A(T', {}_BT_A) \to \mathrm{Hom}_A(A, {}_BT_A) \simeq {}_BT \to 0.$$

(iii) To each A-module M corresponds the canonical sequence

$$0 \to \mathrm{Hom}_A(T, M)\otimes_B T \xrightarrow{\varepsilon_M} M \to \mathrm{Tor}_1^B(\mathrm{Ext}_A^1(T, M), T) \to 0.$$

Indeed, let $0 \to M \to I_0 \to I_1 \to \ldots$ be an injective resolution of M_A, and let $0 \to Q_1 \to Q_0 \to {}_BT \to 0$ be a projective resolution of ${}_BT$. It follows from (i) that we have a short exact sequence of complexes

$$0 \to \mathrm{Hom}_A(T, I_\cdot)\otimes_B Q_1 \to \mathrm{Hom}_A(T, I_\cdot)\otimes_B Q_0 \to \mathrm{Hom}_A(T, I_\cdot)\otimes_B T \to 0.$$

Since injectives are torsion, the last complex is identified to $I_\cdot$ via the evaluation map. Since $I_\cdot$ is an exact complex and $- \otimes_B Q_1$, $- \otimes_B Q_0$ are exact functors, we obtain a long exact sequence

$$0 \to \mathrm{Hom}_A(T, M) \otimes_B Q_1 \xrightarrow{u} \mathrm{Hom}_A(T, M)\otimes_B Q_0 \to M$$
$$\to \mathrm{Ext}_A^1(T, M)\otimes_B Q_1 \xrightarrow{v} \mathrm{Ext}_A^1(T, M)\otimes_B Q_0 \to 0.$$

Hence we have a short exact sequence

$$0 \to \mathrm{Coker}\,u \simeq \mathrm{Hom}_A(T, M)\otimes_B T \to M \to \mathrm{Ker}\,v \simeq \mathrm{Tor}_1^B(\mathrm{Ext}_A^1(T, M), {}_BT) \to 0.$$

By 1.9, this is indeed the canonical sequence of M_A in the torsion theory $(\mathscr{T}(T), \mathscr{F}(T))$ (thus, $M/tM \simeq \mathrm{Tor}_1^B(\mathrm{Ext}_A^1(T, M), T)$).

(iv) $\mathrm{Ext}_A^1(T, M)\otimes_B T = 0$.

This follows from the surjectivity of the morphism v in (iii).

(v) To each B-module N corresponds the canonical sequence

$$0 \to \mathrm{Ext}_A^1(T, \mathrm{Tor}_1^B(N, T)) \to N_B \xrightarrow{\delta_N} \mathrm{Hom}_A(T, N\otimes_B T) \to 0$$

where δ_N: $x \to (t \to x\otimes t)$.

Indeed, let $\ldots \to Q_1 \to Q_0 \to N_B \to 0$ and $0 \to P_1 \to P_0 \to T_A \to 0$ be projective resolutions in mod$B$ and mod$A$ respectively. Since $Q_i \otimes_B T\in \mathscr{T}(T)$ for each i, we have an exact sequence of complexes

$$0 \to \mathrm{Hom}_A(T, Q_\bullet\otimes_B T) \to \mathrm{Hom}_A(P_0, Q_\bullet\otimes_B T) \to \mathrm{Hom}_A(P_1, Q_\bullet\otimes_B T) \to 0$$

where the first complex is identified to $Q_\bullet$ via δ. Hence the long exact sequence

$$0 \to \mathrm{Hom}_A(P_0, \mathrm{Tor}_1^B(N, T)) \xrightarrow{u'} \mathrm{Hom}_A(P_1, \mathrm{Tor}_1^B(N, T)) \to N$$
$$\to \mathrm{Hom}_A(P_0, N\otimes_B T) \xrightarrow{v'} \mathrm{Hom}_A(P_1, N\otimes_B T) \to 0$$

yields the result. (It will follow from the corollary below that $(\mathscr{X}(T), \mathscr{Y}(T))$ is a torsion theory in mod$B$, so that the given short exact sequence is the canonical sequence of N_B in $(\mathscr{X}(T_A), \mathscr{Y}(T_A))$.)

(vi) $\mathrm{Hom}_A(T, \mathrm{Tor}_1^B(N, T)) = 0$.

This follows from the injectivity of the morphism u' in (v).

(vii) *Proof of* (c). Since $N\otimes_B T\in \mathscr{T}(T)$, we have $\mathrm{Ext}_A^1(T, N\otimes_B T) = 0$. The other statements are (i), (iv), (vi).

(viii) *Proof of* (b). Applying (iii) to $M\in \mathscr{T}(T)$ yields $M \simeq \mathrm{Hom}_A(T, M)\otimes_B T$. Similarly, if $N\in \mathscr{Y}(T)$, then $\mathrm{Tor}_1^B(N, T) = 0$ gives $N \simeq \mathrm{Hom}_A(T, N\otimes_B T)$. We prove similarly that $\mathrm{Ext}_A^1(T, \quad)$ and $\mathrm{Tor}_1^B(-, T)$ are mutually inverse on $\mathscr{F}(T)$ and $\mathscr{X}(T)$.

(ix) *Proof of* (a). We already know that $\mathrm{pd}_B T\leqslant 1$. A projective resolution $0 \to P_1 \to P_0 \to T_A \to 0$ gives a short exact sequence

$$0 \to \mathrm{Hom}_A(T, T) \to \mathrm{Hom}_A(P_0, T) \to \mathrm{Hom}_A(P_1, T) \to 0,$$

hence (T3). In order to show (T2), we first observe that

$$D(_BT) \simeq \mathrm{Hom}_k(_BT_A\otimes_A A, k) \simeq \mathrm{Hom}_A(T, DA)\in \mathscr{Y}(T_A),$$

so that, applying (b),

$$\operatorname{Ext}^1_{B^{\mathrm{op}}}({}_BT, {}_BT) \simeq \operatorname{Ext}^1_B(D({}_BT), D({}_BT)) \simeq \operatorname{Ext}^1_B(\operatorname{Hom}_A(T, DA), \operatorname{Hom}_A(T, DA))$$
$$\simeq \operatorname{Ext}^1_A(DA, DA) = 0.$$

Thus ${}_BT$ is a tilting left B-module.

It remains to show that the canonical algebra homomorphism $f\colon A \to \operatorname{End}({}_BT)$ given by $a \to (t \to ta)$ is an isomorphism. By (T3), there exists an injection $A \to T'$ with $T' \in \operatorname{add}(T)$. Therefore f is injective. On the other hand, we have vector space isomorphisms $A \simeq \operatorname{End} DA \simeq \operatorname{End} \operatorname{Hom}_A(T, DA) \simeq \operatorname{End} D({}_BT)$. Hence $\dim_k A = \dim_k \operatorname{End}({}_BT)$, and we have finished.

COROLLARY. *We have* $D\mathscr{X}(T_A) = \mathscr{F}({}_BT)$ *and* $D\mathscr{Y}(T_A) = \mathscr{T}({}_BT)$. *In particular,* $(\mathscr{X}(T_A), \mathscr{Y}(T_A))$ *is a torsion theory in* $\operatorname{mod} B$.

Proof. This follows from the isomorphisms $\operatorname{Tor}^B_i(D({}_BN), T) \simeq D\operatorname{Ext}^i_B(T, N)$ valid for all $i \geqslant 0$ (see [29], 5.1).

DEFINITION. A *tilting triple* (B, T, A) is defined to consist of two finite dimensional algebras A and B and a B-A-bimodule ${}_BT_A$ such that T_A is a tilting module and $B \simeq \operatorname{End} T_A$.

EXAMPLE. Let A be given by the commutative quiver of Fig. 6. Then Γ_A is

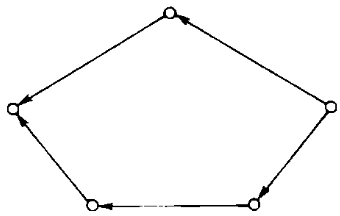

Fig. 6

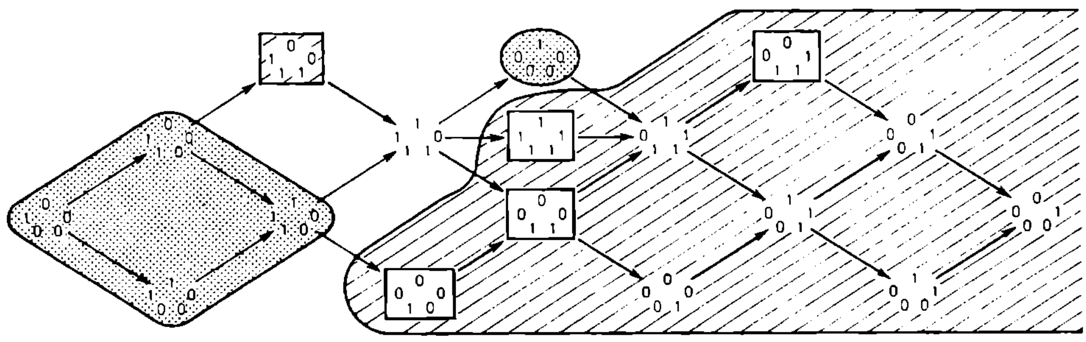

Fig. 7

given by Fig. 7, where the indecomposables are represented by their dimension vectors. It is easily checked that

$$T_A = \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ 1 & & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 1 & & 1 \\ 1 & & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 0 & & 0 \\ 1 & & 0 \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 0 & & 0 \\ 1 & & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ 1 & & 1 \end{smallmatrix}$$

is a tilting module. We have indicated by shading the full subcategory $\mathscr{T}(T)$, and by dotting the full subcategory $\mathscr{F}(T)$. Then $B = \operatorname{End} T_A$ is given by the

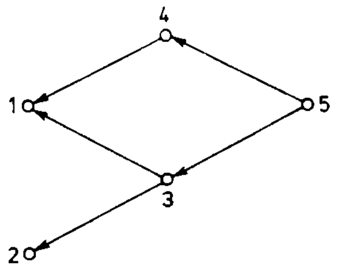

Fig. 8

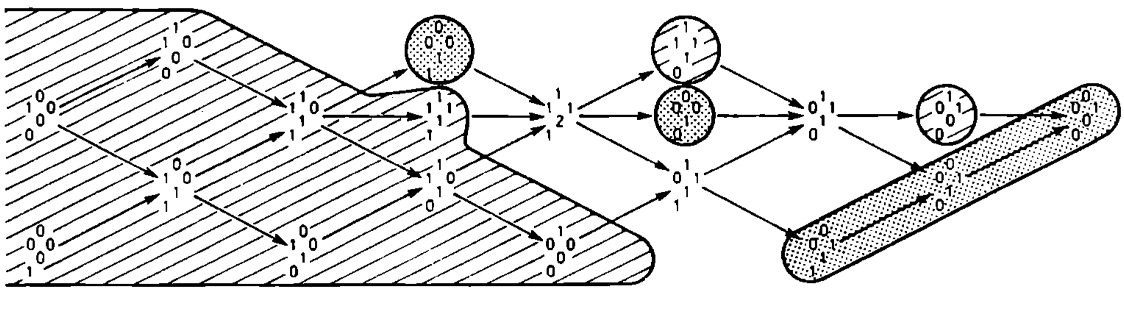

Fig. 9

commutative quiver of Fig. 8, and Γ_B by Fig. 9, where $\mathscr{X}(T)$ is indicated by dotting and $\mathscr{Y}(T)$ by shading. For instance, it is readily verified that

$$\operatorname{Hom}_A\left(T, \begin{smallmatrix} & & 0 & & \\ 0 & & & & 1 \\ & 0 & & 1 & \end{smallmatrix}\right) = \begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \\ 0 & & \end{smallmatrix}, \qquad \operatorname{Hom}_A\left(T, \begin{smallmatrix} & & 0 & & \\ 0 & & & & 1 \\ & 0 & & 0 & \end{smallmatrix}\right) = \begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 0 & \\ 0 & & \end{smallmatrix},$$

and similarly, that

$$\operatorname{Ext}^1_A\left(T, \begin{smallmatrix} & & 1 & & \\ 0 & & & & 0 \\ & 0 & & 0 & \end{smallmatrix}\right) = D\operatorname{Hom}_A\left(\begin{smallmatrix} & & 1 & & \\ 0 & & & & 0 \\ & 0 & & 0 & \end{smallmatrix}, \tau T\right) = \begin{smallmatrix} & 1 & \\ 0 & & 1 \\ & 0 & \\ 0 & & \end{smallmatrix},$$

$$\operatorname{Ext}^1_A\left(T, \begin{smallmatrix} & & 1 & & \\ 1 & & & & 0 \\ & 1 & & 0 & \end{smallmatrix}\right) = D\operatorname{Hom}_A\left(\begin{smallmatrix} & & 1 & & \\ 1 & & & & 0 \\ & 1 & & 0 & \end{smallmatrix}, \tau T\right) = \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & \\ 0 & & \end{smallmatrix}.$$

Observe that $\mathscr{Y}(T)$ always contains the projective B-modules.

2.2. THEOREM. *Let* (B, T, A) *be a tilting triple. Then*

$$|\operatorname{gl.dim} B - \operatorname{gl.dim} A| \leqslant 1.$$

We shall start by proving the following lemma.

LEMMA. *If* $M \in \mathscr{T}(T_A)$, *then* $\operatorname{pd}\operatorname{Hom}_A(T, M) \leqslant \operatorname{pd} M$.

Proof. By induction on $n = \operatorname{pd} M$. If $n = 0$, then M is projective; hence, since $M \in \mathscr{T}(T)$, we must have $M \in \operatorname{add}(T)$. Therefore, $\operatorname{Hom}_A(T, M)$ is projective and we are done.

Next assume $n \geqslant 1$. By 1.8, there exists a short exact sequence

$$(*) \qquad 0 \to K \to T_0 \to M \to 0$$

with $T_0 \in \operatorname{add}(T)$ and $K \in \mathscr{T}(T)$. Therefore we have a short exact sequence

$$0 \to \operatorname{Hom}_A(T, K) \to \operatorname{Hom}_A(T, T_0) \to \operatorname{Hom}_A(T, M) \to 0.$$

We claim that, if $n = 1$, then $K \in \operatorname{add}(T)$. Indeed, applying the functor $\operatorname{Hom}_A(-, N)$ with $N \in \mathscr{T}(T)$ to $(*)$ yields

$$0 = \operatorname{Ext}_A^1(T_0, N) \to \operatorname{Ext}_A^1(K, N) \to \operatorname{Ext}_A^2(M, N) = 0.$$

Therefore $\operatorname{Ext}_A^1(K, -)|_{\mathscr{T}(T)} = 0$, that is, K is Ext-projective in $\mathscr{T}(T)$ and so $K \in \operatorname{add}(T)$. In particular, if $n = 1$, then $\operatorname{Hom}_A(T, K)$ is projective so that $\operatorname{pd} \operatorname{Hom}_A(T, M) \leqslant 1$. Finally, if $n \geqslant 2$, it follows from the sequence $(*)$ and [25], 8.1, Corollaire 2, that $\operatorname{pd} K \leqslant n-1$. Hence it follows from the induction hypothesis that $\operatorname{pd} \operatorname{Hom}_A(T, K) \leqslant n-1$ and so

$$\operatorname{pd} \operatorname{Hom}_A(T, M) \leqslant 1 + \operatorname{pd} \operatorname{Hom}_A(T, K) \leqslant 1 + (n-1) = n.$$

Proof of the theorem. Let N_B be an arbitrary B-module. Then there exists a short exact sequence $0 \to Y_B \to P_B \to N_B \to 0$ with P_B projective. Since $P_B \in \mathscr{Y}(T)$ which is a torsion-free class, $Y \in \mathscr{Y}(T)$ as well. Hence there exists a module $M_A \in \mathscr{T}(T)$ such that $Y_B = \operatorname{Hom}_A(T, M)$ and, by the lemma, $\operatorname{pd} Y_B \leqslant \operatorname{pd} M_A$. Thus

$$\operatorname{pd} N_B \leqslant 1 + \operatorname{pd} Y_B \leqslant 1 + \operatorname{pd} M_A \leqslant 1 + \operatorname{gl.dim} A.$$

We have thus that $\operatorname{gl.dim} B \leqslant 1 + \operatorname{gl.dim} A$. Since ${}_B T$ is a tilting module as well, we also have $\operatorname{gl.dim} A \leqslant 1 + \operatorname{gl.dim} B$ and we are done.

