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REMARK ON SPLINE UNCONDITIONAL BASES IN $H^1(D)$

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In his work [3] Z. Ciesielski has constructed spline unconditional bases in the classical Hardy space $H^1(D)$ of analytic functions on the unit disc in the complex plane. He used systems of knots on R introduced by J. O. Strömberg [5]. Unfortunately, these systems lead to unconditional bases in $H^1(D)$ regarded as a *real* linear space. To construct bases in $H^1(D)$ over C, we have to use other systems. The following systems of knots on R are suitable for this purpose:

1. $\pi(j) = (t_i^{(j)}, i \in Z)$, $j \in Z$, with $t_i^{(j)} = i/2^j$;
2. $\pi(j, k) = (t_i^{(j,k)}, i \in Z)$, $j \in Z$, $j > 0$, $k = 1, \ldots, 2^j-1$, with

$$t_i^{(j,k)} = \begin{cases} n & \text{for } i = n(2^j+k), \\ n+p/2^{j+1} & \text{for } i = n(2^j+k)+p,\ p \in Z,\ 1 \leqslant p < 2\lceil k/2 \rceil, \\ n+(p-\lceil k/2 \rceil)/2^j & \text{for } i = n(2^j+k)+p, \\ & \qquad 2\lceil k/2 \rceil \leqslant p < 2^j+\lceil k/2 \rceil-\lfloor k/2 \rfloor, \\ n+(p+2^j-k)/2^{j+1} & \text{for } i = n(2^j+k)+p, \\ & \qquad 2^j+\lceil k/2 \rceil-\lfloor k/2 \rfloor \leqslant p < 2^j+k, \end{cases}$$

where

$$\lceil x \rceil = \min\{n \in Z:\ n \geqslant x\}, \qquad \lfloor x \rfloor = \max\{n \in Z:\ n \leqslant x\};$$

3. $\pi(j, k, l) = (t_i^{(j,k,l)}, i \in Z)$, $j \in Z$, $j > 0$, $k = 0, \ldots, 2^j-2$, $l \in Z$, with

$$t_i^{(j,k,l)} = \begin{cases} t_{i+l}^{(j,k+1)} & \text{for } i \leqslant l(2^j+k), \\ t_i^{(j,k)} & \text{for } i > l(2^j+k), \end{cases} \qquad l < 0,$$

and

$$t_i^{(j,k,l)} = \begin{cases} t_i^{(j,k+1)} & \text{for } i \leqslant l(2^j+k+1), \\ t_i^{(j,k)} & \text{for } i > l(2^j+k+1), \end{cases} \qquad l \geqslant 0,$$

and for $j \leqslant 0$

$$\pi(j, 0, l) = (t_i^{(j,0,l)}, i \in Z), \qquad l \in Z,$$

with

$$t_i^{(j,0,l)} = \begin{cases} i/2^{j+1} & \text{for } i \leqslant 2l, \\ (2i-l)/2^j & \text{for } i > 2l. \end{cases}$$

Now, by the same method as in [3] we can construct unconditional bases in $H^1(D)$ over C, using the spline functions of order $r > 1$. We give a sketch of the construction.

Let π be one of the partitions 1 – 3, let t_l, $l \in Z$, be the knots of π and let

$$N_{r,l}^{\pi}(s) = (t_{l+r} - t_l)[t_l, \ldots, t_{l+r}; (t-s)_+^{r-1}]$$

be the corresponding B-spline of order r. We begin our construction with the following subspaces of $L^2(R)$:

$$S_\pi^r L^2 := \operatorname{span}_{L^2(R)}(N_{r,l}^{\pi}, l \in Z).$$

It is clear that the codimension of $S_{\pi(j,k,l)}^r$ in $S_{\pi(j,k,l+1)}^r$ is one. Thus, there is $f_{(j,k,l)}^{(r)} \in S_{\pi(j,k,l+1)}^r$ unique up to sign which is orthogonal to $S_{\pi(j,k,l)}^r$ with

$$\|f_{(j,k,l)}^{(r)}\|_{L^2(R)} = 1.$$

We choose the sign so that

$$\operatorname{sgn} f_{(j,k,l)}^{(r)}(s^{(j,k,l)}) = 1,$$

where $s^{(j,k,l)}$ is the knot belonging to $\pi(j, k, l+1)$ but not to $\pi(j, k, l)$. It follows from the definition and the inclusions

$$S_{\pi(j)}^r \subset S_{\pi(j,k)}^r \subset S_{\pi(j,k,l)}^r \subset S_{\pi(j,k,l+1)}^r \subset S_{\pi(j,k+1)}^r \subset S_{\pi(j+1)}^r$$

that the system

$$(f_{(j,k,l)}^{(r)}, j, l \in Z, k = 0, \ldots, \max(0, 2^j - 1))$$

is orthonormal in $L^2(R)$.

