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# ON PRACTICAL STABILITY AND OPTIMAL STABILIZATION OF CONTROLLED MOTION

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The theory of optimal control of the processes in dynamic systems is presented in a number of fundamental monographs [20], [4], [6], [8], [11]. One of the central problems of the theory of optimal control is the problem of construction of the controlling effects, ensuring the realization of the programmed process. This problem is closely connected with that of stability of motion and is a further development of the theory of stability in the control of motion problems.

A number of problems in the theory of optimal control from the point of view of the practical stability of motion theory [5], [7], [10], [15], [18], [24] is successively presented here on the basis of the Lyapunov's function method [12] and the principle of comparison [13]–[17], [23].

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## § 1. Problems analysed

We consider equations of controlled motion in the form

$$\frac{dx}{dt} = X(t, x) + H(t, u), \tag{1.1}$$

where $x \in R^n$ is a vector-parameter characterizing motion, $u(t) \in U \subset R^m$ are the values characterizing the controlling forces. Let $x = \psi(t)$ be the programmed trajectory, the motion along which must be ensured by an appropriate choice of forces $u_1(t), \ldots, u_m(t)$. We shall make a substitution $z = x - \psi(t)$ in system (1.1); the new equation will be the following:

$$\frac{dz}{dt} = F(t, z) + \Delta(\psi, u, t), \tag{1.2}$$

where

$$\begin{aligned} F(t, z) &= X\big(t, z + \psi(t)\big) - X\big(t, \psi(t)\big), \\ \beta(t, \psi) &= \dot{\psi}(t) - X\big(t, \psi(t)\big), \\ \Delta(\psi, u, t) &= H(t, u) - \beta(t, \psi). \end{aligned}$$

It is obvious that $F(t, 0) = 0$ and the solution $z = 0$ corresponds to the unperturbed motion of the system

$$\frac{dz}{dt} = F(t, z). \tag{1.3}$$

Let us suppose that systems (1.1) and (1.3) are defined in the domain $\Omega$:

$$t \geqslant 0, \quad \|x\| < H, \quad H = \text{const} > 0,$$

where $\|\cdot\|$ is the Euclidean norm and there exist solutions at $(t_0, x_0 = z_0) \in \operatorname{int} \Omega$ and they are unique.

We shall consider the practical stability of unperturbed motion of systems (1.3) or (1.2) with respect to domains $S_0(t)$, $S(t)$, $I$. Here $S_0(t)$, $S(t)$ are connected, open, bounded sets of the space $R^n$, and they are continuous at all $t \in I$. Let $\bar{S}(t)$ be the closure, $\partial S(t)$ the boundary of the set $S(t)$; suppose that

$$S_0(t) \subset S(t) \ \& \ \partial S_0 \cap \partial S = \emptyset \qquad \forall t \in I \stackrel{\Delta}{=} [t_0, +\infty)$$

and denote

$$S(t) \setminus S_0(t) = \{x \in R^n, x \in S(t), x \notin S_0(t)\}.$$

So, we shall consider the following two problems [6], [16], [17].

PROBLEM I (*On practical stabilization of programmed motion*). Let the left end $\{z(t_0), t_0\}$ of a trajectory $z(t)$ be fixed in a set $S_0(t_0)$. It is necessary to define the controls $u(t) \in U$ ensuring the right end $\{z(T), T\}$, $t_0 < T \leqslant \infty$, belongs to the terminal set $S(T)$. Here $S_0(t) \subset S(t)\ \forall t \in [t_0, T)$ & $\partial S_0 \cap \partial S = \emptyset$.

PROBLEM II (*On optimal practical stabilization of programmed motion*). Suppose that together with equation (1.2) we are given the functional

$$W\big(z(\cdot), u(\cdot)\big) = \int_0^\infty \big\{\omega\big(t, z(t)\big) + u^* Bu\big\}\, dt, \tag{1.4}$$

where $\omega(t, z)$ is a non-negative function defined in the domain $\Omega$; let $u^* Bu$ be a given positive definite quadratic form. It is necessary to find the control $u^0 \in U$ ensuring:

(a) the uniform practical stability of unperturbed motion of system (1.3);

(b) the minimal value of the functional (1.4) on all trajectories of system (1.3) starting from the set $S_0(t_0)$ and reaching the set $S(t)$.

The solution of the problems mentioned is to be based on the Lyapunov's function method together with the results of the principle of comparison in the integral form.

*1.1. Definitions and additional estimations.* Basing ourselves on papers [7], [10], [15], [18], [25]–[27], [30], [31], we formulate the definitions necessary for further work.

DEFINITION 1.1. The controlled system (1.1) is *practically stable on a program trajectory* $x = \psi(t)$ if under the given assumptions on the domains $S_0(t)$, $S(t)$, $I$ the inclusion $z(t, t_0, z_0) \in \operatorname{int} S(t)$ holds for all $t \in I$ provided $z_0 \in S_0(t_0)$.

DEFINITION 1.2. The controlled system (1.1) is *uniformly practically stable on the programmed trajectory* $x = \psi(t)$ if for each solution $z(t, t_1, z_1)$, starting from the domain $S_0(t_1)$, i.e., $z_1 \in S_0(t_1)$ for $t = t_1$, the inclusion

$$z(t, t_1, z_1) \in \operatorname{int} S(t)$$

holds for all $t \geqslant t_1$, $t, t_1 \in I$.

We consider the Cauchy problem

$$\frac{dy}{dt} = f\big(t, y + \sigma(t)\big), \quad y(t_0) = y_0, \quad y \in R^k. \tag{1.5}$$

Let its solution $y(t, t_0, y_0)$ be defined on $[t_0, \tau)$.