Remark. We have the following other relations between the homological dimensions in $\operatorname{mod} A$ and $\operatorname{mod} B$.

(i) If $M_A \in \mathscr{F}(T)$, then $\operatorname{pd} \operatorname{Ext}_A^1(T, M) \leqslant 1 + \max(1, \operatorname{pd} M)$.
(ii) If $M_A \in \mathscr{T}(T)$, then $\operatorname{id} \operatorname{Hom}_A(T, M) \leqslant 1 + \operatorname{id} M$.
(iii) If $M_A \in \mathscr{F}(T)$, then $\operatorname{id} \operatorname{Ext}_A^1(T, M) \leqslant \operatorname{id} M$.

For a proof of (i), we refer to [65], 4.1, (6), and for a proof of (ii) and (iii) to [21], 1.7.

EXAMPLES. In the example in 1.6 (ii), $\operatorname{gl.dim} A = 2$ while $\operatorname{gl.dim} B = 3$. In the example in 2.1, $\operatorname{gl.dim} A = 2 = \operatorname{gl.dim} B$.

2.3. LEMMA (The Connecting Lemma). *Let (B, T, A) be a tilting triple and let P (respectively, I) be the projective cover (respectively, the injective envelope)* *of the simple A-module S. Then*

$$\tau^{-1}\operatorname{Hom}_A(T, I) \simeq \operatorname{Ext}_A^1(T, P).$$

In particular, $P \in \operatorname{add}(T)$ *if and only if* $\operatorname{Hom}_A(T, I)$ *is an injective B-module.*

Proof. Since T_A is a tilting module, there exists a short exact sequence

$$(*) \qquad 0 \to P_A \to T'_A \xrightarrow{f} T''_A \to 0$$

with $T', T'' \in \operatorname{add}(T)$ (by (T3)). Applying $\operatorname{Hom}_A(-, T)$ yields a short exact sequence

$$0 \to \operatorname{Hom}_A(T'', T) \xrightarrow{\operatorname{Hom}_A(f, T)} \operatorname{Hom}_A(T', T) \to \operatorname{Hom}_A(P, T) \to 0$$

which is a projective resolution for $\operatorname{Hom}_A(P, {}_BT)$. Now the transpose of $\operatorname{Hom}_A(f, T)$ is clearly $\operatorname{Hom}_A(T, f)$. On the other hand, applying $\operatorname{Hom}_A(T, -)$ to $(*)$ yields

$$0 \to \operatorname{Hom}_A(T, P) \to \operatorname{Hom}_A(T, T') \xrightarrow{\operatorname{Hom}_A(T, f)} \operatorname{Hom}_A(T, T'') \to \operatorname{Ext}_A^1(T, P) \to 0.$$

Hence $\operatorname{Ext}_A^1(T, P) = \operatorname{Tr}\operatorname{Hom}_A(P, T)$, by definition of the transpose. But $\operatorname{Hom}_A(P, T) \simeq D\operatorname{Hom}_A(T, I)$. Therefore we have $\operatorname{Ext}_A^1(T, P) \simeq \operatorname{Tr} D \operatorname{Hom}_A(T, I) = \tau^{-1}\operatorname{Hom}_A(T, I)$.

Remarks. 1. It follows from the Connecting Lemma that any $J_B \in \mathscr{Y}(T)$ which is an indecomposable injective B-module is of the form $J_B \simeq \operatorname{Hom}_A(T, I(i))$ with $P(i) \in \operatorname{add}(T)$. Indeed, since $J \in \mathscr{Y}(T)$, we have $J \simeq \operatorname{Hom}_A(T, M)$ with $M \in \mathscr{T}(T)$. Let $f: M \to E$ be an injective envelope of M. Then $\operatorname{Hom}_A(T, f): J_B \to \operatorname{Hom}_A(T, E)$ is a monomorphism, hence splits, because J_B is injective. Therefore $f \simeq \operatorname{Hom}_A(T, f) \otimes_B T$ splits again; that is, M is injective, say $M = I(i)$. But then $P(i) \in \operatorname{add}(T)$ for, otherwise, the Connecting Lemma would contradict the injectivity of $\operatorname{Hom}_A(T, I(i))$.

2. Let now P and I be both in $\operatorname{add}(T)$. Then, clearly, $\operatorname{Hom}_A(T, I)$ is projective-injective. Conversely, any indecomposable projective-injective B-module is of this form. Indeed, since such a module is projective, it is of the form $\operatorname{Hom}_A(T, T(j))$ for some $T(j) \in \operatorname{add}(T)$. In particular, it lies in $\mathscr{Y}(T)$. Since it is injective, the previous remark implies that it is of the form $\operatorname{Hom}_A(T, I(i))$, with $P(i) \in \operatorname{add}(T)$. Therefore $I(i) \simeq T(j) \in \operatorname{add}(T)$ as well.

COROLLARY. *Let P, I be as in the Lemma, with* $P \notin \operatorname{add}(T)$ *and consider the Auslander–Reiten sequence in* $\operatorname{mod} B$

$$0 \to \operatorname{Hom}_A(T, I) \to E_B \to \operatorname{Ext}_A^1(T, P) \to 0.$$

Then the canonical sequence of E_B *in the torsion theory* $(\mathscr{X}(T), \mathscr{Y}(T))$ *is*

$$0 \to \operatorname{Ext}_A^1(T, \operatorname{rad} P) \to E_B \to \operatorname{Hom}_A(T, I/S) \to 0.$$

Proof. We shall consider two cases, according as S is torsion or torsion-free.

(i) If $S \in \mathscr{T}(T)$, then $\operatorname{Ext}_A^1(T, S) = 0$, so the short exact sequence $0 \to S \to I \to I/S \to 0$ induces a short exact sequence

$$0 \to \operatorname{Hom}_A(T, S) \to \operatorname{Hom}_A(T, I) \to \operatorname{Hom}_A(T, I/S) \to 0.$$

On the other hand, since $P \notin \operatorname{add}(T)$, we have $P \notin \mathscr{T}(T)$. It follows that every morphism from T_A to P_A has its image contained in $\operatorname{rad} P$. That is to say, $\operatorname{Hom}_A(T, P) \simeq \operatorname{Hom}_A(T, \operatorname{rad} P)$. Therefore the short exact sequence $0 \to \operatorname{rad} P \to P \to S \to 0$ induces a short exact sequence

$$0 \to \operatorname{Hom}_A(T, S) \to \operatorname{Ext}_A^1(T, \operatorname{rad} P) \to \operatorname{Ext}_A^1(T, P) \to 0.$$

Since the epimorphism $\operatorname{Ext}_A^1(T, \operatorname{rad} P) \to \operatorname{Ext}_A^1(T, P)$ does not split, we have an exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \operatorname{Hom}_A(T, S) & \to & \operatorname{Ext}_A^1(T, \operatorname{rad} P) & \to & \operatorname{Ext}_A^1(T, P) & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \operatorname{Hom}_A(T, I) & \longrightarrow & E & \longrightarrow & \operatorname{Ext}_A^1(T, P) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \operatorname{Hom}_A(T, I/S) & = & \operatorname{Hom}_A(T, I/S) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

and the middle column yields the result.

(ii) If $S \in \mathscr{F}(T)$, then $\operatorname{Hom}_A(T, S) = 0$ and so we have short exact sequences

$$0 \to \operatorname{Ext}_A^1(T, \operatorname{rad} P) \to \operatorname{Ext}_A^1(T, P) \to \operatorname{Ext}_A^1(T, S) \to 0,$$

$$0 \to \operatorname{Hom}_A(T, I) \to \operatorname{Hom}_A(T, I/S) \to \operatorname{Ext}_A^1(T, S) \to 0.$$

Since the monomorphism $\operatorname{Hom}_A(T, I) \to \operatorname{Hom}_A(T, I/S)$ does not split, we have an exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \operatorname{Ext}_A^1(T, \operatorname{rad} P) & = & \operatorname{Ext}_A^1(T, \operatorname{rad} P) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{Hom}_A(T, I) & \longrightarrow & E & \longrightarrow & \operatorname{Ext}_A^1(T, P) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \operatorname{Hom}_A(T, I) & \to & \operatorname{Hom}_A(T, I/S) & \longrightarrow & \operatorname{Ext}_A^1(T, S) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

and again the middle column yields the result.

EXAMPLE. Let us consider, in the example of 2.1, the indecomposable projective A-module $P = \begin{smallmatrix} & 0 & \\ 1 & 0 & 0 \\ 1 & & \end{smallmatrix}$. Then $I = \begin{smallmatrix} & 0 & \\ 0 & 1 & 1 \\ 1 & & \end{smallmatrix}$, $I/S = \begin{smallmatrix} & 0 & \\ 0 & 1 & 1 \\ 0 & & \end{smallmatrix}$, while $\operatorname{rad} P = \begin{smallmatrix} & 0 & \\ 1 & 0 & 0 \\ 0 & & \end{smallmatrix}$. The Auslander–Reiten sequence ending with $\operatorname{Ext}_A^1(T, P)$ in $\operatorname{mod} B$ is

$$0 \to \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \\ 1 & & \end{smallmatrix} \to \begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \\ 1 & & \end{smallmatrix} \to \begin{smallmatrix} & 0 & \\ 0 & 1 & 0 \\ 0 & & \end{smallmatrix} \to 0.$$

Clearly, the canonical sequence for the middle term is

$$0 \to \begin{smallmatrix} & 0 & \\ 0 & 1 & 0 \\ 1 & & \end{smallmatrix} \to \begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \\ 1 & & \end{smallmatrix} \to \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \\ 0 & & \end{smallmatrix} \to 0$$

and it is readily verified that

$$\operatorname{Hom}_A(T, I/S) = \begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \\ 0 & & \end{smallmatrix} \quad \text{while} \quad \operatorname{Ext}_A^1(T, \operatorname{rad} P) = \begin{smallmatrix} & 0 & \\ 0 & 1 & 0 \\ 1 & & \end{smallmatrix}.$$

2.4. Proposition. *Let (B, T, A) be a tilting triple. Then*

$$\mathscr{F}(T_A) = \operatorname{Cogen}(\tau T).$$

Proof. Since $\tau T \in \mathscr{F}(T)$, by 1.4, we have $\operatorname{Cogen}(\tau T) \subseteq \mathscr{F}(T)$. Conversely, let $M_A \in \mathscr{F}(T)$. There exists a B-module $N \in \mathscr{X}(T)$ such that

$$M \simeq \operatorname{Tor}_1^B(N, T) \simeq D \operatorname{Ext}_{B^{\mathrm{op}}}^1(T, DN).$$

Let ${}_BP \to {}_B(DN)$ be a projective cover. Since $\operatorname{pd}({}_BT) \leqslant 1$, we have an epimorphism $\operatorname{Ext}_{B^{\mathrm{op}}}^1(T, P) \to \operatorname{Ext}_{B^{\mathrm{op}}}^1(T, DN)$, which induces a monomorphism

$$M \simeq D \operatorname{Ext}_{B^{\mathrm{op}}}^1(T, DN) \to D \operatorname{Ext}_{B^{\mathrm{op}}}^1(T, P).$$

Thus the torsion-free A-module M is cogenerated by modules of the form $D \operatorname{Ext}_{B^{\mathrm{op}}}^1(T, P(i))$, where ${}_BP(i)$ is indecomposable projective. Let ${}_BI(i)$ denote the corresponding indecomposable injective. By the Connecting Lemma,

$$D \operatorname{Ext}_{B^{\mathrm{op}}}^1(T, P(i)) \simeq D \operatorname{Tr} D \operatorname{Hom}_{B^{\mathrm{op}}}(T, I(i)) = \tau D \operatorname{Hom}_{B^{\mathrm{op}}}(T, I(i))$$

$$\simeq \tau \operatorname{Hom}_{B^{\mathrm{op}}}(P(i), T) \simeq \tau T(i)_A$$

where $T(i)_A$ is the indecomposable summand of T_A corresponding to the projective module ${}_BP(i)$. This completes the proof.

2.5. THEOREM. *Let (B, T, A) be a tilting triple. Then the mapping $f\colon K_0(A) \to K_0(B)$ defined by*

$$f(\underline{\dim}\, M) = \underline{\dim}\, \mathrm{Hom}_A(T, M) - \underline{\dim}\, \mathrm{Ext}_A^1(T, M)$$

is an isomorphism of the Grothendieck groups.

Proof. A short exact sequence $0 \to M' \to M \to M'' \to 0$ in mod A induces an exact sequence

$$0 \to \mathrm{Hom}_A(T, M') \to \mathrm{Hom}_A(T, M) \to \mathrm{Hom}_A(T, M'')$$
$$\to \mathrm{Ext}_A^1(T, M') \to \mathrm{Ext}_A^1(T, M) \to \mathrm{Ext}_A^1(T, M'') \to 0.$$

Thus, f defines a homomorphism $K_0(A) \to K_0(B)$. Let S be a simple B-module. Since $(\mathscr{X}(T), \mathscr{Y}(T))$ is a torsion theory, we have $S \in \mathscr{X}(T)$ or $S \in \mathscr{Y}(T)$. In the former case, $S \simeq \mathrm{Hom}_A(T, S \otimes_B T)$ and in the latter, $S \simeq \mathrm{Ext}_A^1(T, \mathrm{Tor}_1^B(S, T))$. Hence $\underline{\dim}\, S$ is in the image of S. Consequently, f is surjective and $\mathrm{rank}\, K_0(A) \geqslant \mathrm{rank}\, K_0(B)$. Since ${}_BT$ is also a tilting module, $\mathrm{rank}\, K_0(B) \geqslant \mathrm{rank}\, K_0(A)$ and thus f is an isomorphism.

2.6. COROLLARY. *Let $T_A = T_1^{(m_1)} \oplus \ldots \oplus T_t^{(m_t)}$ with the T_i indecomposable modules such that $T_i \not\simeq T_j$ whenever $i \neq j$. Then T_A is a tilting module if and only if T_A is a partial tilting module and satisfies*

(T3′) $t = \mathrm{rank}\, K_0(A)$.

Proof. If T_A is a tilting module and $B = \mathrm{End}\, T_A$, then $t = \mathrm{rank}\, K_0(B)$. It follows from 2.5 that $t = \mathrm{rank}\, K_0(A)$. Suppose conversely that T_A is a partial tilting module satisfying (T3′). By 1.7, there exists X_A such that $T \oplus X$ is a tilting module. But then it follows from the necessity part that $T \oplus X$ has the same number of non-isomorphic indecomposable summands as T. Hence $X \in \mathrm{add}(T)$ and T is indeed a tilting module.

2.7. Another consequence of 2.5 is the invariance of the (homological) quadratic form of an algebra of finite global dimension under the tilting process. Recall that the *Euler characteristic* of an algebra A of finite global dimension is defined to be the bilinear form $\langle\ ,\ \rangle_A$ on $K_0(A)$ given by

$$\langle \underline{\dim}\, M, \underline{\dim}\, N \rangle_A = \sum_{s \geqslant 0} (-1)^s \dim_k \mathrm{Ext}_A^s(M, N)$$

for all A-modules M and N (the sum above being finite due to our hypothesis on A). The *(homological) quadratic form* of A is the form q_A on $K_0(A)$ defined by

$$q_A(\underline{\dim}\, M) = \langle \underline{\dim}\, M, \underline{\dim}\, M \rangle_A.$$

PROPOSITION. *Let (B, T, A) be a tilting triple, with A of finite global dimension, and let $f\colon K_0(A) \to K_0(B)$ be the homomorphism of 2.5. Then we have*

$$\langle \underline{\dim}\, M, \underline{\dim}\, N \rangle_A = \langle f(\underline{\dim}\, M), f(\underline{\dim}\, N) \rangle_B.$$

Proof. Let $T(i)$, $1 \leqslant i \leqslant n$, denote the pairwise non-isomorphic indecomposable summands of T. We claim that the vectors $\underline{\dim}\, T(i)$ form a basis of $K_0(A)$. Indeed, since A is of finite global dimension, the vectors $\underline{\dim}\, P(a)$, $1 \leqslant a \leqslant n$, form a basis of $K_0(A)$. Now, for each a, we have a short exact sequence

$$0 \to P(a) \to T' \to T'' \to 0$$

with $T', T'' \in \mathrm{add}(T)$. Thus $K_0(A)$ is generated by the vectors $\underline{\dim}\, T(i)$. On the other hand, for each i,

$$f(\underline{\dim}\, T(i)) = \underline{\dim}\, \mathrm{Hom}_A(T, T(i)).$$

Since the modules $\mathrm{Hom}_A(T, T(i))$ are just the indecomposable projective B-modules, and B has finite global dimension, the vectors $f(\underline{\dim}\, T(i))$ form a basis of $K_0(B)$, and this implies our claim.