Along with the orthonormal system $(f_{(j,k,l)}^{(r)})$ we are interested in the family of biorthogonal systems

$$(f_{(j,k,l)}^{(r,m)}, f_{(j,k,l)}^{(r,-m)}), \qquad 0 \leqslant m < r, \ m \in Z,$$

which appears in a natural way in the course of construction of spline bases in the Hardy spaces. The function $f_{(j,k,l)}^{(r,m)}$, for $|m| < r$, is defined as follows:

$$f_{(j,k,l)}^{(r,m)} = \begin{cases} D^m f_{(j,k,l)}^{(r)} & \text{for } m \geqslant 0, \\ H^{-m} f_{(j,k,l)}^{(r)} & \text{for } m < 0, \end{cases} \qquad j, l \in Z, \ k = 0, \ldots, \max(0, 2^j - 1),$$

where D is the differentiation operator and $(Hf)(s) = \int_s^\infty f(t)\,dt$.

The following proposition summarizes the properties of the functions $f_{(j,k,l)}^{(r,m)}$ important for further constructions. They can be proved in a similar way to the properties of the function $f_{(j,k)}^{(r,m)}$ in [3].

PROPOSITION 1. *There are constants $C = C(r)$, $q = q(r)$, $0 < q < 1$, such that for $|m| < r$ we have*

$$|f_{(j,k,l)}^{(r,m)}(t)| \leqslant CN^{1/2+m} q^{N|t-s(j,k,l)|}, \qquad N = 2^j,\ t \in R.$$

Moreover,

$$(f_{(j,k,l)}^{(r,m)}, 1)_{L^2(R)} = 0,$$

$$(f_{(j,k,l)}^{(r,m)}, f_{(j',k',l')}^{(r,-m)})_{L^2(R)} = \delta_{(j,k,l),(j',k',l')}.$$

In what follows we identify the one-dimensional torus with the interval $T = \langle -1/2, 1/2)$. For given integers m and r such that $-r < m < r$ we define the periodic spline in the usual way:

$$\mathring{f}_{(j,k,l)}^{(r,m)}(t) = \sum_{n\in Z} f_{(j,k,l)}^{(r,m)}(t-n) \qquad \text{for } j \geqslant 0,\ k = 0, \ldots, \max(0, 2^j - 1).$$

The following proposition is a consequence of Proposition 1 (cf. [3]).

PROPOSITION 2. *There are constants $C = C(r)$, $q = q(r)$, $0 < q < 1$ such that*

$$|\mathring{f}_{(j,k,l)}^{(r,m)}(t)| \leqslant CN^{1/2+m} q^{Nd_T(t,s(j,k,l))}$$

for $t \in R$, $N = 2^j$. Here d_T is the distance on the torus T.

It is a consequence of Proposition 2 that the functions $\mathring{f}_{(j,k,l)}^{(r,m)}$ are well defined. Moreover, by the definition of $f_{(j,k,l)}^{(r,m)}$,

$$f_{(j,k,l)}^{(r,m)}(t-n) = f_{(j,k,l+n)}^{(r,m)}(t), \qquad n \in Z,$$

which implies

$$\mathring{f}_{(j,k,l)}^{(r,m)}(t) = \mathring{f}_{(j,k,l+n)}^{(r,m)}(t), \qquad n, l \in Z.$$

Thus among the functions $\mathring{f}_{(j,k,l)}^{(r,m)}$, for fixed $j \geqslant 0$, there are only 2^j different ones, and we label them as follows

$$f_0^{(r,m)} = 1,$$

$$f_n^{(r,m)} = \mathring{f}_{(j,k,0)}^{(r,m)}, \qquad \text{for } n = 2^j + k,\ j \geqslant 0,\ k = 0, \ldots, 2^j - 1.$$

It follows from the definition and Proposition 1 that the systems

$$(f_n^{(r,m)}, f_n^{(r,-m)})_{n=0}^{\infty}$$

are biorthogonal in $L^2(T)$.

THEOREM 1. *For $|m| < r$, $r \geqslant 1$ the system $(f_n^{(r,m)})_{n=0}^{\infty}$ is an unconditional basis in $L^2(T)$.*

To prove this theorem we have to investigate the orthogonal projections onto finite-dimensional subspaces spanned by $(f_n^{(r,m)})$. All we need is done for other systems of knots by Z. Ciesielski in the second and third parts of [3] but his proofs, with obvious modifications, work in our case too.