Function $f(t, w)$ has the *property of mixed quasimonotonicity* [28] if $f_s(t, w)$, $s = 1, 2, \ldots, l$, neither decreases in $w_\mu$ ($\mu \neq s$, $\mu = 1, 2, \ldots, l$) nor increases in $w_\nu$ ($\nu = l+1, \ldots, k$) and $f_s(t, w)$, $s = l+1, \ldots, k$ neither increases in $w_\mu$ ($\mu = 1, 2, \ldots, l$) nor decreases in $w_\nu$ ($\nu = l+1, \ldots, k$, $\nu \neq s$).

Let $\tilde{y}(t, t_0, y_0)$ be a solution of the Cauchy problem (1.5) such that

$$\begin{aligned} &y_s(t, t_0, y_0) \leqslant \tilde{y}_s(t, t_0, y_0), \quad s = 1, 2, \ldots, l; \\ &y_s(t, t_0, y_0) \geqslant \tilde{y}_s(t, t_0, y_0), \quad s = l+1, \ldots, k \end{aligned} \tag{1.6}$$

for all $t \in [t_0, \tau)$, or

$$\begin{aligned} &y_s(t, t_0, y_0) \geqslant \tilde{y}_s(t, t_0, y_0), \quad s = 1, 2, \ldots, l; \\ &y_s(t, t_0, y_0) \leqslant \tilde{y}_s(t, t_0, y_0), \quad s = l+1, \ldots, k \end{aligned} \tag{1.7}$$

for all $t \in [t_0, \tau)$.

Having fulfilled conditions (1.6), we obtain the solution $\tilde{y}(t, t_0, y_0)$, which is called an *l-maximal solution* of the Cauchy problem (1.5); in case (1.7) the solution $\tilde{y}(t, t_0, y_0)$ is called *l-minimal* or *$(k-l)$-maximal*. Both in case (1.6) and in case (1.7), $\tilde{y}(t, t_0, y_0)$ is called a *minimax solution*.

LEMMA 1.1. *Let the following conditions be fulfilled*:

(1) *function $f(t, w)$ is continuous, defined in an open domain $A \subseteq I \times R^k$ and has the property of mixed quasimonotonicity*;

(2) *the inequalities*

$$\zeta_s(t) \leqslant \sigma_s(t) + \int_{t_0}^{t} f_s\big(t, \zeta(t)\big)\,dt + y_s(t_0), \qquad s = 1, 2, \ldots, l;$$

$$\zeta_s(t) \geqslant \sigma_s(t) + \int_{t_0}^{t} f_s\big(t, \zeta(t)\big)\,dt + y_s(t_0), \qquad s = l+1, \ldots, k$$

*hold*;

(3) *the functions $\zeta(t)$, $\sigma(t)$ are continuous for all $t \in [t_0, \tau)$, $\big(t, \zeta(t)\big) \in A$, $\big(t, \sigma(t)\big) \in A$ and*

$$\zeta_s(t_0) \leqslant y_{s0} = \tilde{y}_s(t_0), \qquad s = 1, 2, \ldots, l;$$

$$\zeta_s(t_0) \geqslant y_{s0} = \tilde{y}_s(t_0), \qquad s = l+1, \ldots, k.$$

*Then for all $t \in [t_0, \tau)$ we have the estimations*

$$\begin{aligned} &\zeta_s(t) \leqslant \tilde{y}_s(t, t_0, y_0) + \sigma_s(t), \qquad s = 1, 2, \ldots, l; \\ &\zeta_s(t) \geqslant \tilde{y}_s(t, t_0, y_0) + \sigma_s(t), \qquad s = l+1, \ldots, k. \end{aligned} \tag{1.8}$$

For the proof see [1], [2], [28], [30].

## § 2. Theorem on practical stability of programmed motion

Together with system (1.3) we shall consider Lyapunov's vector-function $V(t, z)$ with the components $v_j(t, z)$, $j = 1, 2, \ldots, k$; $v_j\colon I \times R^n \to R^1$, locally large [14] and continuously differentiable at $(t, z)$ in the time-varying domain $I \times S(t)$,

$$\|\nabla v_j\| \overset{\Delta}{=} \left\| \frac{\partial v_j}{\partial z} \right\| \leqslant M_j, \qquad M_j = \text{const} > 0, \qquad j = 1, 2, \ldots, k,$$

for all $t \in I$ and $z \in \bar{S}(t) \setminus S_0(t)$.

We define the derivative of the component function $v_j(t, z)$ by means of the formula

$$\frac{dv_j}{dt} = \frac{\partial v_j}{\partial t} + \nabla v_j^* F(t, z), \qquad j = 1, 2, \ldots, k,$$

and we define the majorant functions $f_j(t, v_1, \ldots, v_k)$ in the domain $A \subset I\times\Omega$, $f_j\colon I\times R^k \to R^1$. Let the function $f_j(t, v_1, \ldots, v_k)$ for each $j = 1, 2, \ldots, k$ have the property of mixed quasimonotonicity.

We introduce one more vector-function,

$$\sigma(t) = M\int_{t_0}^{t} \|\Delta(\psi(t), u(t), t)\|dt, \qquad t \geqslant t_0,$$

where $M = \mathrm{colon}(M_1, \ldots, M_k)$, $u(t) \in U$, assuming that $(t, \sigma) \in A$.

Further on we shall use Lemma 1.1 for an estimation of the components of the vector-function $V(t, z)$. Namely, the following assertion holds.