Also, the projectivity of the B-modules $\mathrm{Hom}_A(T, T(i))$ implies that, for any $1 \leqslant i, j \leqslant n$,

$$\begin{aligned} \langle f(\underline{\dim}\, T(i)), f(\underline{\dim}\, T(j))\rangle_B &= \langle \underline{\dim}\, \mathrm{Hom}_A(T, T(i)), \underline{\dim}\, \mathrm{Hom}_A(T, T(j))\rangle_B \\ &= \dim_k \mathrm{Hom}_B(\mathrm{Hom}_A(T, T(i)), \mathrm{Hom}_A(T, T(j))) \\ &= \dim_k \mathrm{Hom}_A(T(i), T(j)) \\ &= \langle \underline{\dim}\, T(i), \underline{\dim}\, T(j)\rangle_A. \end{aligned}$$

The conclusion now follows from our claim above.

COROLLARY. *Let (B, T, A) be a tilting triple with A of finite global dimension. Then the quadratic forms q_A and q_B are $\mathbb{Z}$-congruent.*

2.8. *The BB-tilting modules.* The following construction, due to Brenner and Butler (see [27] or also [56]) further generalises that of Auslander, Platzeck and Reiten (see 1.6, example (i)). Let A be an algebra, and let $S(i)$ be a simple A-module such that:

(a) $\mathrm{pd}\, \tau^{-1} S(i) \leqslant 1$.

(b) $\mathrm{Ext}_A^1(S(i), S(i)) = 0$.

Then the module $T_A = \tau^{-1} S(i) \oplus (\bigoplus_{j \neq i} P(j))$ is a tilting module (called the *BB-tilting module* associated to $S(i)$). Indeed, since (T1) follows from (a), and (T3′) is trivially satisfied, we only need to show (T2). Now, for each $j \neq i$,

$$\mathrm{Ext}_A^1(\tau^{-1} S(i), P(j)) \simeq D\,\mathrm{Hom}_A(P(j), S(i)) = 0.$$

Thus it suffices to prove that

$$\mathrm{Ext}_A^1(\tau^{-1} S(i), \tau^{-1} S(i)) \simeq D\,\mathrm{Hom}_A(\tau^{-1} S(i), S(i)) = 0.$$

It follows from (b) and the Auslander–Reiten formula that $\underline{\mathrm{Hom}}_A(\tau^{-1} S(i), S(i)) = 0$. Thus, any non-zero morphism $f\colon \tau^{-1} S(i) \to S(i)$ factors through the projective cover $p\colon P(i) \to S(i)$, that is, there exists

$g\colon \tau^{-1}S(i) \to P(i)$ such that $f = pg$. We claim that $\operatorname{Im} g \subseteq \operatorname{rad} P(i)$. For, if this is not the case, then g is surjective, hence splits, and so, since $\tau^{-1}S(i)$ is indecomposable, we obtain $\tau^{-1}S(i) \simeq P(i)$, an absurdity. Thus $\operatorname{Im} g \subseteq \operatorname{rad} P(i)$ and consequently $f ={}'pg = 0$.

Observe that, if T_A is as above, then $\operatorname{ind} \mathscr{F}(T) = \{S(i)\}$ (by 2.4). Also, it is not hard to prove that T_A is a BB-tilting module if and only if $_BT$ is a BB-tilting module, and also that a module of the form $\tau^{-1}S(i) \oplus (\bigoplus_{j \neq i} P(j))$ is a tilting module if and only if the simple module $S(i)$ satisfies (a) and (b). We refer to [27] or [74] for the proofs.

Examples of BB-tilting modules are provided by the APR-tilting modules. The following is an example of a BB-tilting module which is not an APR-tilting module. Let A be as in the example in 2.1. The simple A-module $S(i) = \begin{smallmatrix} & & 1 & & \\ 0 & & & & 0 \\ & 0 & & 0 & \end{smallmatrix}$ satisfies (a) and (b). Then

$$T_A = \begin{smallmatrix} & & 0 & & \\ 1 & & & & 0 \\ & 0 & & 0 & \end{smallmatrix} \oplus \begin{smallmatrix} & & 0 & & \\ 1 & & & & 0 \\ & 1 & & 0 & \end{smallmatrix} \oplus \begin{smallmatrix} & & 0 & & \\ 1 & & & & 0 \\ & 1 & & 1 & \end{smallmatrix} \oplus \begin{smallmatrix} & & 1 & & \\ 1 & & & & 1 \\ & 1 & & 1 & \end{smallmatrix} \oplus \begin{smallmatrix} & & 0 & & \\ 0 & & & & 1 \\ & 1 & & 1 & \end{smallmatrix}$$

is the corresponding BB-tilting module. Its endomorphism algebra is given by the quiver

$$\circ \xleftarrow{\ \delta\ } \circ \xleftarrow{\ \gamma\ } \circ \xleftarrow{\ \beta\ } \circ \xleftarrow{\ \alpha\ } \circ$$

bound by $\alpha\beta\gamma\delta = 0$.

3. Torsion-theoretical properties of tilting modules

3.1. Our first task in this section is to characterise those torsion theories $(\mathscr{T}, \mathscr{F})$ in $\operatorname{mod} A$ such that there exists a tilting module T_A with $\mathscr{T} = \mathscr{T}(T_A)$ and $\mathscr{F} = \mathscr{F}(T_A)$. This is quite useful in practice, since in many applications it is easier to start by constructing the torsion theory, then finding the corresponding tilting module. This problem was first considered by Hoshino [49] who gave a sufficient condition. The necessary and sufficient conditions stated here were obtained independently in [3] and [70]. We shall need the following lemma from [16].

LEMMA. (i) *If $\mathscr{T} = \operatorname{Gen}(X_A)$ is a torsion class, then the numbers of isomorphism classes of indecomposable* Ext-*projectives in $\mathscr{T}$ and of indecomposable* Ext-*injectives in $\mathscr{T}$ are finite and equal.*

(ii) *If $\mathscr{F} = \operatorname{Cogen}(Y_A)$ is a torsion-free class, then the numbers of isomorphism classes of indecomposable* Ext-*projectives in $\mathscr{F}$ and of indecomposable* Ext-*injectives in $\mathscr{F}$ are finite and equal.*

Proof of (i). Since X_A is clearly faithful as an $A/\operatorname{ann}(X)$-module, and we have the embeddings $\mathscr{T} \to \operatorname{mod}(A/\operatorname{ann}(X)) \to \operatorname{mod} A$, it suffices to prove the statement in case X is faithful. By 1.3 and 1.5, the module X_A is a partial tilting module and is Ext-projective in $\mathcal{T}$. Moreover, since X is faithful, all the indecomposable injective A-modules are torsion and so, by 1.4, coincide with the indecomposable Ext-injectives in $\mathcal{T}$.

Let $u_1, \ldots, u_d$ be a basis of the k-vector space $\mathrm{Hom}_A(A, X)$ and consider the morphism $u = [u_1, \ldots, u_d]^t\colon A_A \to X^{(d)}$. Since X is faithful, u is injective and so we have a short exact sequence

$$(*) \qquad 0 \to A_A \xrightarrow{u} X^{(d)} \xrightarrow{v} Y \to 0$$

where $Y = \mathrm{Coker}(u)$. Observe that $Y \in \mathcal{T}$. Also, by [25], 8.1, Corollaire 2, $\mathrm{pd}\, Y_A \leqslant 1$. We shall now prove that Y is Ext-projective in $\mathcal{T}$. Let $M \in \mathcal{T}$ and apply the functor $\mathrm{Hom}_A(-, M)$ to $(*)$. The resulting sequence

$$0 \to \mathrm{Hom}_A(Y, M) \to \mathrm{Hom}_A(X^{(d)}, M) \xrightarrow{\mathrm{Hom}_A(u,M)} \mathrm{Hom}_A(A, M) \to \mathrm{Ext}_A^1(Y, M) \to 0$$

is exact. We claim that $\mathrm{Hom}_A(u, M)$ is surjective. Since $M \in \mathcal{T}$, there exists an epimorphism $g\colon X_0 \to M$ with $X_0 \in \mathrm{add}(X)$. Since A_A is projective, the morphism $\mathrm{Hom}_A(A, g)\colon \mathrm{Hom}_A(A, X_0) \to \mathrm{Hom}_A(A, M)$ is surjective. On the other hand, since $X_0 \in \mathrm{add}(X)$, it follows from the definition of u that the morphism $\mathrm{Hom}_A(u, X_0)\colon \mathrm{Hom}_A(X^{(d)}, X_0) \to \mathrm{Hom}_A(A, X_0)$ is surjective. Therefore, the composition $\mathrm{Hom}_A(u, g)\colon \mathrm{Hom}_A(X^{(d)}, X_0) \to \mathrm{Hom}_A(A, M)$ is surjective. Since

$$\mathrm{Hom}_A(u, g) = \mathrm{Hom}_A(u, M)\mathrm{Hom}_A(X^{(d)}, g),$$

we infer that $\mathrm{Hom}_A(u, M)$ is surjective. Therefore $\mathrm{Ext}_A^1(Y, M) = 0$ and so Y is Ext-projective in $\mathcal{T}$.

We deduce that $T_A = X \oplus Y$ is a tilting module. Indeed, $\mathrm{pd}\, T_A \leqslant 1$, and the Ext-projectivity of both X and Y implies that $\mathrm{Ext}_A^1(T, T) = 0$, while $(*)$ is the short exact sequence of (T3). Since $X, Y \in \mathcal{T}$, we have $\mathcal{T}(T) \subseteq \mathcal{T}$, while if $M \in \mathcal{T}$, then $\mathrm{Ext}_A^1(T, M) = 0$ (because T is Ext-projective in $\mathcal{T}$), thus $M \in \mathcal{T}(T)$. Therefore $\mathcal{T} = \mathcal{T}(T)$. Since T_A is a tilting module, it follows from 1.8 that the non-isomorphic indecomposable Ext-projectives in $\mathcal{T}$ coincide with the non-isomorphic indecomposable direct summands of T. Therefore, by 2.6, their number equals the rank of $K_0(A)$ and thus equals the number of non-isomorphic indecomposable Ext-injectives in $\mathcal{T}$.

3.2. DEFINITION. A torsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathrm{mod}\, A$ is called a *tilting torsion theory* if there exists a tilting module T_A such that $\mathcal{T} = \mathcal{T}(T_A)$ and $\mathcal{F} = \mathcal{F}(T_A)$.

THEOREM. *Let* $(\mathcal{T}, \mathcal{F})$ *be a torsion theory in* $\mathrm{mod}\, A$. *The following conditions are equivalent*:

(i) $(\mathcal{T}, \mathcal{F})$ *is a tilting torsion theory.*
(ii) $\mathcal{T} = \mathrm{Gen}(M)$, *for some A-module M, and* $\mathcal{T}$ *contains the injectives.*
(iii) $\mathcal{F} = \mathrm{Cogen}(N)$, *for some A-module N, and* $\mathcal{T}$ *contains the injectives.*

Proof. (i) $\Rightarrow$ (ii) is obvious, while (i) $\Rightarrow$ (iii) follows from 2.4.

(ii)$\Rightarrow$(i). Let $T_1, \ldots, T_t$ be a complete set of non-isomorphic indecomposable Ext-projectives in $\mathscr{T}$, and let $T_A = \bigoplus_{i=1}^t T_i$. Then T_A is a tilting module. Indeed, (T2) is immediate, while (T1) follows from the fact that

$$\mathrm{Hom}_A(DA, \tau T) = \bigoplus_{i=1}^{t} \mathrm{Hom}_A(DA, \tau T_i) = 0$$

(because τT_i is zero or torsion-free, while $DA \in \mathscr{T}$ by hypothesis). Finally, by 3.1, t equals the number of non-isomorphic indecomposable Ext-injectives in $\mathscr{T}$, thus the number of non-isomorphic indecomposable injective A-modules. Therefore $t = \operatorname{rank} K_0(A)$ and so T_A is a tilting module.

Since M_A is itself Ext-projective in $\mathscr{T}$, its indecomposable direct summands are summands of T. Therefore $\mathscr{T} \subseteq \mathscr{T}(T)$. Since $T \in \mathrm{Gen}(M_A)$, we also have $\mathscr{T}(T) \subseteq \mathscr{T}$ and so $\mathscr{T} = \mathscr{T}(T_A)$.

(iii)$\Rightarrow$(i). Let $N_1, \ldots, N_r$ be a complete set of non-isomorphic indecomposable Ext-injective modules in $\mathscr{F}$. Since no N_i is injective, $\tau^{-1} N_i$ is non-zero and Ext-projective in $\mathscr{T}$. On the other hand, let $P_1, \ldots, P_s$ be a complete set of non-isomorphic indecomposable Ext-projectives in $\mathscr{T}$ which are also projective A-modules. Let

$$T_A = \Big(\bigoplus_{i=1}^{r} \tau^{-1} N_i\Big) \oplus \Big(\bigoplus_{j=1}^{s} P_j\Big).$$

Clearly, T_A is Ext-projective in $\mathscr{T}$ and is in fact the direct sum of a complete set of non-isomorphic indecomposable Ext-projectives in $\mathscr{T}$. We claim that T_A is a tilting module. Indeed, (T2) is clear, while (T1) follows from the fact that

$$\mathrm{Hom}_A(DA, \tau T) = \bigoplus_{i=1}^{r} \mathrm{Hom}_A(DA, N_i) = 0$$

(because $DA \in \mathscr{T}$ and $N_i \in \mathscr{F}$ for all i). There remains to show that $r+s = n$ (where $n = \operatorname{rank} K_0(A)$). Now, by 3.1 and 1.4, r equals the number of non-isomorphic indecomposable Ext-projectives in $\mathscr{F}$, that is, of the modules of the form P/tP for P_A indecomposable projective not in $\mathscr{T}$. Hence $r = n-s$ and T_A is a tilting module.

Finally, since N is Ext-injective in $\mathscr{F}$, its indecomposable summands are summands of $\bigoplus_{i=1}^r N_i = \tau T$. Hence $\mathscr{F} \subseteq \mathscr{F}(T)$. Since $\tau T \in \mathscr{F}$, we infer that $\mathscr{F} = \mathscr{F}(T)$ and so $\mathscr{T} = \mathscr{T}(T)$.

Remark. The condition that $\mathscr{T}$ contains the injectives may clearly be replaced in (ii) by the condition that M is faithful, and in (iii) by the condition that $\mathrm{pd}(\tau^{-1}N) \leqslant 1$.

COROLLARY (Hoshino). *Let $(\mathscr{T}, \mathscr{F})$ be a torsion theory such that $\mathscr{T}$ contains the injectives and either $\mathscr{T}$ or $\mathscr{F}$ contains only finitely many non-isomorphic indecomposable modules. Then $(\mathscr{T}, \mathscr{F})$ is a tilting torsion theory. In particular, if*

A is representation-finite, then a torsion theory $(\mathscr{T}, \mathscr{F})$ *in* mod A *is a tilting torsion theory if and only if* $\mathscr{T}$ *contains the injectives.*

3.3. An important property of tilting torsion theories is the possibility of approximating an arbitrary module by torsion modules.