Let us introduce the following operator acting on functions on T:

$$(Hf)(t) = \int_t^{1/2} f(s)\,ds - \int_{-1/2}^{1/2} \int_u^{1/2} f(s)\,ds\,du.$$

We define the system of functions

$$g_n^{(r)} = \begin{cases} (f_n^{(r)}(t)+\check{f}_n^{(r)}(t))\|f_n^{(r)}+\check{f}_n^{(r)}\|_{L^2(T)}^{-1} & \text{for } n = 0, 1, \\ (f_{2(n-1)}^{(r)}(t)+\check{f}_{2(n-1)}^{(r)}(t))\|f_{2(n-1)}^{(r)}+\check{f}_{2(n-1)}^{(r)}\|_{L^2(T)}^{-1} & \text{for } n = 2, 3, \ldots, \end{cases}$$

where $\check{f}(t) = f(-t)$,

$$g_n^{(r,m)} = \begin{cases} D^m g_n^{(r)} & \text{for } 0 \leqslant m < r, \\ H^{-m} g_n^{(r)} & \text{for } -r < m < 0 \end{cases}$$

for $n > 0$, and

$$g_0^{(r,m)} = 1 \quad \text{for each } m.$$

PROPOSITION 3. (a) *The system $(g_n^{(r)})_{n=0}^{\infty}$ is an orthonormal system of even functions and it is complete in the subspace $L_+^2(T)$ of even functions in $L^2(T)$.*

(b) *If m is even then $(g_n^{(r,m)}, g_n^{(r,-m)})_{n=0}^{\infty}$ is a complete biorthogonal system in $L_+^2(T)$.*

If m is odd then $(g_n^{(r,m)}, g_n^{(r,-m)})_{n=1}^{\infty}$ is a complete biorthogonal system in the subspace $L_-^2(T)$ of odd functions in $L^2(T)$.

Proof. Let $S_n = \operatorname{span}(f_i^{(r)}, i \leqslant 2n-1)$ for $n = 1, 2, \ldots$ Let S_n^+ be the subspace of even functions in S_n. Since the system of knots corresponding to S_n is symmetric with respect to the origin, S_n^+ is the image of S_n under the orthogonal projection onto $L_+^2(T)$. It is clear that $f_n^{(r)}$ is not odd for n even, therefore $g_n^{(r)} \not\equiv 0$ for each n. Moreover, $g_n^{(r)} \in S_n^+$ but $g_n^{(r)} \notin S_{n-1}^+$. The codimension of S_{n-1}^+ in S_n^+ is one and $L_+^2(T) = \overline{\bigcup_n S_n^+}$. Thus

$$L_+^2(T) = \operatorname{span}_{L^2(T)}(g_n^{(r)}, n = 0, 1, 2, \ldots).$$

The orthogonality of the functions $(g_n^{(r)})$ is a consequence of the construction of $g_n^{(r)}, f_n^{(r)}, f_{(j,k,l)}^{(r)}$, and particularly of the fact that

$$\int_R f_{(j,k,l)}^{(r)}(t) f_{(j',k',l')}^{(r)}(-t)\,dt = 0 \quad \text{for } (j, k) \neq (j', k'),\ k, k' \text{ even}.$$

Part (b) follows easily from part (a).

THEOREM 2. *If m, $|m| < r$, is even then the system $(g_n^{(r,m)})_{n=0}^{\infty}$ is an unconditional basis in $L_+^2(T)$.*

If m, $|m| < r$, is odd then the system $(g_n^{(r,m)})_{n=1}^{\infty}$ is an unconditional basis in $L_-^2(T)$.

For the proof we need the following well-known fact (cf. [4], Th. 16.1, Ch. II).

LEMMA. *Let X be a Banach space and $(x_n, f_n)_{n=1}^{\infty}$ an X-complete biorthogonal system. Then $(x_n)_{n=1}^{\infty}$ is an unconditional basis in X iff the series*

$$\sum_{n=1}^{\infty} |f_n(x)|\,|f(x_n)|$$

converge for each $x \in X$ and $f \in X^*$.

Proof of Theorem 2. The proof is quite easy. Let us assume that $m \geqslant 0$ is even and let $x(t),\ y(t) \in L_+^2(T)$. We have

$$\begin{aligned}\sum_{n=0}^{\infty} |(x, g_n^{(r,-m)})|\,|(g_n^{(r,m)}, y)| \leqslant{}& \sum_{n=0}^{\infty} |(x, c_n H^m f_n^{(r)})|\,|(y, c_n D^m f_n^{(r)})| \\ &+ \sum_{n=0}^{\infty} |(x, c_n H^m f_n^{(r)})|\,|(y, c_n D^m f_n^{(r)})| \\ &+ \sum_{n=0}^{\infty} |(x, c_n H^m f_n^{(r)})|\,|(y, c_n D^m f_n^{(r)})| \\ &+ \sum_{n=0}^{\infty} |(x, c_n H^m f_n^{(r)})|\,|(y, c_n D^m f_n^{(r)})|\end{aligned}$$

where

$$c_n = \|f_n^{(r)} + \check{f}_n^{(r)}\|_{L^2(T)}^{-1}.$$

One can easily verify that

$$f_n^{(r,m)} = \begin{cases} D^m f_n^{(r)} & \text{for } m \geqslant 0, \\ H^{-m} f_n^{(r)} & \text{for } m < 0 \end{cases}$$

and

$$(H(\check{f}))(t) = -(Hf)\check{\ }(t), \quad (x, Hf)_{L^2(T)} = -(x, H\check{f})_{L^2(T)}, \quad f \in L^2(T).$$

Now, after simple computations, it follows from Theorem 1 that each sum above is finite. Thus by the lemma $(g_n^{(r,m)})_{n=0}^{\infty}$ is an unconditional basis in $L_+^2(T)$.