LEMMA 2.1. *Assume that*

(1) *the above-mentioned functions $V(t, z)$, $f(t, v)$, $\sigma(t)$ exist and*

(a) $$v_s\big(t, z(t)\big) \leqslant \sigma_s(t) + \int_{t_1}^{t} f_s\Big(t, V\big(t, z(t)\big)\Big)dt + y_s(t_1), \qquad s = 1, 2, \ldots, l;$$

(b) $$v_s\big(t, z(t)\big) \geqslant \sigma_s(t) + \int_{t_1}^{t} f_s\Big(t, V\big(t, z(t)\big)\Big)dt + y_s(t_1), \qquad s = l+1, \ldots, k;$$

(2) *the functions $v_s\big(t, z(t)\big)$, $\sigma_s(t)$ are continuous at all $t \in [t_1, \tau)$, $(t, v_s) \in A$, $(t, \sigma_s) \in A$ and*

(a) $$v_s\big(t_1, z(t_1)\big) \leqslant y_{s0} = v^{S\setminus S_0}_{s\,\sup}(t_1) - \sigma_s(t_1), \qquad s = 1, 2, \ldots, l,$$

(b) $$v_s\big(t_1, z(t_1)\big) \geqslant y_{s0} = v^{S\setminus S_0}_{s\,\inf}(t_1) - \sigma_s(t_1), \qquad s = l+1, \ldots, k.$$

*Then for the components of the vector-function the following estimations hold:*

$$\text{(2.1)}\qquad \begin{aligned} &\text{(a)}\quad v_s\big(t, z(t)\big) \leqslant \tilde{y}_s(t, t_1, y_{s0}) + \sigma_s(t), \qquad s = 1, 2, \ldots, l;\\ &\text{(b)}\quad v_s\big(t, z(t)\big) \geqslant \tilde{y}_s(t, t_1, y_{s0}) + \sigma_s(t), \qquad s = l+1, \ldots, k \end{aligned}$$

*for all $t \in I$ and $z(t) \in \bar{S}(t)\setminus S_0(t)$, where $\tilde{y}_s(t, t_1, y_{s0})$ are the minimax solutions of the Cauchy problem*

$$\text{(2.2)}\qquad \frac{dy}{dt} = f\big(t, y+\sigma(t)\big), \qquad y(t_1) = y_0, \qquad t_1 \geqslant t_0.$$

*Proof.* We obtain conditions (1) (a)–(b) of Lemma 2.1 from the supposition of the realization of the differential inequalities

$$\frac{dv_s}{dt} \leqslant f_s(t, V) + M_s\|\Delta(\psi, u, t)\|, \qquad s = 1, 2, \ldots, l;$$

$$\frac{dv_s}{dt} \geqslant f_s(t, V) + M_s\|\Delta(\psi, u, t)\|, \qquad s = l+1, \ldots, k,$$

which estimate the full derivatives of the components $v_j(t, z)$, $j = 1, 2, \ldots, k$, of the vector-function $V(t, z)$ in virtue of system (1.2). Passing on from these inequalities to the integral ones, we obtain

$$\zeta_s(t) \leqslant \zeta_s(t_1) - \sigma_s(t_1) + \int_{t_1}^{t} f_s\big(t, \zeta(t)\big)\, dt + \zeta_s(t), \qquad s = 1, \ldots, l$$

and

$$\zeta_s(t) \geqslant \zeta_s(t_1) - \sigma_s(t_1) + \int_{t_1}^{t} f_s\big(t, \zeta(t)\big)\, dt + \zeta_s(t), \qquad s = l+1, \ldots, k,$$

where $\zeta_s(t) = v_s\big(t, z(t, t_1, z_0)\big)$, $s = 1, 2, \ldots, k$. We now obtain the estimations (2.1) using Lemma 1.1.

THEOREM 2.1. *Suppose that*

(1) *the above-mentioned functions $V(t, z)$ and $f(t, V)$ exist and*

(a) $$\frac{dv_s}{dt} \leqslant f_s(t, V) \quad \text{for} \quad s = 1, 2, \ldots, l;$$

(b) $$\frac{dv_s}{dt} \geqslant f_s(t, V) \quad \text{for} \quad s = l+1, \ldots, k$$

*for all $t \in I$, $z \in \bar{S}(t) \setminus S_0(t)$;*

(2) *on a convex-compact bounded set $U$ the function $\sigma(t)$ is bounded on each finite interval $I_1 \subset [t_0, +\infty)$;*

(3) *a minimax solution $\tilde{y}(t, t_0, y_0)$ of the Cauchy problem*

$$\frac{dy}{dt} = f\big(t, y + \sigma(t)\big), \qquad y(t_0) = y_0$$

*is defined for all $t \geqslant t_0$.*

*Then for any vector of control $u^0 \in U$ ensuring the inequalities*

(2.3)
(a) $\sigma_s(t) > v_{s\inf}^{\partial S}(t) - \tilde{y}_s(t, t_0, y_{s0})$, $\quad s = 1, 2, \ldots, l;$
(b) $\sigma_s(t) < v_{s\sup}^{\partial S}(t) - \tilde{y}_s(t, t_0, y_{s0})$, $\quad s = l+1, \ldots, k,$

*the controlled system* (1.1) *is practically stable on a programmed trajectory.*

*Proof.* Let us consider case (b) in condition (1) and, consequently, inequality (b) in (2.3). In this case the Cauchy problem (from condition (3) of the theorem) is the following:

$$\frac{dy_s}{dt} = f_s\big(t, y_s + \sigma_s(t)\big).$$

$$y_s(t_0) = y_{s0} = v_{s\inf}^{\partial S_0}(t_0) - \sigma_s(t_0), \qquad s = l+1, \ldots, k.$$

Let the left end $\{z(t_0), t_0\}$ of the trajectory $z(t)$ of system (1.2) be fixed in the domain $S_0(t_0)$, i.e., $z(t_0) \in \operatorname{int} S_0(t_0) \subset S(t_0) \& \partial S_0 \cap \partial S = \emptyset$ $\forall t \in I$. We assume that the trajectory $z(t)$ reaches the boundary of the domain $S(t)$ at $t = t_1$, i.e., $z(t_1) \in \partial S(t_1)$, $t_1 \in [t_0, +\infty)$. Then the right end $\{z(t), t\}$ of the trajectory $z(t)$ does not leave the inside of the domain $S(t)$ for $t \in [t_0, t_1)$. Let us verify if it is possible for the trajectory $z(t)$ to reach the boundary of the domain while fulfilling the conditions of the theorem. In virtue of conditions (1)–(2) of the theorem we have an estimation

$$v_s(t, z(t)) \geqslant \bar{y}_s(t, t_0, y_{s0}) + \sigma_s(t) \quad \text{for all} \quad t \geqslant t_0,\ s = l+1, \ldots, k.$$

The control $u^0 \in U$ (satisfying condition (2.3)) for a given value $t_1 \in [t_0, +\infty)$ leads to the inequality

$$v_{s\sup}^{\partial S}(t_1) < v_s(t_1, z(t_1)), \quad s = l+1, \ldots, k,$$

which contradicts the existence of $t_1 \in I$ for which $z(t_1) \in \partial S(t_1)$. This fact proves that the controlled system (1.1) is practically stable on the programmed trajectory. We have a similar scheme of the proof of stability in the case of inequality (a) in condition (1) of the theorem; that is why it is not necessary to repeat it here.