DEFINITION (Tachikawa–Wakamatsu). Let $\mathscr{T} = \mathrm{Gen}(T_A)$ be a torsion class. Given an A-module M, a short exact sequence of the form

$$0 \to M \xrightarrow{u_0} U_0 \xrightarrow{v_0} T_0 \to 0,$$

with $U_0 \in \mathscr{T}$ and $T_0 \in \mathrm{add}(T)$, is called a *torsion resolution* for M.

Clearly, if a module M_A admits a torsion resolution as above, then it is not unique since, for each $T_0' \in \mathrm{add}(T)$, we have another resolution

$$0 \to M \xrightarrow{u} U_0 \oplus T_0' \xrightarrow{v} T_0 \oplus T_0' \to 0,$$

with $u = \begin{bmatrix} u_0 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} v_0 & 0 \\ 0 & 1 \end{bmatrix}$. On the other hand, if a torsion class $\mathscr{T} = \mathrm{Gen}(T_A)$ is such that every A-module M has a torsion resolution, then $\mathscr{T}$ is necessarily the torsion class of a tilting torsion theory (for, the torsion resolution for DA_A splits, therefore $DA_A \in \mathscr{T}$; we then apply 3.2). The next theorem asserts the existence of minimal torsion resolution for tilting torsion theories. We shall follow the proof in [75].

THEOREM (Tachikawa–Wakamatsu). *Let* (B, T, A) *be a tilting triple. Then, for every module* M_A, *there exists a torsion resolution in* $(\mathscr{T}(T), \mathscr{F}(T))$

$$0 \to M \xrightarrow{u_M} U(M) \xrightarrow{v_M} T(M) \to 0$$

such that $T(M) = P \otimes_B T$ *where* P_B *is a projective cover of* $\mathrm{Ext}_A^1(T, M)$, *and moreover such that, for any other torsion resolution of M*

$$0 \to M \xrightarrow{u'} U' \xrightarrow{v'} T' \to 0,$$

there exists $T'' \in \mathrm{add}(T)$ *such that we have an exact commutative diagram*

$$\begin{array}{ccccccccc} 0 \to & M & \xrightarrow{u'} & U' & \xrightarrow{v'} & T' & \longrightarrow & 0 \\ & \| & & \| & & \| & & \\ 0 \to & M & \xrightarrow{u} & U(M) \oplus T'' & \xrightarrow{v} & T(M) \oplus T'' & \to & 0 \end{array}$$

with $u = \begin{bmatrix} u_M \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} v_M & 0 \\ 0 & 1 \end{bmatrix}$.

Proof. (a) *Existence.* (i) If $M \in \mathscr{T}(T)$, then we set $U(M) = M$ and $T(M) = 0$.

(ii) Suppose $M \in \mathscr{F}(T)$ and let p: $P_B \to \mathrm{Ext}_A^1(T, M)$ be a projective cover with kernel K. Applying $- \otimes_B T$ to the sequence

$$0 \to K \to P \xrightarrow{p} \mathrm{Ext}_A^1(T, M) \to 0$$

we obtain an exact sequence

$$0 \to \operatorname{Tor}_1^B(\operatorname{Ext}_A^1(T, M), T) \to K \otimes_B T \to P \otimes_B T \to \operatorname{Ext}_A^1(T, M) \otimes_B T \to 0.$$

The first term is M, the last is zero (by 2.1) and since $P \otimes_B T \in \operatorname{add}(T)$, we put $U(M) = K \otimes_B T \in \mathscr{T}(T)$ and we have the wanted sequence.

(iii) Let M be arbitrary and consider its canonical sequence $0 \to tM \to M \to M/tM \to 0$ as an element e of $\operatorname{Ext}_A^1(M/tM, tM)$. Since $M/tM \in \mathscr{F}(T)$, (ii) gives an exact sequence

$$0 \to M/tM \to U(M/tM) \to T(M/tM) \to 0$$

with $T(M/tM) = P \otimes_B T$, where P_B is a projective cover of $\operatorname{Ext}_A^1(T, M/tM) \simeq \operatorname{Ext}_A^1(T, M)$ (because $\operatorname{Ext}_A^1(T, tM) = 0$). Applying the functor $\operatorname{Hom}_A(-, tM)$ to this sequence yields an isomorphism

$$\operatorname{Ext}_A^1(U(M/tM), tM) \simeq \operatorname{Ext}_A^1(M/tM, tM)$$

and hence an exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & tM & = & tM & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & M & \longrightarrow & \bar{U} & \longrightarrow & T(M/tM) & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & M/tM & \to & U(M/tM) & \to & T(M/tM) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where the first column is e. Since tM and $U(M/tM)$ are torsion, so is $\bar{U}$ and we have the required sequence.

(b) *Minimality*. Let $0 \to M \xrightarrow{u'} U' \xrightarrow{v'} T' \to 0$ be another torsion resolution of M. Since U' and $U(M)$ are torsion, we have $\operatorname{Ext}_A^1(T(M), U') = 0$ and $\operatorname{Ext}_A^1(T', U(M)) = 0$. Hence, applying respectively the functors $\operatorname{Hom}_A(T(M), -)$ to the previous torsion resolution of M, and $\operatorname{Hom}_A(T', -)$ to the torsion resolution of M obtained in (a), we get the following exact sequences:

$$0 \to \operatorname{Hom}_A(T(M), M) \to \operatorname{Hom}_A(T(M), U') \to \operatorname{Hom}_A(T(M), T') \to \operatorname{Ext}_A^1(T(M), M) \to 0,$$

$$0 \to \operatorname{Hom}_A(T', M) \to \operatorname{Hom}_A(T', U(M)) \to \operatorname{Hom}_A(T', T(M)) \to \operatorname{Ext}_A^1(T', M) \to 0.$$

Considering our two torsion resolutions of M as extensions, we see that they are induced one from the other via morphisms $T(M) \to T'$ and $T' \to T(M)$. We thus have an exact commutative diagram

$$\begin{array}{ccccccccc} 0 \to & M & \xrightarrow{u_M} & U(M) & \xrightarrow{v_M} & T(M) & \to 0 \\ & \| & & \downarrow f & & \downarrow g & \\ 0 \to & M & \xrightarrow{u'} & U' & \xrightarrow{v'} & T' & \to 0 \\ & \| & & \downarrow f' & & \downarrow g' & \\ 0 \to & M & \xrightarrow{u_M} & U(M) & \xrightarrow{v_M} & T(M) & \to 0 \end{array}$$

Applying $\mathrm{Hom}_A(T, -)$, we deduce a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_A(T, T(M)) & \xrightarrow{p} & \mathrm{Ext}_A^1(T, M) \to 0 \\ h \downarrow & & \| \\ \mathrm{Hom}_A(T, T(M)) & \xrightarrow{p} & \mathrm{Ext}_A^1(T, M) \to 0 \end{array}$$

with $h = \mathrm{Hom}_A(T, g'g)$. Since $\mathrm{Hom}_A(T, T(M))$ is a projective cover, $\mathrm{Hom}_A(T, g'g)$ is an isomorphism. Therefore so is $g'g$ and hence so is $f'f$. Thus f and g are sections while f' and g' are retractions.

Remarks. Let $f: M \to N$ be a morphism. Then there exist $f_U: U(M) \to U(N)$ and $f_T: T(M) \to T(N)$ such that we have an exact commutative diagram

$$\begin{array}{ccccccc} 0 \to & M & \xrightarrow{u_M} & U(M) & \xrightarrow{v_M} & T(M) & \to 0 \\ & f\downarrow & & f_U\downarrow & & f_T\downarrow & \\ 0 \to & N & \xrightarrow{u_N} & U(N) & \xrightarrow{v_N} & T(N) & \to 0 \end{array}$$

Indeed, applying $\mathrm{Hom}_A(-, U(N))$ to the minimal torsion resolution of M yields an exact sequence (with $w = \mathrm{Hom}_A(u_M, U(N))$)

$$0 \to \mathrm{Hom}_A(T(M), U(N)) \to \mathrm{Hom}_A(U(M), U(N)) \xrightarrow{w} \mathrm{Hom}_A(M, U(N)) \to 0.$$

We take f_U to be a preimage of $u_N f$ under w, and f_T follows by passing to the cokernels.

EXAMPLE. Let A be given by the quiver of Fig. 10 bound by

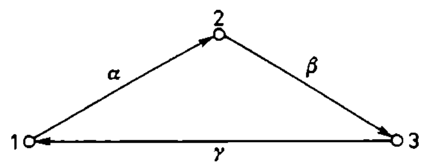

Fig. 10

$\gamma\alpha\beta = 0$, $\alpha\beta\gamma\alpha = 0$. Then Γ_A is as shown in Fig. 11, where indecomposables are represented by their Loewy series, the horizontal dotted lines denote the Auslander–Reiten translations, and we identify along the vertical dotted lines. For the BB-tilting module T corresponding to $S(3)$ (see 2.8), the summands of T are indicated by squares, the subcategory $\mathscr{T}(T)$ by the shaded regions and

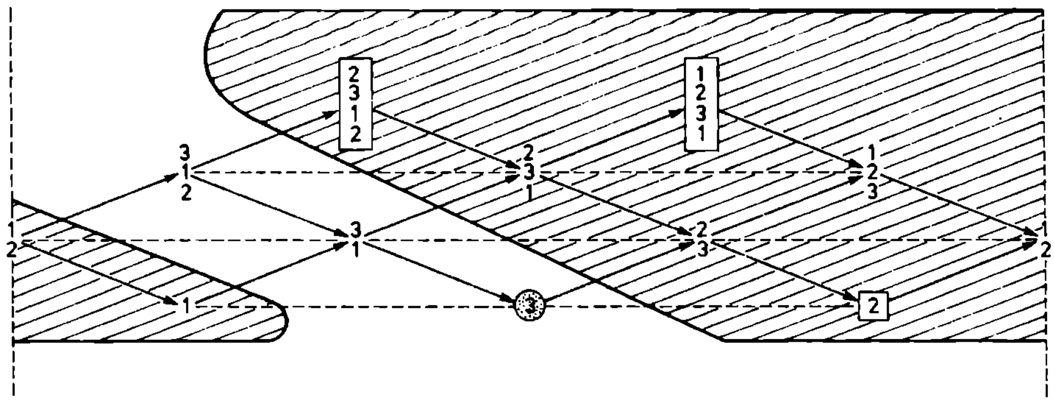

Fig. 11

the subcategory $\mathscr{F}(T)$ by the dotted one. The torsion resolutions of the non-torsion indecomposable A-modules are:

$$0 \to 3 \to \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \to 2 \to 0,$$

$$0 \to \begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \to \begin{smallmatrix} 2 \\ 3 \\ 1 \end{smallmatrix} \to 2 \to 0,$$

$$0 \to \begin{smallmatrix} 3 \\ 1 \\ 2 \end{smallmatrix} \to \begin{smallmatrix} 2 \\ 3 \\ 1 \\ 2 \end{smallmatrix} \to 2 \to 0.$$

3.4. We shall see in 3.6 a simple application of the torsion resolutions. The most important, however, deals with the trivial extension algebras, which have played a prominent rôle in the classification of the self-injective algebras of polynomial growth (see [69]). Recall that the *trivial extension* $A \ltimes DA$ of an algebra A by its minimal injective cogenerator bimodule ${}_ADA_A$ is the algebra whose additive structure is that of the group $A \oplus DA$, and whose multiplication is defined by

$$(a, f)(b, g) = (ab, ag + fb)$$

(for $a, b \in A$ and $f, g \in {}_ADA_A$). Then $A \ltimes DA$ is a self-injective and, actually, symmetric algebra. Tachikawa and Wakamatsu have proved the following theorem.

THEOREM [75]. *Let* (B, T, A) *be a tilting triple, Then there exists a stable equivalence* S: $\underline{\mathrm{mod}}(A \ltimes DA) \simeq \underline{\mathrm{mod}}(B \ltimes DB)$ *such that* $S|_{\mathscr{T}(T_A)} = \mathrm{Hom}_A(T, -)$.

In the proof, the torsion resolutions allow one to construct explicitly the stable functor S and its quasi-inverse. An important consequence is as follows.

Let $\hat{A}$ be the repetitive algebra of A, that is, let $\hat{A}$ be the infinite matrix algebra (see [52])

$$\hat{A} = \begin{bmatrix} \ddots & & & & 0 \\ & A_{i-1} & & & \\ & \ddots & & & \\ & & E_{i-1} & A_i & \\ & & & E_i & A_{i+1} \\ 0 & & & & \ddots & \ddots \end{bmatrix}$$

where matrices have only finitely many non-zero coefficients, $A_i = A$ and $E_i = DA$ for all $i \in \mathbf{Z}$, addition is the usual addition of matrices, and multiplication is induced from the canonical bimodule structure of DA and the zero map $DA \otimes_A DA \to 0$. This is a self-injective, locally finite dimensional algebra without identity. The identity maps $A_i \to A_{i-1}$, $E_i \to E_{i-1}$, for each i, induce an automorphism ν_A of $\hat{A}$ and clearly, $\hat{A}$ is a Galois covering of $A \ltimes DA$ with the infinite cyclic group generated by ν_A. In particular, mod $\hat{A}$ is equivalent to the category of $\mathbf{Z}$-graded $(A \ltimes DA)$-modules. Now Wakamatsu has shown that the stable functor S: $\underline{\text{mod}}(A \ltimes DA) \to \underline{\text{mod}}(B \ltimes DB)$ is compatible with the grading, so that we have:

THEOREM [79]. *Let (B, T, A) be a tilting triple. Then* $\underline{\text{mod}}\,\hat{A} \simeq \underline{\text{mod}}\,\hat{B}$.

If A has finite global dimension, this also follows from Happel's results on the derived category of a finite dimensional algebra (see 4.3 below).

3.5. DEFINITION. Let (B, T, A) be a tilting triple. Then T_A is said to be a *separating* (respectively, *splitting*) *tilting module* if the torsion theory $(\mathscr{T}(T_A), \mathscr{F}(T_A))$ splits in mod A (respectively, the torsion theory $(\mathscr{X}(T_A), \mathscr{Y}(T_A))$ splits in mod B).

Thus a tilting module T_A is separating (respectively, splitting) if and only if the tilting module $_BT$ is splitting (respectively, separating). Also, the tilting module T_A is separating (respectively, splitting) if and only if the torsion-free class $\mathscr{F}(T)$ (respectively, $\mathscr{Y}(T)$) is closed under the action of the Auslander–Reiten translation τ, or, equivalently, if and only if the torsion class $\mathscr{T}(T)$ (respectively, $\mathscr{X}(T)$) is closed under the action of τ^{-1}.

Examples of separating tilting modules are the APR-tilting modules. An example of a splitting tilting module is given in 3.6 below. Construction procedures for separating tilting modules are given in [3].

Separating and splitting tilting modules are particularly useful as they allow one to keep a good measure of control on the tilting process. Indeed, if T_A is separating, then all the indecomposable A-modules are mapped into indecomposable B-modules by the functors $\text{Hom}_A(T, -)$ and $\text{Ext}_A^1(T, -)$ (so that B has at least as many non-isomorphic indecomposables as A). On the other hand, if T_A is splitting, then all the indecomposable B-modules are images under the same functors of indecomposable A-modules (so that B has at most as many non-isomorphic indecomposables as A). In this latter case, we also have a complete description of the Auslander–Reiten sequences in $\operatorname{mod} B$.

THEOREM. *Let T_A be a splitting tilting module, and $B = \operatorname{End} T_A$. Every Auslander–Reiten sequence in* $\operatorname{mod} B$ *either lies completely in $\mathscr{X}(T)$, or lies completely in $\mathscr{Y}(T)$, or is of the form*

$$0 \to \operatorname{Hom}_A(T, I) \to \operatorname{Hom}_A(T, I/S) \oplus \operatorname{Ext}_A^1(T, \operatorname{rad} P) \to \operatorname{Ext}_A^1(T, P) \to 0$$

where P_A is an indecomposable projective A-module not in $\operatorname{add}(T)$, *S_A its simple top and I_A the injective envelope of S.* (Such a sequence is called a *connecting sequence*).