Using the technique developed in [2] we can prove the following standard estimates for $g_n^{(r,m)}$:

$$|g_n^{(r,m)}(t)| \leqslant C n^{1/2+m} q^{n\delta(t;s_n)},$$

$$|g_n^{(r,m)}(t) - g_n^{(r,m)}(s)| \leqslant C n^{3/2+m} d_T(t,s)\, q^{\delta(t,s;s_n)},$$

$$\|g_n^{(r,m)}\|_{L^2(T)} \sim n^m,$$

where $C = C(r)$, $q = q(r)$, $0 < q < 1$, $t, s \in T$, $s = (2k-1)/2^{j+1}$ for $n = 2^j + k$, $j = 0, 1, \ldots,$ $k = 1, \ldots, 2^j$, and $\partial(t_1, \ldots, t_i; s) = \min(d_T(t_l; s), d_T(-t_l; s)$: $1 \leqslant l \leqslant i)$.

Now we are ready to consider the Hardy spaces. Let $\tilde{g}$ denote the trigonometric conjugate of g. We define $H^1(T)$ in the following way:

$$H^1(T) = \{g \in L^1(T) \colon \tilde{g} \in L^1(T)\},$$

with the norm

$$\|g\|_{H^1(T)} = \|g\|_{L^1(T)} + \|\tilde{g}\|_{L^1(T)}.$$

LEMMA. *If $r \geqslant 2$ and $|m| \leqslant r-2$ then there is $C = C(r)$ such that*

$$\|g_n^{(r,m)}\|_{H^1(T)} \leqslant Cn^{-1/2+m}.$$

This lemma was proved for spline orthonormal systems in [1] (see also [6]). In the nonorthogonal case it can be proved with minor modifications only, therefore we omit the proof.

THEOREM 3. *Let $r \geqslant 2$ and $|m| \leqslant r-2$. Then:*

(a) *For m even the system $(g_n^{(r,m)})_{n=0}^{\infty}$ is an unconditional basis in the subspace $H^1_+(T)$ of even functions in $H^1(T)$. For m odd the system $(g_n^{(r,m)})_{n=1}^{\infty}$ is an unconditional basis in the subspace $H^1_-(T)$ of odd functions in $H^1(T)$.*

(b) *The system $\{1\} \cup \{g_n^{(r,m)}\}_{n=1}^{\infty} \cup \{\tilde{g}_n^{(r,m)}\}_{n=1}^{\infty}$ is an unconditional basis in $H^1(T)$.*

(c) *The system $\{1\} \cup \{G_n^{(r,m)}(\varrho e^{it})\}_{n=1}^{\infty}$ is an unconditional basis in $H^1(D)$. Here $G_n^{(r,m)}(\varrho e^{it}) = g_n^{(r,m)} + i\tilde{g}_n^{(r,m)}(\varrho e^{it})$ where $g_n^{(r,m)}$ and $\tilde{g}_n^{(r,m)}$ are extended to D via the Poisson formula.*

We sketch the proof only. Since both $H^1_+(T)$ and $H^1_-(T)$ are isomorphic to $H^1(I)$, the atomic H^1 space over $I = \langle 0, 1 \rangle$ (see e.g. [1]), we can consider both cases together. Thus to prove (a) it is sufficient to prove that the system $h_n^{(r,m)}(t) = g_n^{(r,m)}(t/2)$, $t \in I$, $n = 0, 1, \ldots$ for m even and $n = 1, 2, \ldots$ for m odd, is an unconditional basis in $H^1(I)$. Analogously to P. Wojtaszczyk]([6]) we can prove that if $f = \sum_n a_n h_n^{(r,m)}$ then

$$\|f\|_{BMO} \sim \sup_n \Big(n \sum_{(l) \subset (n)} l^{2m} |a_l|^2\Big)^{1/2},$$

where (n) denotes the dyadic interval corresponding to the function $h_n^{(r,m)}$. This proves (a). Part (b) follows from the fact that the trigonometric conjugation operator $\tilde{}$ is an isomorphism of $H^1(T)$ (mod constants), and $\tilde{}$ maps even functions to odd ones and vice versa. Part (c) is obvious.

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