## § 3. Theorem on optimal stabilization

In the applied problems of dynamics of controlled systems, parallel with the requirement of practical stability of unperturbed motion, there arise problems of optimization of the transient. In a great number of cases they can be expressed in the form of minimality of an integral

$$W(x(\cdot), u(\cdot)) = \int_{t_0}^{\infty} \omega(t, v(t, x(\cdot)), x(\cdot), u(\cdot))\,dt, \tag{3.1}$$

where $\omega(t, v, x, u)$ is a non-negative function; its other properties will be defined more exactly later on. As in [6] by the symbol $u[t]$ we denote the controls which are realized in the system

$$\frac{dx}{dt} = X(t, x, u), \quad x \in R^n, \tag{3.2}$$

where $X$: $I \times \Omega \times U \to R^n$. By the symbol $x[t]$ we denote the motions of system (3.2) corresponding to the controls $u[t]$. If the motion is generated by any fixed value of control, then this fact will be marked by a superscript $*$, i.e., $x^*[t]$, $u^*[t]$.

The choice of the function $\omega(t, v, x, u)$ in expression (3.1) is realized every time in conformity with the conditions of the problem considered.

However, as a rule, the following three principles remain unchanged:

(A) The condition of minimum of integral (3.1) must ensure the quickest possible return of the solution into the domain $S_0(t)$ ($S_0(t) \subset S(t)$ $\forall t \in I$) in case of damping of motion or the slowest possible transition from the domain $S_0(t)$ into that of $S(t)$ in case of an increase of motion.

(B) Function $\omega$ must be, on the one hand, such as to ensure the applicability of the corresponding theorems on differential inequalities and, on the other hand, such as to provide the solution of the Cauchy problem appearing here in an explicit form or to solve the question of stability of the zero solution of the system of comparison effectively.

(C) The value of the integral $W$ must satisfactorily estimate the outlay spent on the realization of the motion needed.

Thus, let the quality criterion (3.1) be chosen and let the convex-compact bounded set $U$ be given. It is necessary to determine the controlling effects $u^0(t, x) \in U$ which ensure the practical stability of unperturbed motion of system (3.2) in the sense of one of the definitions 1.1, 1.2 and, for any other values of the vector $u^*(t, x) \in U$ which solve the problem of stabilization, the inequality

$$\int_{t_0}^{\infty} \omega\big(t, v(t), x^0[t], u^0[t]\big)\,dt \leqslant \int_{t_0}^{\infty} \omega\big(t, v(t), x^*[t], u^*[t]\big)\,dt$$

must be fulfilled for all initial conditions $(t_0, x_0) \in \operatorname{int}\big(I \times S_0(t_0)\big)$.

Let us call attention to the fact which is inherent in the whole theory of practical stability of motion: namely, the domain $S_0(t)$ must be formed of the technical conditions of the functioning of the controlled system (3.2) and the domain $S(t)$ must be defined from the quality conditions of the synthesized controls. It has been accepted to call the vector-function $u = u^0(t, x)$, which solves the problem of optimal stabilization, the optimal control.

We compose the expressions [6]

$$B_s[V; t, x, u, \omega] = \frac{dv_s}{dt} + \omega_s(t, V, x, u), \qquad s = 1, 2, \ldots, k,$$

where the full derivative of the function $v_s(t, x)$ with respect to system (3.2) is denoted by the symbol $dv_s/dt$.

Let the inequalities

$$B_s[V; t, x, u, \omega] \leqslant 0, \qquad s = 1, 2, \ldots, k \tag{3.3}$$

be fulfilled for a certain choice of components of the vector-function $V(t, x)$ and a vector of controls $u^* \in U$ in the domain $I \times S(t)$. This means that for a given vector of controls $u^* \in U$ the variation of the components of the vector-function $V(t, x)$ on the solution $x^*[t]$ can be estimated by the inequalities

$$\frac{dv_s}{dt} \leqslant -\omega_s(t, V, x), \quad s = 1, 2, \ldots, k. \tag{3.4}$$

On the basis of Chaplygin's theorem on differential inequalities we find the estimations

$$v_s\big(t, x(t)\big) \leqslant \tilde{y}_s(t, t_0, y_0), \quad s = 1, 2, \ldots, k, \tag{3.5}$$

where $\tilde{y}_s(t, t_0, y_0)$ is the $y$-upper solution of the Cauchy problem

$$\frac{dy}{dt} + \omega(t, y, x) = 0, \quad y(t_0) = y_0, \tag{3.6}$$

defined in $[t_0, \tau)$, where

$$v_s\big(t_0, x(t_0)\big) \leqslant y_{s0}, \quad s = 1, 2, \ldots, k.$$

In addition, the function $\omega(t, y, x)$ must satisfy Ważewski's condition on the domain $A$:

$$\omega_s(t, y', x) \leqslant \omega_s(t, y'', x) \quad \text{for} \quad y'_s = y''_s, \; y'_\nu \leqslant y''_\nu$$
$$(\nu \neq s, \; \nu, s = 1, 2, \ldots, k).$$

We summarize the above by the following assertion.