Proof. Let $0 \to E'_B \to E_B \to E''_B \to 0$ be an Auslander–Reiten sequence in $\operatorname{mod} B$. Since the torsion theory $(\mathscr{X}(T), \mathscr{Y}(T))$ splits, we have one of the following cases.

(a) $E'' \in \mathscr{Y}(T)$. Then both E' and E belong to $\mathscr{Y}(T)$ as well, and the sequence lies completely in $\mathscr{Y}(T)$.

(b) $E' \in \mathscr{X}(T)$. Then both E and E'' belong to $\mathscr{X}(T)$ as well, and the sequence lies completely in $\mathscr{X}(T)$.

(c) $E' \in \mathscr{Y}(T)$ and $E'' \in \mathscr{X}(T)$. In this case, let $M_A = E' \otimes_B T$ and let I_A denote the injective envelope of M. Then, since $M \in \mathscr{T}(T)$,

$$\begin{aligned}\operatorname{Ext}_A^1(I/M, M) &\simeq \operatorname{Ext}_B^1(\operatorname{Hom}_A(T, I/M), \operatorname{Hom}_A(T, M)) \\ &\simeq \operatorname{Ext}_B^1(\operatorname{Hom}_A(T, I/M), E') \\ &\simeq D\,\underline{\operatorname{Hom}}_B(\tau^{-1}E', \operatorname{Hom}_A(T, I/M)) = 0\end{aligned}$$

since $\tau^{-1}E' = E'' \in \mathscr{X}(T)$, while $\operatorname{Hom}_A(T, I/M) \in \mathscr{Y}(T)$. Therefore, M_A is a direct summand of I_A, so that $I_A = M_A$. In particular, I is indecomposable and $\operatorname{Hom}_A(T, I) \simeq E'_B$. By the Connecting Lemma 2.3, $E''_B \simeq \operatorname{Ext}_A^1(T, P)$, where P_A is the projective cover of $S = \operatorname{soc} I_A$. Also, $P \notin \operatorname{add}(T)$. Thus the given Auslander–Reiten sequence is of the form

$$0 \to \operatorname{Hom}_A(T, I) \to E_B \to \operatorname{Ext}_A^1(T, P) \to 0$$

and by 2.3, Corollary, the canonical sequence for E_B in $(\mathscr{X}(T), \mathscr{Y}(T))$ is

$$0 \to \operatorname{Ext}_A^1(T, \operatorname{rad} P) \to E_B \to \operatorname{Hom}_A(T, I/S) \to 0.$$

Since $(\mathscr{X}(T), \mathscr{Y}(T))$ splits, $E_B \simeq \operatorname{Hom}_A(T, I/S) \oplus \operatorname{Ext}_A^1(T, \operatorname{rad} P)$ and we are done.

3.6. The following criterion, due to Hoshino [51], allows one to decide whether a tilting module is separating, splitting, or not. The proof given below follows [76]. Another proof is given in [40], III, 4.12.

THEOREM. *Let (B, T, A) be a tilting triple.*

(i) *T_A is separating if and only if, for each $X_B \in \mathscr{X}(T)$,* $\operatorname{pd} X_B = 1$.
(ii) *T_A is splitting if and only if, for each $M_A \in \mathscr{F}(T)$,* $\operatorname{id} M_A = 1$.

Proof of (i). The torsion theory $(\mathscr{T}(T), \mathscr{F}(T))$ is splitting if and only if, for each $M_A \in \mathscr{F}(T)$, we have $\operatorname{Ext}_A^1(M, -)|_{\mathscr{T}(T)} = 0$. Consider thus the minimal torsion resolution of $M \in \mathscr{F}(T)$

$$(*) \qquad 0 \to M \to U(M) \to T(M) \to 0.$$

Apply the functor $\operatorname{Ext}_A^1(-, N)$, where $N \in \mathscr{T}(T)$, to this sequence. Since $\operatorname{Ext}_A^1(T(M), N) = 0$, we have an isomorphism $\operatorname{Ext}_A^1(M, N) \simeq \operatorname{Ext}_A^1(U(M), N)$. Thus $\operatorname{Ext}_A^1(M, -)|_{\mathscr{T}(T)} = 0$ if and only if $\operatorname{Ext}_A^1(U(M), -)|_{\mathscr{T}(T)} = 0$. Since $U(M) \in \mathscr{T}(T)$, this amounts to saying that $U(M)$ is Ext-projective in $\mathscr{T}(T)$ or, equivalently, that $U(M) \in \operatorname{add}(T)$.

Applying the functor $\operatorname{Hom}_A(T, -)$ to $(*)$ yields an exact sequence

$$0 \to \operatorname{Hom}_A(T, U(M)) \to \operatorname{Hom}_A(T, T(M)) \to \operatorname{Ext}_A^1(T, M) \to 0$$

with a projective middle term. It follows from the above remarks that T_A is separating if and only if, for each $M \in \mathscr{F}(T)$, $U(M) \in \operatorname{add}(T)$. Now this is the case if and only if, for each $M \in \mathscr{F}(T)$, $\operatorname{pd} \operatorname{Ext}_A^1(T, M) = 1$. Since each $X_B \in \mathscr{X}(T)$ can be written as $X = \operatorname{Ext}_A^1(T, M)$, for some $M \in \mathscr{F}(T)$, we are done.

Remark and examples. The above criterion is particularly easy to check if T_A is a BB-tilting module: indeed, the BB-tilting module corresponding to the simple module $S(i)_A$ is splitting if and only if $\operatorname{id} S(i) = 1$. For instance, if A is as in the example in 2.1 and T_A is the BB-tilting module corresponding to the simple module $\begin{smallmatrix} & & 1 & & \\ 0 & & & & 0 \\ & 0 & & 0 & \end{smallmatrix}$, then T_A is a splitting tilting module. On the other hand, if A and T_A are as in the example in 3.3, then $\operatorname{id} S(3) = 4$ and so T_A is not splitting.

COROLLARY. *Let A be a hereditary algebra. Then every tilting A-module is splitting.*

3.7. It is natural to ask whether the converse of the above Corollary is true, that is, whether an algebra such that every tilting module is splitting is necessarily hereditary. While this statement is clearly false in general (as is shown by the case of the self-injective algebras, see also the example in 1.6, remark 3), we shall show that it is true if the algebra is *triangular* (that is, its ordinary quiver has no oriented cycles).

THEOREM [4]. *Let A be a triangular algebra such that every separating tilting module is splitting. Then A is hereditary.*

Proof. We shall suppose that A is not hereditary, and construct a separating tilting module which is not splitting. Since A is triangular, we may order the points of its quiver as $\{1, 2, \dots, n\}$ so that $\mathrm{Hom}_A(P(s), P(t)) \neq 0$ implies $s \leqslant t$. If A is not hereditary, there exists a smallest i such that $\mathrm{id}\, S(i) > 1$. Letting $P = \bigoplus_{j=1}^{i} P(j)$, we define

$$T_A = \tau^{-1} P \oplus (\bigoplus_{a>i} P(a)).$$

Observe first that P is a hereditary projective: indeed, if $j < i$, we have a short exact sequence

$$0 \to S(j) \to I(j) \to \bigoplus_{a \to j} I(a) \to 0.$$

Hence, for any $h \neq j$, $\mathrm{Hom}_A(P(h), I(j)) \simeq \bigoplus_{a\to j} \mathrm{Hom}_A(P(h), I(a))$, or equivalently, $\mathrm{Hom}_A(P(j), P(h)) \simeq \bigoplus_{a \to j} \mathrm{Hom}_A(P(a), P(h))$. That is, j is not the terminal point of a relation on the quiver of A. This clearly implies that P is a hereditary projective and its indecomposable submodules are just $P(1), \dots, P(i)$. Therefore $\mathrm{Hom}_A(DA, \tau T) = \mathrm{Hom}_A(DA, P) = 0$ and so $\mathrm{pd}\, T \leqslant 1$. On the other hand, if $j \leqslant i$ and $a > i$, we have

$$\mathrm{Ext}_A^1(\tau^{-1} P(j), P(a)) \simeq D\,\mathrm{Hom}_A(P(a), P(j)) = 0$$

and also, if $h, j \leqslant i$, we have $\mathrm{Ext}_A^1(\tau^{-1}P(j), \tau^{-1}P(h)) = 0$, which gives $\mathrm{Ext}_A^1(T, T) = 0$. Since the number of non-isomorphic indecomposable summands of T equals n, T is indeed a tilting module.

Next, T_A is separating. Indeed, $\mathscr{F}(T)$ is cogenerated by $\tau T \simeq P$ and thus $\mathrm{ind}\, \mathscr{F}(T) = \{P(1), \dots, P(i)\}$. On the other hand, the isomorphisms

$$\mathrm{Ext}_A^1(T, M) \simeq D\,\mathrm{Hom}_A(M, \tau T) \simeq D\,\mathrm{Hom}_A(M, P)$$

show that, for an indecomposable module M_A, $M \notin \mathscr{T}(T)$ if and only if $\mathrm{Hom}_A(M, P) \neq 0$, that is to say, if and only if $M \in \mathscr{F}(T)$.

To show that T_A is not splitting, it suffices to show that the injective dimension of an indecomposable torsion-free module is larger than one. Now, since $P(i)$ is a hereditary projective, its radical $\mathrm{rad}\, P(i)$ is projective and a simple inductive argument shows that $\mathrm{id}\, \mathrm{rad}\, P(i) \leqslant 1$. Thus, applying the functor $\mathrm{Hom}_A(M, -)$ to the short exact sequence

$$0 \to \mathrm{rad}\, P(i) \to P(i) \to S(i) \to 0$$

yields $\mathrm{Ext}_A^2(M, P(i)) \simeq \mathrm{Ext}_A^2(M, S(i))$. The proof is now complete.

4. Tilting-cotilting equivalence

4.1. We shall now apply the preceding results to the study of the representation theory of certain classes of finite dimensional algebras. The idea is to start with one class whose representation theory is known, then to enlarge it by applying finitely many times the tilting process. Since we have a large measure of control on this process, we can study the wider class using our knowledge of the first one. It is useful, for many practical purposes, to be able to reverse the cotilting process while staying within the same class. We are thus led to define the concept dual to that of a tilting module.

DEFINITION. Let A be an algebra. A module T_A is called a *cotilting module* if it satisfies the following conditions:

(T1*) $\operatorname{id} T_A \leqslant 1$.
(T2) $\operatorname{Ext}^1_A(T, T) = 0$.
(T3′) The number of non-isomorphic indecomposable summands of T_A equals the rank of $K_0(A)$.

Thus, if A is hereditary, every tilting module is a cotilting module and conversely. More generally, if A is any algebra, T_A is a cotilting A-module if and only if ${}_A(DT)$ is a tilting A^{op}-module. We may now make the following definition.

DEFINITION. Let A and B be two algebras. Then A and B are said to be *tilting-cotilting equivalent* if there exist a sequence of algebras $A = A_0, A_1, \ldots, A_m = B$ and a sequence of tilting or cotilting modules $T^{(i)}_{A_i}$, $0 \leqslant i < m$, such that $A_{i+1} = \operatorname{End} T^{(i)}_{A_i}$ for each i. This is evidently an equivalence relation.

4.2. The following procedure due to Hughes and Waschbüsch [52] is useful to construct examples of tilting-cotilting equivalent algebras. Let A be a triangular algebra, i a sink in its ordinary quiver, and e_i the corresponding primitive idempotent. Consider the one-point extension

$$A[I(i)_A] = \begin{bmatrix} A & 0 \\ I(i)_A & k \end{bmatrix}.$$

Clearly, the bound quiver of this algebra contains the bound quiver of A as a full bound subquiver and exactly one additional vertex, which is a source corresponding to the new idempotent. We define the *reflection of A at the sink i* to be the quotient of $A[I(i)]$ by the two-sided ideal generated by the idempotent e_i, that is,

$$S_i^+ A = A[I(i)]/\langle e_i \rangle.$$

Dually, starting with a source j, we can define the reflection $S_j^- A$ at the source j. Observe that, if i is a sink (respectively, j is a source) in the quiver of A, then the repetitive algebras of A and $S_i^+ A$ (respectively, A and $S_j^- A$) are isomorphic. Also, it is easily seen that the trivial extensions of A and $S_i^+ A$ (respectively, A and $S_j^- A$) are isomorphic. The following result is due to Tachikawa and Wakamatsu (see [76]).

PROPOSITION. *Let A be triangular and let i be a sink (respectively, j be* *a source) in the ordinary quiver of A. Then A and S_i^+A (respectively, A and S_j^-A) are tilting-cotilting equivalent.*

EXAMPLE. Let A be given by the fully commutative quiver of Fig. 12. The

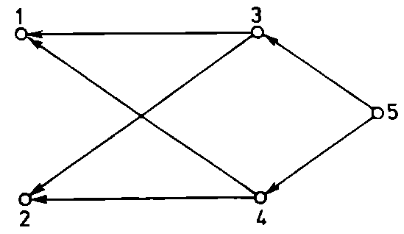

Fig. 12

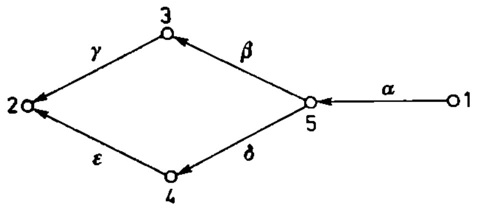

Fig. 13

reflection S_1^+A of A at the sink 1 is given by the quiver of Fig. 13 bound by $\beta\gamma = \delta\varepsilon$ and $\alpha\beta\gamma = 0$. On the other hand, $S_5^-A \simeq A^{op}$.

Observe that the reflection procedure is easier to carry out than tilting, as it only requires the knowledge of the bound quiver, while computing the endomorphism ring of a tilting module requires some knowledge about the module category.

4.3. Let now $D^b(A)$ denote the derived category of bounded complexes over the abelian category mod A, in the sense of Verdier [78]. The following theorem was first shown by Happel [39] in the case where the global dimension of A is finite, a restriction later removed by Cline, Parshall and Scott [30].

THEOREM. *Let A and B be tilting-cotilting equivalent. Then $D^b(A) \simeq D^b(B)$ as triangulated categories.*

Note that, using this theorem, one can give alternative proofs for most of the fundamental results of tilting theory (see [40], III).

Also, Happel has given a concrete description of $D^b(A)$, for A of finite global dimension. Let $\hat{A}$ be the repetitive algebra of A. Then, since $\hat{A}$ is self-injective, the stable category $\underline{\text{mod}}\,\hat{A}$ can be given the structure of a triangulated category. We have:

THEOREM. *Let A be a finite dimensional algebra of finite global dimension. Then $D^b(A) \simeq \underline{\text{mod}}\, \hat{A}$ as triangulated categories.*

For the proof, we refer the reader to [39] or [40]. As a consequence of this theorem and the previous one, if A and B are tilting-cotilting equivalent with A of finite global dimension, then $\underline{\text{mod}}\, \hat{A} \simeq \underline{\text{mod}}\, \hat{B}$ as triangulated categories (compare with 3.4).

As a consequence of the invariance under tilting of the derived category, Happel obtains the invariance of the Hochschild cohomology of a finite dimensional algebra. Let A be an algebra. We shall denote by $H^i(A)$ the ith Hochschild cohomology group of A with coefficients in the bimodule $_AA_A$ (see [29]).

THEOREM [41]. *Let A and B be tilting-cotilting equivalent. Then $H^i(A) \simeq H^i(B)$, for all i.*

4.4. It is conjectured in particular that the first Hochschild cohomology group $H^1(A)$ is trivial if and only if the algebra A is simply connected. If true, this statement and the preceding theorem would imply that an algebra tilting-cotilting equivalent to a simply connected algebra is also simply connected. Recall from [9] that a triangular algebra A is *simply connected* if, for any presentation $A \simeq kQ/I$ of A as a bound quiver algebra, the fundamental group $\pi_1(Q, I)$ of the bound quiver (Q, I) (in the sense of Martínez and de la Peña [57]) is trivial. Equivalently, a triangular algebra A is simply connected if and only if it has no proper Galois covering. If A is representation-finite, this notion of simple connectedness is equivalent to that introduced by Bongartz and Gabriel in [24], that is, a representation-finite, basic and connected algebra is simply connected if and only if the geometric realisation of its Auslander–Reiten quiver is simply connected as a simplicial complex. In this case, we have the following partial result from [2].

PROPOSITION. *Let A be a representation-finite simply connected algebra and let T_A be a splitting tilting module. Then $B = \text{End}\, T_A$ is simply connected.*

Proof. In this proof, we shall denote by $[X]$ the point of the Auslander–Reiten quiver corresponding to the indecomposable module X. Let us assume that B is not simply connected. Then the Auslander–Reiten quiver Γ_B of B must contain a closed walk w which is not contractible. We can of course assume w to be of minimal length. Observe that Γ_B contains no oriented cycles: indeed, such a cycle cannot lie completely in $\mathscr{X}(T_A)$ or in $\mathscr{Y}(T_A)$, by the simple connectedness of A, and thus it must contain a module $X_B \in \mathscr{X}(T_A)$ and a module $Y_B \in \mathscr{Y}(T_A)$ such that $\text{Hom}_B(X, Y) \neq 0$, an absurdity. In particular, the absence of oriented cycles implies the existence of sources and sinks on w. One can suppose that every sink corresponds to a projective module. Indeed, if the sink $[U]$ is such that U_B is not projective, then we can replace $[U]$ by $[\tau U]$, and each arrow α: $[V] \to [U]$ on w by the corresponding arrow $\sigma\alpha$: $[\tau U] \to [V]$. This process does not affect the length of w. On the other hand, applying repeatedly this procedure, we cannot reach another point of the original walk w: this follows from the minimality of w, and the fact that it is not contractible.

Let now $[Y]$ be a point on w which is not a sink. There exists a path on w from $[Y]$ to some sink $[P]$. But P_B is projective, hence $P \in \mathscr{Y}(T_A)$. Thus $Y \in \mathscr{Y}(T_A)$ as well. This shows that all the modules on w belong to $\mathscr{Y}(T_A)$.

Let $[Z]$ be a source on w and put $N_A = Z \otimes_B T_A$. By 3.5, if N_A is not injective, then the Auslander–Reiten sequence starting with $Z_B = \mathrm{Hom}_A(T, N)$ lies entirely in $\mathscr{Y}(T_A)$. We may thus replace $[Z]$ by $[\tau^{-1}Z]$, and each arrow β: $[Z] \to [Y]$ on w by $\sigma^{-1}\beta$: $[Y] \to [\tau^{-1}Z]$, thus obtaining a new path, homotopic to w, of the same length and still lying in $\mathscr{Y}(T_A)$. Applying this process as many times as necessary, we obtain a new walk w' in $\mathscr{Y}(T_A)$ homotopic to w, of the same length and such that, for any source $[Z_B]$ on w', the A-module $N_A = Z \otimes_B T_A$ is injective. Observe that w' may have sinks which do not correspond to projectives; what is important for our purposes, however, is that w' still lies in $\mathscr{Y}(T_A)$.

Applying the functor $-\otimes_B T_A$, we obtain a closed walk v' in Γ_A corresponding to the walk w' in Γ_B (because A is representation-finite). Since A is simply connected, v' is contractible. Now let $[Z]$ be a source on w'. Then the corresponding A-module $N_A = Z \otimes_B T_A$ is injective. Hence there exists on v' a single arrow $[N] \to [M]$ with source $[N]$. If $M \in \mathscr{T}(T_A)$, then $\mathrm{Hom}_A(T, M) \in \mathscr{Y}(T_A)$ and so $[\mathrm{Hom}_A(T, M)]$ lies on w', which contradicts the fact that w' has minimal length. Therefore $M \notin \mathscr{T}(T_A)$. On the other hand, the arrow $[N] \to [M]$ corresponds, because N is injective, to a surjection of N on a direct summand of $N/\mathrm{soc}\, N$. But $N \in \mathscr{T}(T_A)$, hence $M \in \mathscr{T}(T_A)$, a contradiction which completes the proof.

5. Tilted algebras

We shall always assume in the sequel our algebras to be basic and connected, and our tilting and cotilting modules to be minimal in the sense of 1.2.

5.1. In two cases, we have a good knowledge of the module category, namely if the algebra is hereditary, or if it is tubular canonical in the sense of [65]. It is thus natural to study algebras which are tilting-cotilting equivalent to a hereditary algebra or to a tubular canonical algebra. The first, simplest, case deals with those algebras which are obtained from a hereditary algebra by a single tilt.

DEFINITION. Let $\vec{\Delta}$ be a finite connected quiver without oriented cycles. An algebra A is called a *tilted algebra of type* $\vec{\Delta}$ if there exists a tilting module T_B over the path algebra $B = k\vec{\Delta}$ such that $A = \mathrm{End}\, T_B$.

Observe that, by 3.6, such a tilting module is necessarily splitting. Equivalently, the algebra A is tilted of type $\vec{\Delta}$ if there exists a (necessarily separating) tilting A^{op}-module ${}_AU$ such that $\operatorname{End}({}_AU) = k\vec{\Delta}$. Actually, as shown by Happel, a tilted algebra may be obtained using only partial tilting modules:

THEOREM. *Let B be a hereditary algebra, and let T_B be a partial tilting module. Then $A = \operatorname{End} T_B$ is a tilted algebra.*

The proof, for which we refer to [40], III, 6.5, uses the technique of perpendicular categories of [36].

It follows directly from the definition and 2.2 that tilted algebras always have global dimension at most two (but it is easy to find algebras of global dimension two which are not tilted).

EXAMPLES. The algebras of examples in 1.6(ii) and 2.1 are tilted algebras, respectively of type $\tilde{\mathbf{D}}_5$ and $\mathbf{D}_5$. Indeed, let B be the path algebra of the quiver of Fig. 14 and $T_B = P(1) \oplus M \oplus I(1) \oplus \tau I(2) \oplus \tau I(5) \oplus \tau I(6)$, where M is the

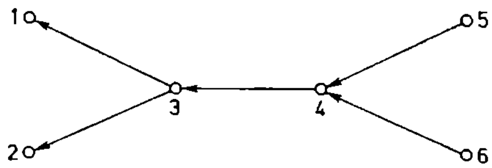

Fig. 14

unique indecomposable B-module with dimension vector $\underline{\dim}\, M = \begin{smallmatrix}1 & & & 1\\ & 1 & 1 & \\ 1 & & & 1\end{smallmatrix}$ (thus, M is simple regular non-homogeneous). Then T_B is a tilting module having as endomorphism ring the algebra of 1.6(ii).

Similarly, let C be the path algebra of the quiver of Fig. 15 and

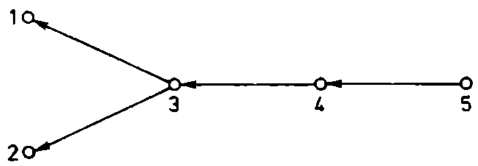

Fig. 15

$U_C = P(1) \oplus P(5) \oplus \tau^{-1}P(2) \oplus \tau^{-2}P(1) \oplus I(1)$. Then U_C is a tilting module having as endomorphism ring the algebra of 2.1.

5.2. In both of the above examples, the Auslander–Reiten quiver of the tilted algebra contains a full connected subquiver which is sectional (that is, does not factor through an Auslander–Reiten sequence) and which is isomorphic to the quiver of the original hereditary algebra. Such a subquiver is called a complete slice and its existence characterises tilted algebras. The following definition is due to Ringel [65]. For earlier attempts, we refer the reader to [43], [21] and [50].

DEFINITION. A class $\mathcal{S}$ of non-isomorphic indecomposable A-modules is called a *complete slice* in $\operatorname{mod} A$ if it satisfies the following axioms:

(CS1) $U = \bigoplus_{M\in\mathcal{S}} M$ is a *sincere* module (that is, $\operatorname{Hom}_A(P, U) \neq 0$ for any projective A-module P).

(CS2) If $M_0 \to M_1 \to \dots \to M_m$ is a sequence of non-zero non-isomorphisms in $\operatorname{mod} A$ with $M_0, M_m \in \mathcal{S}$, then $M_i \in \mathcal{S}$ for all $0 < i < m$.

(CS3) If $0 \to L \to M \to N \to 0$ is an Auslander–Reiten sequence, then at most one of L and N lies in $\mathcal{S}$. Furthermore, if an indecomposable summand of M lies in $\mathcal{S}$, then either L or N lies in $\mathcal{S}$.

Observe that, if $\mathcal{S}$ is a complete slice, then $U = \bigoplus_{M\in\mathcal{S}} M$ is a separating tilting (and also cotilting) A-module. It is called the *slice module* of $\mathcal{S}$. It is a consequence of the following theorem that the endomorphism ring of a slice module is always hereditary.

THEOREM [65]. *If B is hereditary, and T_B is a tilting module with* $A = \operatorname{End} T_B$, *then the class of all indecomposable A-modules of the form* $\operatorname{Hom}_B(T, I)$, *with I_B indecomposable injective, is a complete slice in* $\operatorname{mod} A$. *Conversely, if $\mathcal{S}$ is a complete slice in* $\operatorname{mod} A$, *then $U_A = \bigoplus_{M\in\mathcal{S}} M$ is a tilting module with $B = \operatorname{End} U_A$ hereditary and thus $\mathcal{S}$ is isomorphic to a complete slice of the previous form.*

Examples of complete slices which do not occur in a preprojective or a preinjective component can be found among the tame one-relation algebras obtained by glueings, as described in [64], 2.7.

Remark and example. The previous theorem allows one not only to recover the hereditary algebra from which a given tilted algebra derives, but also the corresponding tilting module. Indeed, consider the following easy example: let A be given by the quiver of Fig. 16 bound by $\alpha\beta = \gamma\delta$, $\mu\beta = 0$ and $\lambda\gamma = 0$. Then

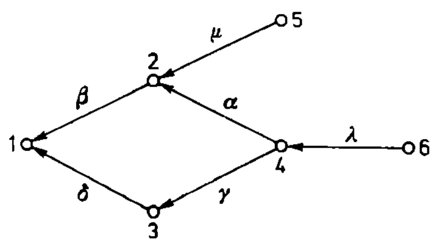

Fig. 16

Γ_A is illustrated in Fig. 17. As shown, it contains a complete slice $\mathcal{S}$ of type $\mathbf{E}_6$. The endomorphism ring of the slice module of $\mathcal{S}$ is the hereditary algebra B given by the quiver of Fig. 18 and Γ_B is shown in Fig. 19. Now it is known

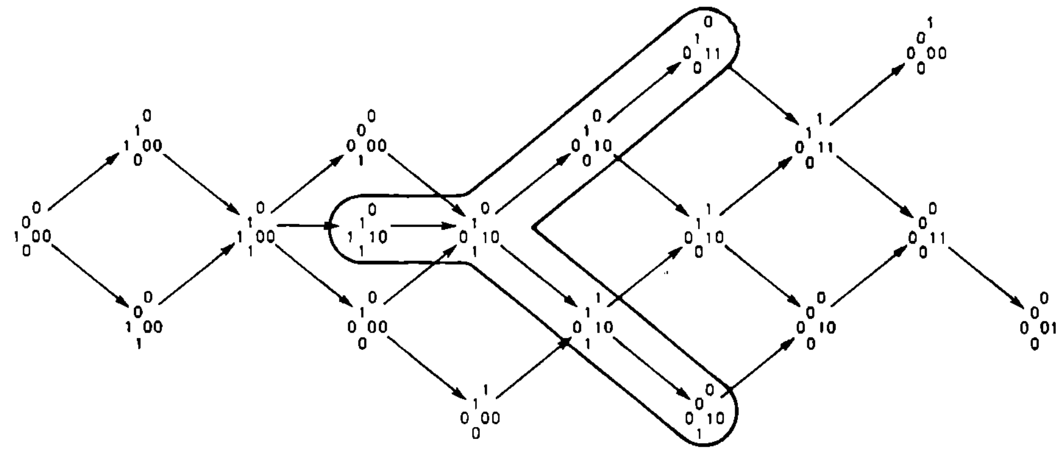

Fig. 17

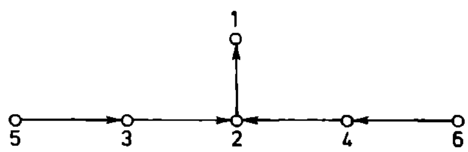

Fig. 18

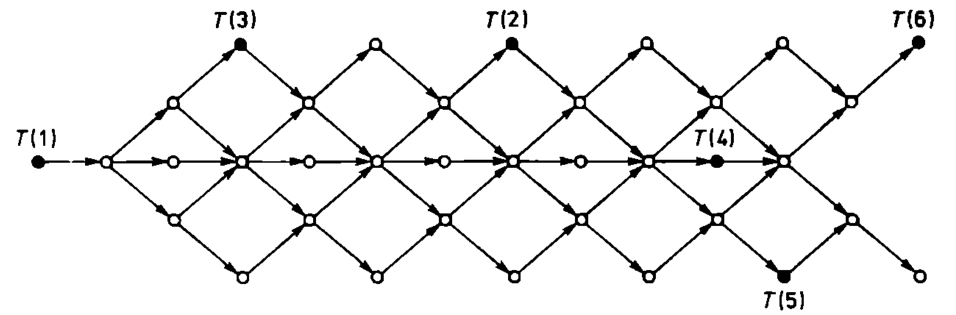

Fig. 19

that $\mathcal{S}$ is of the form $\mathrm{Hom}_B(T, I(i))$, for i a point in the quiver of B. Thus

$$\mathrm{Hom}_B(T, I(1)) = \begin{smallmatrix} & 0 & & \\ 1 & 1 & 1 & 0 \\ & 1 & & \end{smallmatrix}, \quad \mathrm{Hom}_B(T, I(4)) = \begin{smallmatrix} & 0 & & \\ 0 & 1 & 1 & 0 \\ & 0 & & \end{smallmatrix},$$

$$\mathrm{Hom}_B(T, I(2)) = \begin{smallmatrix} & 0 & & \\ 0 & 1 & 1 & 0 \\ & 1 & & \end{smallmatrix}, \quad \mathrm{Hom}_B(T, I(5)) = \begin{smallmatrix} & 0 & & \\ 0 & 0 & 1 & 0 \\ & 1 & & \end{smallmatrix},$$

$$\mathrm{Hom}_B(T, I(3)) = \begin{smallmatrix} & & 1 & \\ 0 & 1 & 1 & 0 \\ & 1 & & \end{smallmatrix}, \qquad \mathrm{Hom}_B(T, I(6)) = \begin{smallmatrix} & & 0 & \\ 0 & 1 & 1 & 1 \\ & 0 & & \end{smallmatrix},$$

and so the indecomposable summands $T(j)$ of T_B (where $T(j)$ corresponds to the point j in the quiver of A) are given by

$$T(1) = \begin{smallmatrix} & & 1 & & \\ 0&0&0&0&0 \end{smallmatrix}, \qquad T(4) = \begin{smallmatrix} & & 1 & & \\ 1&1&1&1&1 \end{smallmatrix},$$
$$T(2) = \begin{smallmatrix} & & 1 & & \\ 0&1&1&1&1 \end{smallmatrix}, \qquad T(5) = \begin{smallmatrix} & & 0 & & \\ 0&1&0&0&0 \end{smallmatrix},$$
$$T(3) = \begin{smallmatrix} & & 1 & & \\ 1&1&1&0&0 \end{smallmatrix}, \qquad T(6) = \begin{smallmatrix} & & 0 & & \\ 0&0&0&0&1 \end{smallmatrix}.$$

The above theorem may be applied in particular to the following case. Let us recall from [65] that a *cycle* in $\mathrm{mod}\, A$ is a sequence of non-zero non-isomorphisms $X_0 \to X_1 \to \ldots \to X_t = X_0$ between indecomposable A-modules. We say that $\mathrm{mod}\, A$ is *directed* if no indecomposable A-module lies on a cycle. It is easily seen that, if $\mathrm{mod}\, A$ is directed, then A is representation-finite (see [65], 2.4, (9′)). We have

COROLLARY [43], [65]. *Let A be such that* $\mathrm{mod}\, A$ *is directed, and assume that A has a sincere indecomposable module. Then A is a tilted algebra.*

For the proof, we fix a sincere indecomposable module M_A, and consider the set $\mathscr{S}$ of all indecomposable A-modules X such that there exists a sequence of non-zero non-isomorphisms $X \to \ldots \to M$, and moreover, no such sequence is of the form $X \to \ldots \to \tau N \to * \to N \to \ldots \to M$. We then check that $\mathscr{S}$ satisfies the axioms of a complete slice.