LEMMA 3.1. *Suppose that for system* (3.2) *there exist Lyapunov's vector-function* $V(t, x)$ *and a majorizing function* $\omega(t, V, x)$ *and*

(a) *inequalities* (3.3) *hold for* $u^* \in U$;

(b) *the y-upper solution of the Cauchy problem is defined for all* $t \geqslant t_0$.

*The estimation* (3.5) *holds in the joint domain of the existence of the y-upper solution of the Cauchy problem* (3.6) *and the fulfilment of inequalities* (3.4).

Now we pass to the main theorem on optimal stabilization.

THEOREM 3.1. *If the differential equations of perturbed motion* (3.2) *are of such a type that we can find a vector-function* $V(t, x)$ *together with positive definite locally large components* $v_s(t, x)$, $s = 1, 2, \ldots, k$ *and functions* $u^0(t, x) \in U$ *of such a type that*:

(1) *the function* $\omega(t, V, x) = \omega(t, v_1, \ldots, v_k, x^0[t], u^0[t])$ *satisfies Ważewski's condition in the domain* $A$ *for* $(t, x) \in I \times S(t)$ *and is non-negative*;

(2) *there exists a solution of the system* $\dot{x} = X(t, x, u^0)$ *and it is unique and continuous for all* $t \geqslant t_0$;

(3) *the inequalities*

$$B_s[V; t, x, u^0, \omega] \leqslant 0, \quad s = 1, 2, \ldots, k$$

*are valid*;

(4) *the inequalities*

$$B_s[V; t, x, u, \omega] \geqslant 0, \quad s = 1, 2, \ldots, k$$

*are valid for any meanings of* $u \in U$;

(5) *the y-upper solution of the Cauchy problem is such that*

(a) $$\tilde{y}_s(t, t_0, y_0) < v_{s\inf}^{\partial S}(t) \quad \forall t \geqslant t_0;$$

(b) $$\lim_{t\to\infty} \tilde{y}_s(t, t_0, y_0) = 0, \quad s = 1, 2, \ldots, k,$$

*then the vector* $u^0(t, x) \in U$ *ensures the solution of the problem of optimal stabilization of controlled motion. In this case the motion* $x^0[t] = 0$ *is practically stable and the inequality*

$$\int_{t_0}^{\infty} \omega_s(t, V, x^0[t], u^0[t])\,dt = \min \int_{t_0}^{\infty} \omega_s(t, V, x^*[t], u^*[t])\,dt \leqslant \leqslant v_s\big(t_0, x(t_0)\big), \quad s = 1, 2, \ldots, k, \tag{3.7}$$

*holds.*

*Proof.* In order to determine the validity of the theorem it is necessary to verify two facts. Firstly, the functions $u^0 \in U$ must ensure the practical stability of unperturbed motion of system (3.2); secondly, relation (3.7) must be fulfilled. The practical stability of unperturbed motion can be derived from Lemma 1.1 and conditions (1), (2), (4), (a) of Theorem 3.1. Indeed, suppose that, for $t = t_0$, the left end of the trajectory $\{x(t_0), t_0\}$ is fixed in a set $S_0(t_0)$, i.e., $x(t_0) \in S_0(t_0) \subset S(t_2)$. If we assume that there exists a moment $t_2 \in I$ for which $x(t_2, t_0, x_0) \in \partial S(t_2)$, then

$$v_s\big(t_2, x(t_2, t_0, x_0)\big) \leqslant \tilde{y}_s(t_2, t_0, y_0) < v_{s\inf}^{\partial S}(t_2), \quad s = 1, 2, \ldots, k.$$

This inequality contradicts the supposition of the existence of $t_2 \in I$ for which $x(t_2, t_0, x_0) \in \partial S(t_2)$. This proves practical stability.

Now, let us verify the relation (3.7). In virtue of conditions (4), (b) and estimations (3.5) we have

$$\lim_{t\to\infty} v_s(t, x^0[t]) = 0, \quad s = 1, 2, \ldots, k. \tag{3.8}$$

Taking into account (3.8), we find the estimations

$$v_s\big(t_0, x(t_0)\big) \geqslant \int_{t_0}^{\infty} \omega_s(t, V, x^0[t], u^0[t])\,dt, \quad s = 1, 2, \ldots, k \tag{3.9}$$

from inequality (3.4). Let us now solve the problem of optimal stabilization by means of the vector $u^* \in U$. There are two possible variants:

(1) the motion $x^*[t]$ of system (3.2) does not leave the domain $S_0(t)$ at any $t \geqslant t_0$;

(2) motion $x^*[t]$ eventually leaves the domain $S_0(t)$.

According to condition (4) of the theorem we have

$$\frac{dv_s}{dt} \geqslant -\omega_s(t, V, x^*[t], u^*[t]), \quad s = 1, 2, \ldots, k. \tag{3.10}$$

Taking into account the relations

$$\lim_{t\to\infty} v_s(t, x^*[t]) = 0, \quad s = 1, 2, \ldots, k \tag{3.11}$$

once more, we obtain

$$v_s\big(t_0, x(t_0)\big) \leqslant \int_{t_0}^{\infty} \omega_s(t, V, x^*[t], u^*[t])\,dt, \quad s = 1, 2, \ldots, k. \tag{3.12}$$

If $\tau \in I$, $t_0 < \tau$, is the last moment of time at which the motion $x^*[\tau] \in \operatorname{ext} S_0(\tau)$ (for all $t \geqslant t_0$, $x^*[t] \in \operatorname{int} S(t)$), then we find from (3.10) that

$$v_s\big(\tau, x^*(\tau)\big) \leqslant \int_{\tau}^{\infty} \omega_s(t, V, x^*[t], u^*[t])\,dt, \quad s = 1, 2, \ldots, k. \tag{3.13}$$