A tilted algebra as in the Corollary is called a *sincere directed* algebra. This class of tilted algebras has been extensively studied. We refer the reader to [65], 6, for more results and comments.

5.3. The following characterisation of tilted algebras is due to Bakke [19].

THEOREM. *Let A be an algebra. The following conditions are equivalent*:

(i) *A is a tilted algebra.*

(ii) *There exists a separating tilting module T_A such that, for all $M \in \mathscr{T}(T)$,* $\mathrm{Hom}_A(\tau^{-1}M, T) = 0$.

(iii) *There exists a separating tilting module T_A such that, for all $M \in \mathscr{T}(T)$,* $\mathrm{id}\, M \leqslant 1$.

(iv) *There exists a sincere A-module M such that there is no chain of non-zero morphisms between indecomposable modules of the form*

$$M' \to \ldots \to N \to * \to \tau^{-1}N \to \ldots \to M''$$

with M' and M'' in $\mathrm{add}(M)$.

Proof. (iii) $\Rightarrow$ (ii). Since $\operatorname{id} M \leqslant 1$ and $M \in \mathscr{T}(T)$, we have

$$\operatorname{Hom}_A(\tau^{-1}M, T) = D\operatorname{Ext}_A^1(T, M) = 0.$$

(ii) $\Rightarrow$ (i). We claim that $B = \operatorname{End} T_A$ is hereditary, and for this, it suffices to show that any $Y_B \in \mathscr{Y}(T)$ such that $\operatorname{Hom}_B(Y, B) \neq 0$ is projective. This amounts to proving that any $M \in \mathscr{T}(T)$ such that $\operatorname{Hom}_A(M, T) \neq 0$ lies in $\operatorname{add}(T)$. This is clear if M is projective. So suppose it is not. Then $\tau M \notin \mathscr{T}(T)$ since otherwise we have a contradiction to $\operatorname{Hom}_A(M, T) = \operatorname{Hom}_A(\tau^{-1}(\tau M), T) = 0$. Therefore, since T is separating, we have $\tau M \in \mathscr{F}(T)$. But then, by 1.4, M is Ext-projective in $\mathscr{T}(T)$, so it lies in $\operatorname{add}(T)$.

(i) $\Rightarrow$ (iv). Let $\mathscr{S}$ be a complete slice in $\operatorname{mod} A$. Then $M = \bigoplus_{N \in \mathscr{S}} N$ obviously satisfies (iv).

(iv) $\Rightarrow$ (iii). We first define a splitting torsion theory $(\mathscr{T}, \mathscr{F})$ in $\operatorname{mod} A$, then show it is induced by a tilting module. Let $\mathscr{T}$ be the full additive subcategory of $\operatorname{mod} A$, generated by the indecomposable modules N such that there is a chain of nonzero morphisms between indecomposable modules of the form $M' \to \ldots \to N$, and let $\mathscr{F}$ be generated by the remaining indecomposables. Clearly, $(\mathscr{T}, \mathscr{F})$ is a torsion theory, and is splitting by definition. Moreover, since M is sincere, $DA \in \mathscr{T}$.

Observe that, by the given condition, we have for all $N \in \mathscr{T}$

$$\operatorname{Ext}_A^1(M, N) \simeq D\underline{\operatorname{Hom}}_A(\tau^{-1}N, M) = 0.$$

Therefore M is Ext-projective in $\mathscr{T}$. Now, any Ext-projective T_0 in $\mathscr{T}$ is a partial tilting module: indeed, $\operatorname{Ext}_A^1(T_0, T_0) = 0$ by definition, while $\tau T_0 \in \mathscr{T}$ and $DA \in \mathscr{T}$ yield $\operatorname{Hom}_A(DA, \tau T_0) = 0$, so that $\operatorname{pd} T_0 \leqslant 1$. This implies that the number of non-isomorphic indecomposable Ext-projectives in T is finite and does not exceed the rank n of $K_0(A)$. For, otherwise, there exists an Ext-projective module T_0 with $t > n$ non-isomorphic indecomposable summands. Since T_0 is a partial tilting module, by 1.7, there exists a module X such that $T_0 \oplus X$ is a tilting module, thus has n indecomposable summands, a contradiction.

Let thus T_A denote the direct sum of a complete set of non-isomorphic indecomposable Ext-projectives in $\mathscr{T}$. Since T is a partial tilting module, there exists, by 1.7, a short exact sequence

$$(*) \qquad 0 \to A_A \xrightarrow{f} E_A \xrightarrow{g} T_A^{(d)} \to 0,$$

where $d = \dim_k \operatorname{Ext}_A^1(T, A)$. Since $(\mathscr{T}, \mathscr{F})$ is a splitting torsion theory, $E = X \oplus Y$ with $X \in \mathscr{T}$ and $Y \in \mathscr{F}$.

We claim that $Y = 0$. Indeed, assume $Y \neq 0$. Then $\operatorname{Hom}_A(Y, T) \neq 0$. For, if the restriction to Y of the morphism g of $(*)$ is zero, then Y_A is projective and so $\operatorname{Hom}_A(Y, M) \neq 0$ (because M is sincere), thus $\operatorname{Hom}_A(Y, T) \neq 0$. On the other hand,

$$\operatorname{Hom}_A(Y, \tau T) \simeq D\operatorname{Ext}_A^1(T, Y) = 0.$$

Let thus v_1: $Y \to T_1$ be a non-zero morphism, with $T_1 \in \mathrm{add}(T)$ indecomposable. We shall obtain inductively morphisms v_i: $Y \to T_i$ and irreducible morphisms u_i: $T_{i+1} \to T_i$ with $T_i \in \mathrm{add}(T)$ indecomposables such that $u_1 u_2 \ldots u_{i-1} v_i \neq 0$. Indeed, since v_i is not a retraction, it factors through the minimal almost split morphism $K_i \oplus L_i \to T_i$, where $K_i \in \mathscr{T}$ and $L_i \in \mathscr{F}$. Observe that $L_i \in \mathrm{add}(\tau T)$; for, $L_i \in \mathscr{F}$ is not injective, so $\mathrm{Hom}_A(T_i, \tau^{-1} L_i) \neq 0$ implies $\tau^{-1} L_i \in \mathscr{T}$, thus $\tau^{-1} L_i$ is Ext-projective in $\mathscr{T}$, that is, $\tau^{-1} L_i \in \mathrm{add}(T)$. Since $\mathrm{Hom}_A(Y, \tau T) = 0$, we infer that v_i: $Y \to T_i$ factors through u_i: $K_i \to T_i$. On the other hand, $K_i \in \mathrm{add}(T)$. This is clear if K_i is projective, so suppose it is not: if T_i is not projective, then $\mathrm{Hom}_A(\tau K_i, \tau T_i) \neq 0$ implies that $\tau K_i \in \mathscr{F}$ so that K_i is Ext-projective in $\mathscr{T}$, while, if T_i is projective, the sincerity of M implies that $\mathrm{Hom}_A(T_i, M) \neq 0$, so, if $\tau K_i \in \mathscr{F}$, we would get a chain of non-zero morphisms between indecomposable modules of the form

$$M' \to \ldots \to \tau K_i \to * \to K_i \to T_i \to M''$$

with $M', M'' \in \mathrm{add}(M)$, a contradiction which shows that $\tau K_i \in \mathscr{F}$ and so, again, that K_i is Ext-projective in $\mathscr{T}$. We thus set $T_{i+1} = K_i$ and let v_{i+1}: $Y \to T_{i+1}$ be such that $v_i = u_i v_{i+1}$. This yields the wanted morphisms, and therefore a contradiction, since all the u_i belong to $\mathrm{rad}\,\mathrm{End}(T)$, which is a nilpotent ideal. Thus $Y = 0$.

This implies that the short exact sequence (*) is of the form

$$0 \to A_A \to X_A \to T_A^{(d)} \to 0$$

with $X_A \in \mathscr{T}$. Applying $\mathrm{Ext}_A^1(-, N)$, with $N \in \mathscr{T}$, to this sequence yields

$$0 = \mathrm{Ext}_A^1(T^{(d)}, N) \to \mathrm{Ext}_A^1(X, N) \to \mathrm{Ext}_A^1(A, N) = 0$$

and so X is Ext-projective in $\mathscr{T}$, thus lies in $\mathrm{add}(T)$. This shows that T_A is indeed a tilting module.

We now show that $(\mathscr{T}, \mathscr{F}) = (\mathscr{T}(T), \mathscr{F}(T))$. For, let $N \in \mathscr{T}$; since T_A is Ext-projective in $\mathscr{T}$, we have $\mathrm{Ext}_A^1(T, N) = 0$ and so $N \in \mathscr{T}(T)$. Conversely, if $N \in \mathscr{T}(T)$, then there is an epimorphism $T' \to N$ with $T' \in \mathrm{add}(T) \subseteq \mathscr{T}$, thus $N \in \mathscr{T}$, because $\mathscr{T}$ is a torsion class. Hence $\mathscr{T} = \mathscr{T}(T)$, and so $\mathscr{F} = \mathscr{F}(T)$.

It remains to prove that $\mathrm{id}\, N \leqslant 1$ for all $N \in \mathscr{T}$. If this is not the case for some $N \in \mathscr{T}$, then $\mathrm{Hom}_A(\tau^{-1} N, P) \neq 0$ for some indecomposable projective A-module P. Since M is sincere, there exists an indecomposable summand $M'' \in \mathrm{add}(M)$ such that $\mathrm{Hom}_A(P, M'') \neq 0$. On the other hand, since $N \in \mathscr{T}$, there exists a chain of non-zero morphisms between indecomposable modules of the form

$$M' \to \ldots \to N \to * \to \tau^{-1} N \to P \to M''$$

with $M', M'' \in \mathrm{add}(M)$, a contradiction which completes the proof.

Remark. Corollary 5.2 above follows directly from the implication (iv) $\Rightarrow$ (i).

5.4. Classification theorems are known for various classes of tilted algebras. If $\vec{\Delta}$ is a Dynkin quiver, so that $k\vec{\Delta}$ is a representation-finite hereditary algebra, a complete classification exists in the cases where $\Delta = \mathbf{A}_n$ (see [1], [31]) and $\Delta = \mathbf{D}_n$ (see [31]).

Let now $\vec{\Delta}$ be an Euclidean quiver, so that $B = k\vec{\Delta}$ is a representation-infinite tame hereditary algebra. The essential properties of the tilting B-modules are given in [44]. In particular, it is shown that no tilting module is regular (also, no regular homogeneous module can be a summand of a tilting module). Thus, any tilting module must contain a non-zero preprojective or preinjective direct summand, so we have, up to duality, the following possibilities for a tilting module:

(i) T_B is preprojective.

(ii) $T_B = T_1 \oplus T_2$ with $T_1 \neq 0$ preprojective and $T_2 \neq 0$ regular.

(iii) $T_B = T_1 \oplus T_2 \oplus T_3$ with $T_1 \neq 0$ preprojective, T_2 regular and $T_3 \neq 0$ preinjective.

It is shown in [44] that the tilted algebra $A = \operatorname{End} T_B$ is representation-finite if and only if T_B is of the form (iii) above. The cases (i) and (ii) correspond to representation-infinite tilted algebras. The case (i) is the case of the so-called tame concealed algebras: an algebra A is called *tame* (respectively, *wild*) *concealed* if it is the endomorphism ring of a preprojective (or preinjective) tilting module over a tame (respectively, wild) hereditary algebra. Concealed algebras are characterised as follows:

THEOREM. *An algebra A is concealed if and only if there exist two different components of the Auslander–Reiten quiver of A containing a complete slice.*

For the proof of the necessity, we refer the reader to [65], 4.3, and for the proof of the sufficiency, to [40], **III**, (7.2).

The tame concealed algebras were classified by Happel and Vossieck [47] and also, independently, by Bongartz [23]; these algebras can be used to give an effective criterion allowing one to decide whether an algebra is representation-finite or not [22].

A constructive procedure allowing one to construct all representation-infinite tilted algebras of Euclidean type is given in [65], 4.9. A complete classification of the tilted algebras of type $\vec{\Delta}$, for $\Delta = \tilde{\mathbf{A}}_n$, including the representation-finite case, has been obtained by Roldán [67] (see also [8]). The calculation of the torsion and torsion-free classes if the tilting module has non-zero regular summands is given in [61].

The situation is known to be completely different if $\vec{\Delta}$ is a wild quiver. In fact, one has the following theorem, due to Ringel.

THEOREM [66]. *Let B be a wild hereditary algebra with at least three non-isomorphic simple modules. Then there exists a tilting module T_B with only regular direct summands.*

A constructive procedure for such regular tilting modules is given in [18] using the technique of perpendicular categories [36]. Using the same technique, Strauss has shown in [72] that, if a tilted algebra of wild type is not wild, then the tilting module has a non-zero preprojective and a non-zero preinjective direct summand. He also shows that a tilted algebra of wild type has a projective component if the tilting module has no preinjective direct summand. In [53], Kerner shows how to reduce the general case to the latter case, then, in [54], he studies the preprojective components of the tilted algebras arising in this way. His reduction also allows him to obtain characterisations of the tame and wild tilted algebras of wild type [53]. Certain classes of wild concealed algebras were classified in [77] and also, independently, in [55]. The wild concealed algebras always have a representation-infinite concealed factor algebra:

THEOREM [46]. *Let A be a concealed algebra of type $\vec{\Delta}$, where $\vec{\Delta}$ is a wild quiver having more than 2 vertices. Then there exists a primitive idempotent e of A such that $A/\langle e\rangle$ is representation-infinite concealed of type $\vec{\Delta}'$, and $\vec{\Delta}'$ is a full connected subquiver of $\vec{\Delta}$.*

6. Iterated tilted algebras

6.1. We shall now embed the class of tilted algebras into a wider class, closed under the tilting process.

DEFINITION [5]. Let $\vec{\Delta}$ be a finite connected quiver without oriented cycles. An algebra A is called an *iterated tilted algebra of type* $\vec{\Delta}$ if there exist a sequence of algebras $A = A_0, A_1, \ldots, A_m = k\vec{\Delta}$ and a sequence $T_{A_i}^{(i)}$ $(0 \leqslant i < m)$ of separating tilting modules such that $A_{i+1} = \operatorname{End} T_{A_i}^{(i)}$ for each i.

It follows directly from the definition that, if $\vec{\Delta}$ is a Dynkin quiver, then an iterated tilted algebra of type $\vec{\Delta}$ is necessarily representation-finite (moreover, by 4.4, it is even simply connected). Similarly, if $\vec{\Delta}$ is an Euclidean quiver, then an iterated tilted algebra of type $\vec{\Delta}$ is either representation-finite or tame representation-infinite (and in the latter case, it is even domestic and 1-parametric). Also, by 2.7, the quadratic form of an iterated tilted algebra of type $\vec{\Delta}$ is congruent to that of $k\vec{\Delta}$. Consequently, it is either positive definite (if $\vec{\Delta}$ is a Dynkin quiver), positive semidefinite of corank one (if $\vec{\Delta}$ is an Euclidean quiver), or indefinite (if $\vec{\Delta}$ is a wild quiver). Finally, by 2.2, iterated tilted algebras have finite global dimension.