In virtue of the estimation

$$\sup\big(v_s(t, x) \text{ for } x \in \partial S_0(t),\ s = 1, 2, \ldots, k\big) < \\ < \inf\big(v_s(t, x) \text{ for } x \in \partial S(t),\ s = 1, 2, \ldots, k\big),$$

where $\dot{S}_0(t) \subset S(t) \& \partial S_0 \cap \partial S = \emptyset$, it is not difficult to determine that

$$v_s\big(t_0, x(t_0)\big) < v_s\big(\tau, x^*(\tau)\big), \quad s = 1, 2, \ldots, k. \tag{3.14}$$

Since the function $\omega$ is non-negative, we have

$$\int_{\tau}^{\infty} \omega_s(t, V, x^*[t], u^*[t])\,dt < \int_{t_0}^{\infty} \omega_s(t, V, x^*[t], u^*[t])\,dt, \\ s = 1, 2, \ldots, k. \tag{3.15}$$

Combining estimations (3.9), (3.12), (3.15), we determine

$$\int_{t_0}^{\infty} \omega_s(t, V, x^0[t], u^0[t])\,dt \leqslant v_s\big(t_0, x(t_0)\big) \leqslant \\ \leqslant \int_{t_0}^{\infty} \omega_s(t, V, x^*[t], u^*[t])\,dt, \quad s = 1, 2, \ldots, k.$$

And (3.7) follows from this fact. ■

## § 4. Controlled systems with a neutral part

We consider a set of equations of perturbed motion

$$\frac{dx}{dt} = f(t, x) + g(t, x)u, \quad x \in R^n, \tag{4.1}$$

where $f\colon I \times R^n \to R^n$; $g(t, x)$ is an $n \times m$-matrix, $u$ is an $m$-dimensional vector of control.

The zero-solution of the uncontrolled system

$$\frac{dx}{dt} = f(t, x) \tag{4.2}$$

is uniformly practically stable, which is ensured by the existence of a positive definite locally large function $v(t, x)$ whose derivative is, in virtue of system (4.2),

$$\frac{\partial v}{\partial t} + \nabla v^* f(t, x) \equiv w(t, x) \leqslant 0, \quad x \in \bar{S}(t) \setminus S_0(t)$$

and

$$\sup_{x \in \partial S_0} v(t_1, x) < \inf_{x \in \partial S} v(t_2, x), \quad x \in \bar{S}(t) \setminus S_0(t), \quad t_2 > t_1.$$

We determine the functional of quality in the form of

$$W = \int_0^T \{\omega(t, x(t)) + u^* Bu\} dt + \gamma(x[T]), \quad 0 < T < +\infty, \tag{4.3}$$

where a non-negative function $\omega$ must be determined, $u^*Bu$ is a given positive definite quadratic form with the symmetric matrix ($B^* = B$), and $\gamma(x[T])$ is a continuous function.

The problem of analysis is the following. There is given a convex-compact bounded set $U$. It is necessary to define a sub-domain $U^0 \subset U$ or vector $u^0(t, x) \in U$ such that

(1) the unperturbed motion of system (4.1) will be practically stable;

(2) the functional (4.3) will take the minimal value on the solutions of system (4.1) with the left end fixed in the domain $S_0(t_0)$.

We shall base the solution of the problem on the consideration of the function $v(t, x)$ given by the equation

$$v(t, x) = \sum_{s=1}^{k} a_s v_s(t, x), \quad a_s = \text{const} > 0, \tag{4.4}$$

where $v_s(t, x)$ are the components of Lyapunov's vector-function. By means of N. N. Krasovski's method [6] we compose an expression

$$B[v; t, x, u] = \frac{dv}{dt} + \omega(t, x) + u^* Bu$$
$$= w(t, x) + \omega(t, x) + u^* Bu.$$

We define the vector $u^0 \in U$ from the condition of the minimum of the function $B$:

$$B[v; t, x, u] = 0 \quad \text{at} \quad u = u^0$$

and

$$\frac{\partial}{\partial u} B[v; t, x, u] = 0 \quad \text{at} \quad u = u^0.$$

And we obtain the equation

$$g^*(t, x) \nabla_x v(t, x) + 2Bu^0 = 0,$$

from which we have

$$u^0(t, x) = -\tfrac{1}{2} B^{-1} g^*(t, x) \nabla_x v(t, x). \tag{4.5}$$

If we substitute expression (4.5) (instead of $u(t, x)$) into the equation of perturbed motion (4.1), then we shall obtain the system

$$\frac{dx}{dt} = F(t, x), \tag{4.6}$$

where

$$F(t, x) \equiv f(t, x) - \tfrac{1}{2} g(t, x) B^{-1} g^*(t, x) \nabla_x v(t, x).$$

Practical stability analysis of a system of type (4.6) is made on the basis of the Lyapunov's function method or the method of comparison.

Now we pass to problem (2), namely the problem of minimization of the functional (4.3). With this aim in view we notice that [19]

$$\nabla_x v^* g(t, x) u + u^* Bu = -2u^{0*} Bu^0 + u^* Bu \tag{4.7}$$
$$= (u - u^0)^* B (u - u^0) - u^{0*} Bu^0.$$

We obtain

$$w(t, x) + \omega(t, x) - u^{0*} Bu^0 = 0$$

by substituting the value (4.5) into the expression of the function $B$, taking into account (4.7); hence

$$\omega(t, x) = -w(t, x) + u^{0*} Bu^0$$

and the criterion of quality takes the form

$$W = \int_0^T [-w(t, x) + u^{0*}Bu^0 + u^*Bu]dt + \gamma(x[T]). \tag{4.8}$$

THEOREM 4.1. *Suppose that*

(1) *there exists a positive definite locally large Lyapunov's function* $v(t, x)$;

(2) $S_0(t) \subset S(t) \& \partial S_0 \cap \partial S = \emptyset$ *holds for all* $t \in [t_0, \infty)$;

(3) *there exists a definite differentiable and non-decreasing function* $\eta(t)$ *and*

(a) $$\frac{\partial v}{\partial t} + \nabla v^* F(t, x) < \frac{d\eta}{dt} \quad \textit{for} \quad x \in \bar{S}(t) \setminus S_0(t), \quad t \in I;$$

(b) $$\eta(t_0) \leqslant v_{\sup}^{\partial S_0}(t_0);$$

(c) $$\eta(t) \leqslant v_{\inf}^{\partial S}(t) \quad \textit{for all} \quad t \geqslant t_0;$$

(4) $$\min_{u \in U} B[v; t, x, u] = \left(\frac{dv}{dt}\right)_{(4.6)} + \omega(t, x) + u^{0*}Bu^0 = 0.$$

*Then the unperturbed motion of system* (4.1) *is practically stable for the controlling effects* (4.5) *and on trajectories* $x^0[t]$ *the functional*

$$W = \int_0^T [-w(t, x) + u^{0*}Bu^0 + u^*Bu]dt + \gamma(x[T])$$

*takes the minimal value, where $T$ is an arbitrarily large finite number.*

*Proof.* Having fulfilled conditions (1)–(3), we obtain practically stable unperturbed motion. Let us prove the optimality. We define from condition (4)

$$v\big(t_0, x(t_0)\big) = \gamma(x[T]) + \int_0^T [\omega(t, x[t]) + u^{0*}Bu^0]dt, \tag{4.9}$$

where

$$\gamma(x[T]) \equiv v\big(T, x(T)\big).$$

Suppose that there exists another optimal control $u^1(t, x)$, solving the problem of optimal stabilization. We have from condition (4)

$$\left(\frac{dv}{dt}\right)_{(4.6)} + \omega(t, x) + u^{1*}Bu^1 \geqslant 0.$$

Hence

$$v\big(t_0, x(t_0)\big) \leqslant \gamma(x^1[T]) + \int_0^T [\omega(t, x^1[t] + u^{1*}Bu^1]dt.$$

Now, by repeating the argument of the proof of Theorem 3.1 it is not difficult to verify the second assertion of Theorem 4.1.

## § 5. Controlled systems with integrable approximation

We mean here the following sets of equations of perturbed motion:

$$\frac{dx}{dt} = f(t, x) + g(t, x)u + \mu R(t, x), \tag{5.1}$$

where the sense of $f$, $g$, $u$ is the same as in § 4, $\mu > 0$ is a small parameter; $R(t, x)$ is a vector-function of a known structure.

By preserving the assumption of § 4 we obtain

$$\frac{dx}{dt} = F(t, x) + \mu R(t, x) \tag{5.2}$$

instead of system (4.6).

Theorem 4.1 extends to system (5.1) under the assumption that system (4.6) is integrable, i.e., the solution $x = \bar{x}(t, t_0, x_0)$ of the Cauchy problem

$$\frac{dx}{dt} = F(t, x), \qquad x(t_0, t_0, x_0) = x_0 \tag{5.3}$$

is known.

We define the function $\omega(t, x, \mu)$ by the formula

$$\omega(t, x, \mu) = -w(t, x) + u^{0*}Bu^0 - \mu\nabla_x v^* R(t, x).$$

Thus, the structure of the quality criterion is the following:

$$W = \int_0^{t^*} \{-w(t, x) - \mu\nabla_x v^* R(t, x) + u^{0*}Bu^0 + u^* Bu\} dt, \qquad 0 < t^* < \infty.$$

Let $\bar{x}(t, t_0, x_0)$ be the integral curves of the Cauchy problem (5.3) for the values $(t_0, x_0) \in I \times S_0(t_0)$ (or $I \times S(t_0)$). We denote $\varphi(t, x) = \nabla_x v^* R(t, x)$ and consider the mean

$$\psi_0(t_0, x_0) = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \varphi\big(t, \bar{x}(t, t_0, x_0)\big) dt. \tag{5.4}$$

We define the distance from a point $x \in R^n$ to a set $M \subset R^n$ by the formula

$$\varrho(x, M) = \inf[\|x - x_0\|,\ x_0 \in M].$$

Let $V^*(x)$ be a non-negative scalar definite function, continuous in the domain $S(t)$ for all $t \geqslant t_0$. The number of points $x \in S(t)$ for which $V^*(x) = 0$ is denoted by $E(V^* = 0)$.

DEFINITION 5.1. *$\psi_0(t_0, x_0) < 0$ is definite in the set $E(V^* = 0)$* if for the given $S_0(t)$, and $S(t)$, $S_0(t) \subset S(t) \forall t \in I$, we can define $r(S_0, S)$ and $\delta = \delta(S_0, S)$ such that $\psi_0(t_0, x_0) < -\delta$ for $x_0 \in S(t) \setminus S_0(t)$, $\varrho(x_0, E(V^* = 0)) < r$ for all $t_0 \in [0, +\infty)$.

THEOREM 5.1. *Let the following conditions be fulfilled:*

(1) *for system* (5.3) *there exists a positive definite locally large function $v(t, x)$ which has an infinitely small high limit, and the function $V^*(x)$ such that*

$$\frac{\partial v}{\partial t} + \nabla_x v^* F(t, x) \leqslant V^*(x) \leqslant 0 \quad \textit{in the domain } S(t);$$

$$(2) \qquad \min_{u \in U} B[v; t, x, u] = \frac{\partial v}{\partial t} + \omega(t, x, \mu) + u^{0*} B u^0 = 0;$$

(3) *there exist integrable functions $K(t)$, $F(t)$ and $N(t)$, constants $k_0, f_0$ and $n_0$, and also a nondecreasing function $\chi(\gamma)$, $\lim_{\gamma \to 0} \chi(\gamma) = 0$ such that*

$$\|R(t, x)\| \leqslant K(t), \qquad \int_{t_1}^{t_2} K(t)\,dt \leqslant k_0(t_2 - t_1);$$

$$|\varphi(t, x') - \varphi(t, x'')| \leqslant \chi(\|x' - x''\|) F(t);$$

$$\int_{t_1}^{t_2} F(t)\,dt \leqslant f_0(t_2 - t_1)$$

*in the domain $S(t)$ on any finite interval $[t_1, t_2]$;*

(4) *the inequality*

$$\varphi(t, x) \leqslant N(t), \qquad \int_{t_1}^{t_2} N(t)\,dt \leqslant n_0(t_2 - t_1)$$