EXAMPLE. The algebra A given by the quiver of Fig. 20 bound by $\alpha\beta = \gamma\delta$, $\mu\lambda = 0$, $\lambda\alpha = 0$, $\lambda\gamma = 0$, $\nu\zeta = 0$ and $\zeta\delta = 0$ is an iterated tilted algebra of type $\tilde{\mathbf{E}}_7$ (actually, A is representation-finite of global dimension four, as is easy to check). Indeed, letting $A = A_0$ and putting

$$T_{A_0}^{(0)} = \tau^{-2}\{P(1)\oplus P(2)\oplus P(3)\}\oplus\tau^{-1}P(5)\oplus P(4)\oplus P(6)\oplus P(7)\oplus P(8)$$

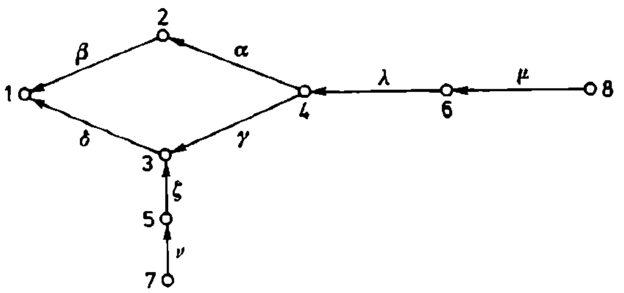

Fig. 20

we see at once that $T^{(0)}_{A_0}$ is a separating tilting module and that $A_1 = \operatorname{End} T^{(0)}_{A_0}$ is given by the quiver of Fig. 21 bound by $\alpha\beta\gamma = \delta\varepsilon$, $\mu\gamma = 0$, $\nu\alpha = 0$, $\nu\delta = 0$.

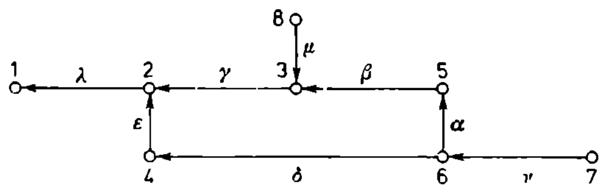

Fig. 21

Next, let

$$T^{(1)}_{A_1} = \tau^{-5}P(1) \oplus \tau^{-4}\{P(2) \oplus P(3) \oplus P(5)\} \oplus \tau^{-3}P(4) \oplus P(6) \oplus P(7) \oplus \tau^{-2}P(8).$$

Then $T^{(1)}_{A_1}$ is again a separating tilting module and its endomorphism algebra $A_2 = \operatorname{End} T^{(1)}_{A_1}$ is given by the commutative quiver of Fig. 22. Finally, the

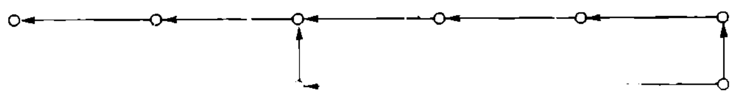

Fig. 22

Auslander–Reiten quiver of A_2 contains a complete slice of type $\tilde{\mathbf{E}}_7$, so that A_2 is a tilted algebra of that type.

6.2. Let A be an iterated tilted algebra of type $\vec{\Delta}$. Then it follows from 4.3 that $D^b(A) \simeq D^b(k\vec{\Delta})$ as triangulated categories. This made it possible to use our knowledge of the structure of the derived category of a hereditary algebra, described by Happel in [39], in order to obtain information on the iterated tilted algebras.

PROPOSITION. *Let A be an iterated tilted algebra. Then A is triangular. If moreover A is representation-finite, then* $\operatorname{mod} A$ *is directed.*

For the proof, we refer to [39]. Another proof of the first part, using the repetitive algebra and covering techniques, is due to Skowroński [68].

It was natural to ask whether, conversely, an algebra A such that $D^b(A) \simeq D^b(k\vec{\Delta})$ as triangulated categories is necessarily iterated tilted of type $\vec{\Delta}$. This was proved by Happel in [39] in the case where $\vec{\Delta}$ is a Dynkin quiver, and in [38] in the case where $\vec{\Delta}$ is an Euclidean quiver. It also follows from the proof of the latter result that, if A is iterated tilted of type $\vec{\Delta}$, where $\vec{\Delta}$ is a Dynkin (respectively, a Euclidean) quiver, then A may be transformed to a hereditary algebra of Dynkin (respectively, Euclidean) type by a finite sequence of APR-tilting modules (respectively, a finite sequence of APR-tilting modules followed by a finite sequence of APR-cotilting modules). In the general case, we have the following result of Happel, Rickard and Schofield [42].

THEOREM. *Let $\vec{\Delta}$ be a finite connected quiver without oriented cycles, and let A be an algebra. The following conditions are equivalent*:

(i) *A is an iterated tilted algebra of type $\vec{\Delta}$.*
(ii) *A is tilting-cotilting equivalent to $k\vec{\Delta}$.*
(iii) *$D^b(A) \simeq D^b(k\vec{\Delta})$ as triangulated categories.*

The proof is done by induction, using perpendicular categories. In particular, it follows from this theorem that the class of iterated tilted algebras is closed under the tilting process. Also, an algebra A is iterated tilted of type $\vec{\Delta}$ if and only if the opposite algebra A^{op} is iterated tilted of type $\vec{\Delta}$.

6.3. In [39], Happel shows that one can define for a triangulated category a notion of Auslander–Reiten triangles which extends, in an obvious way, the notion of Auslander–Reiten sequences in mod A. He then proves that $D^b(A)$ has Auslander–Reiten triangles. We may thus define the *quiver* of $D^b(A)$ to have as vertices the isomorphism classes $[P_\bullet]$ of the indecomposable complexes P in $D^b(A)$ and to have an arrow $[P_\bullet] \to [Q_\bullet]$ whenever there is an irreducible morphism $P_\bullet \to Q_\bullet$ in $D^b(A)$. A component Γ of the quiver of $D^b(A)$ is called a *tube* (see [65]) if it has no multiple arrows, contains a cyclic path and its geometric realization $|\Gamma| = S^1 \times \mathbf{R}_0^+$ (where S^1 is the unit circle and $\mathbf{R}_0^+$ the set of non-negative real numbers). Finally, we shall say that $D^b(A)$ is *cycle-finite* if, for any sequence of non-zero non-isomorphisms $P_\bullet^0 \to P_\bullet^1 \to \dots \to P_\bullet^t = P_\bullet^0$ between indecomposable objects in $D^b(A)$, the objects $P_\bullet^j$ lie in one tube of $D^b(A)$.

It follows from the structure of the quiver of $D^b(C)$ for C hereditary of Dynkin or Euclidean type (see [39]), or for C tubular canonical (see [45]) that in both of these cases, $D^b(C)$ is cycle-finite. Therefore, if A is tilting-cotilting equivalent to an algebra C as before, then $D^b(A)$ is also cycle-finite. Surprisingly, the converse is also true.

THEOREM [10]. *Let A be an algebra. The following conditions are equivalent*:

(i) *$D^b(A)$ is cycle-finite.*

(ii) *There exists an algebra C which is either hereditary of Dynkin or Euclidean type, or else tubular canonical, such that $D^b(A) \simeq D^b(C)$ as triangulated categories.*

(iii) *There exists an algebra C which is either hereditary of Dynkin or Euclidean type, or else tubular canonical, such that A and C are tilting-cotilting equivalent,* An easy and natural corollary of this theorem and 6.2 is

COROLLARY. *Let C be a tubular canonical algebra. Then* $D^b(A) \simeq D^b(C)$ *as triangulated categories if and only if A and C are tilting-cotilting equivalent.*

6.4. Another characterisation of the same classes of algebras uses repetitive algebras. Recall from 3.4 that the repetitive algebra $\hat{A}$ of A is a Galois covering of the trivial extension $A \ltimes DA$ with the infinite cyclic group generated by the automorphism ν_A, and $\mathrm{mod}\,\hat{A}$ is equivalent to the category of $\mathbf{Z}$-graded $(A \ltimes DA)$-modules. We shall say that $\hat{A}$ is *exhaustive* provided the pushdown functor $\mathrm{mod}\,\hat{A} \to \mathrm{mod}(A \ltimes DA)$ associated with the covering $\hat{A} \to A \ltimes DA$ (see [24]) is dense, that is, every $(A \ltimes DA)$-module is gradable. We may now state

THEOREM [13]. *Let A be an algebra. The following conditions are equivalent:*

(i) $\hat{A}$ *is tame and exhaustive.*

(ii) *There exists an algebra B which is tilted of Dynkin type or representation-infinite tilted of Euclidean type, or tubular, such that* $\hat{A} \simeq \hat{B}$.

(iii) *There exists an algebra C which is either hereditary of Dynkin type, or of Euclidean type, or tubular canonical, such that A and C are tilting-cotilting equivalent.*

6.5. Again, many classes of iterated tilted algebras are completely classified. The following result shows that, if $\vec{\Delta}$ is a Dynkin or an Euclidean quiver, then the classification splits into two cases, the simply connected case, and the case $\tilde{\mathbf{A}}_n$.

THEOREM [9]. *Let A be an algebra such that* $D^b(A)$ *is cycle-finite. Then A is simply connected if and only if A is not an iterated tilted algebra of type* $\tilde{\mathbf{A}}_n$.

The same statement holds true if the assumption that $D^b(A)$ is cycle-finite is replaced by the assumption that $\hat{A}$ is tame and exhaustive [13]. This theorem shows that, if A is iterated tilted of Dynkin or Euclidean type $\neq \tilde{\mathbf{A}}_n$, then A is simply connected. The iterated tilted algebras of type $\tilde{\mathbf{A}}_n$ are completely classified in [8]. A constructive procedure allowing one to construct the representation-infinite iterated tilted algebras of Euclidean type, and the representation-infinite algebras which are tilting-cotilting equivalent to a tubular canonical algebra, is given in [10]. The following handy criterion allows one to decide whether or not a representation-finite algebra is iterated tilted of Dynkin type or of Euclidean type $\neq \tilde{\mathbf{A}}_n$.

THEOREM [11]. *Let A be a representation-finite algebra.*

(i) *A is iterated tilted of Dynkin type if and only if A is simply connected and its quadratic form* q_A *is positive definite.*

(ii) *A is iterated tilted of Euclidean type* $\neq \tilde{\mathbf{A}}_n$ *if and only if A is simply connected and* q_A *is positive semidefinite of corank one.*

Part (i) was also obtained independently by Happel (private communication). No similar characterisation is presently known for the representation-finite algebras which are tilting-cotilting equivalent to a tubular canonical algebra. The iterated tilted algebras of Dynkin type $\mathbf{A}_n$ were classified in [5], those of Dynkin type $\mathbf{D}_n$ were classified in terms of their bound quivers in [12] and in terms of their Auslander–Reiten quivers in [82]. Also, S. Brenner has obtained a simple and effective combinatorial construction which determines if an algebra can be tilted to a hereditary algebra by a finite sequence of APR-tilting modules (thus, in particular, if an algebra is iterated tilted of Dynkin type) and, if this is the case, allows one at the same time to determine the type of the iterated tilted algebra [26]. For other characterizations of the iterated tilted algebras of Dynkin type, we refer the reader to [37] and [39]. Almost nothing is known about the iterated tilted algebras of wild type.

The following conjecture, due to Roldán, would allow one, if true, to reduce the study of the representation-finite iterated tilted algebras of Euclidean or wild type to the study of the representation-finite tilted algebras of that type. Let A be a representation-finite iterated tilted algebra of type $\vec{\Delta}$, where $\vec{\Delta}$ is not a Dynkin quiver. Then the conjecture states that there exists a sequence of tilts $A = A_0, A_1, \ldots, A_{m-1}, A_m = k\vec{\Delta}$, as in Definition 6.1, but with A_{m-1} representation-finite.

6.6. Tilted and iterated tilted algebras are applied naturally in the classification of the self-injective algebras of polynomial growth [69], and in particular of the trivial extension algebras. Recall first that the trivial extension $A \ltimes DA$ of A is a symmetric algebra. Thus, by Riedtmann's theorem [63], if $A \ltimes DA$ is representation-finite, then the stable part of its Auslander–Reiten quiver is isomorphic to $\mathbf{Z}\Delta/G$, where Δ is a Dynkin diagram (called the *Cartan class* of $A \ltimes DA$), and G is an admissible group of automorphisms of $\mathbf{Z}\Delta$. Then we have

THEOREM. *Let A be an algebra. The following conditions are equivalent*:

(i) *$A \ltimes DA$ is representation-finite of Cartan class Δ.*

(ii) *There exists a tilted algebra B of Dynkin type $\vec{\Delta}$ such that $A \ltimes DA \xrightarrow{\sim} B \ltimes DB$.*

(iii) *A is an iterated tilted algebra of Dynkin type $\vec{\Delta}$.*

Here, $\vec{\Delta}$ is an arbitrary orientation of the Dynkin diagram Δ. Indeed, since the Dynkin diagrams are trees, all orientations of Δ give rise to the same strip $\mathbf{Z}\Delta$.

The equivalence of (i) and (ii) was shown in [52], [28], [48], and the equivalence of (i) and (iii) in [6].

Proof of (i)$\Leftrightarrow$(iii). If A is an iterated tilted algebra of Dynkin type $\vec{\Delta}$ and $H = k\vec{\Delta}$, then we have $\underline{\mathrm{mod}}(A \ltimes DA) \xrightarrow{\sim} \underline{\mathrm{mod}}(H \ltimes DH)$ by 3.4. Since $H \ltimes DH$

is representation-finite of Cartan class Δ, by [73], so is $A \ltimes DA$. Thus (iii) implies (i).

Conversely, let $A \ltimes DA$ be representation-finite of Cartan class Δ. Then A is also representation-finite. Let $n(A)$ denote the number of isomorphism classes of indecomposable A-modules. It follows from [81] that A is triangular. Therefore, by 3.7, if A is not hereditary, then there exists a separating tilting module T_A which is not splitting. Let $B = \operatorname{End} T_A$. Since we have $\underline{\mathrm{mod}}(A \ltimes DA) \simeq \underline{\mathrm{mod}}(B \ltimes DB)$ by 3.4, $B \ltimes DB$ is representation-finite, and hence so is B. Since T_A is separating but not splitting, we have $n(B) > n(A)$. Repeating this operation, we arrive, by induction, at a hereditary algebra H which is necessarily representation-finite. (Here, we are using the fact that there are only finitely many non-isomorphic representation-finite algebras with the same number of simple modules.) Finally, since the Cartan class of $H \ltimes DH$ equals that of $A \ltimes DA$, namely Δ, H is hereditary of type $\vec{\Delta}$ for some orientation of Δ.

6.7. It was natural to expect similar results in the tame representation-infinite case. Actually, Tachikawa showed that, if H is hereditary of Euclidean type, then $H \ltimes DH$ is domestic and even 2-parametric [73]. The converse, however, is not true. Indeed, let A be given by the quiver $\circ \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \circ$ bound by $\alpha\beta = 0$ and $\beta\alpha = 0$. Then it is easily seen that $A \ltimes DA \simeq H \ltimes DH$, where H is the path algebra of the Kronecker quiver $\circ \rightrightarrows \circ$. In particular, $A \ltimes DA$ is 2-parametric, but A is not iterated tilted, because it is not triangular (see 6.2). We thus have to split the classification into two cases, according as A is simply connected or not. In the simply connected case, the following two theorems characterise the trivial extension algebras of polynomial growth.

THEOREM [7]. *Let A be a simply connected algebra. The following conditions are equivalent:*

(i) *$A \ltimes DA$ is domestic and representation-infinite.*

(ii) *$A \ltimes DA$ is 2-parametric.*

(iii) *There exists a representation-infinite tilted algebra B of type $\tilde{\mathbf{D}}_n$ or $\tilde{\mathbf{E}}_p$ such that $A \ltimes DA \simeq B \ltimes DB$.*

(iv) *A is iterated tilted of type $\tilde{\mathbf{D}}_n$ or $\tilde{\mathbf{E}}_p$.*

THEOREM [60]. *Let A be a simply connected algebra. The following conditions are equivalent:*

(i) *$A \ltimes DA$ is non-domestic of polynomial growth.*

(ii) *There exists a tubular algebra B such that $A \ltimes DA \simeq B \ltimes DB$.*

(iii) *There exists a tubular canonical algebra C such that A and C are tilting-cotilting equivalent.*

Finally, the polynomial growth trivial extensions of non-simply connected algebras were classified by J. Nehring [59].

Acknowledgement

The author gratefully acknowledges support from the Natural Sciences and Engineering Research Council of Canada.

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