*holds for $x \in S(t) \setminus E(V^* = 0)$ and $t \in [t_0, \infty)$;*

(5) *there exists a mean* (5.4) *uniform with respect to $(t_0, x_0) \in I \times S(t_0)$;*

(6) *in the set $E(V^* = 0)$ there is definite $\psi_0(t_0, x_0) < 0$.*

*Then for the given domains $S_0(t)$ and $S(t)$ we can define $\mu_0(S) > 0$ such that*

(a) *the unperturbed motion of system* (5.3) *is practically stable for* $0 < \mu < \mu_0$;

(b) *on the trajectories of system* (5.1) *the functional*

$$W = \int_0^{t^*} \{-w(t, x[t]) - \mu \nabla_x v^* R(t, x) + u^{0*} B u^0 + u^* B u\} dt$$

*takes the minimal value, where* $t^*$ *is an arbitrarily large finite number.*

*Proof.* Conditions (1), (3)–(6) ensure the practical $\mu$-stability of unperturbed motion of system (5.2). Condition (2) ensures the minimum of the functional $W$; this fact can be verified in the same way as in the proof of Theorem 3.1.

## References


[1] Н. В. Азбелев, З. Б. Цалюк, *Об интегральных неравенствах*, I, Мат. сборник 56 (1962), 325–342.

[2] —, *О дифференциальных и интегральных неравенствах*, в: Тр. IV Мат. съезда, т. 2, Наука, 1964, 384–391.

[3] В. И. Зубов, *Теория колебаний*, Высшая школа, Москва 1979.

[4] —, *Проблема устойчивости процессов управления*, Судостроение, Ленинград 1980.

[5] К. А. Карачаров, А. Г. Пилютик, *Введение в техническую теорию устойчивости движения*, ГФМЛ, Москва.

[6] Н. Н. Красовский, Дополнение IV в книге И. Г. Малкин, *Теория устойчивости движения*, Наука, Москва 1968.

[7] —, *Некоторые задачи теории устойчивости движения*, Физматгиз, Москва 1959.

[8] —, *Теория управления движением*, Наука, Москва 1968.

[9] В. И. Косолапов, *К устойчивости движения в нейтральном случае*, Докл. АН УССР, Сер. А 1 (1979), 27–31.

[10] Ж. Ла-Сааль, С. Лефшец, *Исследование устойчивости прямым методом Ляпунова*, Мир, Москва 1964.

[11] Э. Б. Ли, Л. Маркус, *Основы теории оптимального управления*, пер. с англ., Наука, Москва 1972.

[12] А. М. Ляпунов, *Общая задача об устойчивости движения*, ОНТИ, Ленинград–Москва 1935.

[13] В. М. Матросов, Л. Ю. Анапольский, С. Н. Васильев, *Метод сравнения в математической теории систем*, Наука, Новосибирск 1980.

[14] А. А. Мартынюк, *Метод усреднения и принцип сравнения в технической теории устойчивости движения*, Прикл. механика 7, 9 (1971), 64–69.

[15] —, *Техническая устойчивость в динамике*, Техника, Киев 1973.

[16] —, *О технической стабилизации управляемых движений*, Мат. физика 24 (1980), 22–27.

[17] А. А. Мартынюк, Р. Гутовски, *Интегральные неравенства и устойчивость движения*, Наук. думка, Киев 1979.

[18] Г. И. Мельников, *Некоторые вопросы прямого метода Ляпунова*, Докл. АН СССР 110, 3 (1956), 326–329.

[19] С. М. Молдабаев, *Об оптимальной стабилизации нелинейных систем*, в: *Динамика управляемых систем*, Наука, Новосибирск 1979, 192–198.

[20] Л. С. Понтрягин, В. Г. Болтянский, Р. В. Гамкрелидзе, Е. Ф. Мищенко, *Математическая теория оптимальных процессов*, Наука, Москва 1969.
[21] Е. Л. Тонков, *Некоторые вопросы управления периодическими движениями*, в: *Динамика управляемых систем*, Наука, Новосибирск 1979, 286–293.
[22] С. Л. Черноусько, Л. Д. Акуленко, Б. Н. Соколов, *Управление колебаниями*, Наука, Москва 1980.
[23] С. А. Чаплыгин, *Новый метод приближенного интегрирования дифференциальных уравнений*, в: *Избр. тр.*, Москва 1976, 307–360.
[24] Н. Г. Четаев, *Об одной мысли Пуанкаре*, Сб. научн. тр. Казанск. авиац. ин-та 3 (1935).
[25] —, *О некоторых вопросах, относящихся к задаче об устойчивости неустановившихся движений*, Прикл. математика и механика 24, 1 (1960), 6–19.
[26] W. Bogusz, *Stateczność techniczna*, PWN, Warszawa 1972.
[27] L. T. Grujič, *On practical stability*, Internat. J. Control 17, 4 (1973), 881–887.
[28] V. Lakshmikantham, S. Leela, *Differential and integral inequalities — Theory and Applications*, vol. 1, Academic Press, New York 1969.
[29] A. N. Michel, D. W. Porter, *Practical stability and finite-time stability of discontinuous systems*, IEEE Trans. Circuit Theory CT-19 2 (1972), 123–129.
[30] C. P. Tsokos, Rama Mohana Rao, *Finite time stability of control systems and integral inequalities*, Bul. Inst. Politehn. Iaşi 15, 1–2 (1969), 105–112.
[31] L. Weiss, E. F. Infante, *On the stability of systems defined over a time interval*, Proc. Nat. Acad. Sci. USA 54, 1 (1965), 44–48.