TOPICS IN ALGEBRA
BANACH CENTER PUBLICATIONS, VOLUME 26, PART 2
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1990

SOME APPLICATIONS OF THE TRACE MAPPING FOR DIFFERENTIALS

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In memoriam Erich Platte

In this paper we study the trace mapping for differentials of (complex) analytic algebras, which has already been used in [4] and [5]. We give new proofs of essential results in the papers [17], [18], [19] and [20] by E. Platte, moreover we generalize some of these results.

The base field is assumed to be the field $\mathbf{C}$ of complex numbers only for clearness and simplicity. But it is important to notice that the characteristic of the field is zero.

(1) Notations. By a *complex analytic algebra A* we understand a residue algebra of a formal or convergent power series algebra $\mathbf{C}\langle\langle X_1, \ldots, X_n\rangle\rangle$ over $\mathbf{C}$. In the following A is additionally assumed to be reduced and pure dimensional. $\Omega_A^{\boldsymbol{\cdot}} = (\Omega_A^{\boldsymbol{\cdot}}, d)$ denotes the complex of Kähler or holomorphic differentials, and $M_A^{\boldsymbol{\cdot}} = Q(A)\otimes_A \Omega_A^{\boldsymbol{\cdot}} = (M_A^{\boldsymbol{\cdot}}, d)$ denotes the complex of meromorphic differentials. $\Omega_A^{\boldsymbol{\cdot}}$ is the exterior algebra of the universally finite differential module $D_{\mathbf{C}}(A) = \Omega_A^1$ of A, $M_A^{\boldsymbol{\cdot}}$ is the exterior algebra of the free module $Q(A)\otimes_A \Omega_A^1 = M_A^1$ over the total quotient ring $Q(A)$ of A. The rank of Ω_A^1, i.e. the rank of M_A^1 is $\dim A$. See also [13] and [23].

The kernel of the canonical complex homomorphism $\Omega_A^{\boldsymbol{\cdot}} \to M_A^{\boldsymbol{\cdot}}$ is the torsion subcomplex $\mathrm{t}\Omega_A^{\boldsymbol{\cdot}}$ of $\Omega_A^{\boldsymbol{\cdot}}$. We always identify $\tilde{\Omega}_A^{\boldsymbol{\cdot}} := \Omega_A^{\boldsymbol{\cdot}}/\mathrm{t}\Omega_A^{\boldsymbol{\cdot}}$ with its image in $M_A^{\boldsymbol{\cdot}}$ and hence as a subcomplex and a subalgebra of $M_A^{\boldsymbol{\cdot}}$.

(2) The trace mapping. Let $A \to B$ be a finite extension of reduced pure-d-dimensional analytic $\mathbf{C}$-algebras with a well defined rank r, i.e. $Q(B)$ is a free extension of $Q(A)$ with rank r. Then $M_B^{\boldsymbol{\cdot}} = Q(B)\otimes_{Q(A)} M_A^{\boldsymbol{\cdot}}$, and the trace mapping $\mathrm{Sp}\colon Q(B) \to Q(A)$ induces an $M_A^{\boldsymbol{\cdot}}$-linear trace mapping $\mathrm{Sp}\otimes_{Q(A)} M_A^{\boldsymbol{\cdot}}$

This paper is in final form and no version of it will be submitted for publication elsewhere.

from $M^{\cdot}_B$ to $M^{\cdot}_A$, which we also denote by

$$\mathrm{Sp}^B_A\colon M^{\cdot}_B \to M^{\cdot}_A.$$

Obviously $\mathrm{Sp}^B_A | M^{\cdot}_A = r\,\mathrm{id}_{M^{\cdot}_A}$. This trace mapping commutes with the exterior derivatives d on $M^{\cdot}_A$ and $M^{\cdot}_B$. For this it is sufficient to verify that the diagram

$$\begin{array}{ccc} Q(B) & \xrightarrow{d} & M^1_B \\ {\scriptstyle \mathrm{Sp}}\downarrow & & \downarrow{\scriptstyle \mathrm{Sp}} \\ Q(A) & \xrightarrow{d} & M^1_A \end{array}$$

is commutative. This may be done explicitly for elements $y \in Q(B)$ by using representations $yx_\sigma = \sum_\varrho a_{\varrho\sigma} x_\varrho$, $dx_\sigma = \sum_\varrho \omega_{\varrho\sigma} x_\varrho$ with a $Q(A)$-base $x_1, \ldots, x_r$ of $Q(B)$ and coefficients $a_{\varrho\sigma} \in Q(A)$, $\omega_{\varrho\sigma} \in M^1_A$ (see also [5] and the more general discussions in [7], § 4 and [15], § 4). In the convergent case one can use alternatively a finite (ramified) covering $Y \to X$ of complex space germs which represents the extension $A \to B$. For a given $\omega \in M^p_Y$ one removes the ramification points of the covering and the singular points of ω, the problem is then reduced to the trivial case of a holomorphic form and a finite unramified covering.

We remark that for $p \in \mathbf{Z}$ the canonical diagram

$$\begin{array}{ccc} M^p_B & \xrightarrow{\mathrm{Sp}} & M^p_A \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{Q(B)}(M^{d-p}_B, M^d_B) & \longrightarrow & \mathrm{Hom}_{Q(A)}(M^{d-p}_A, M^d_A) \end{array}$$

is commutative, where the homomorphism of the bottom row is the composition of the trace mapping on M^d_B and the restriction to M^{d-p}_A.

As a first application of the existence of the trace mapping we prove the following result which answers a question raised by G. Scheja and has been proved for the first time by E. Platte.

(3) Proposition (Platte [18]). *Let $P := \mathbf{C}\langle\langle X_1, \ldots, X_n\rangle\rangle$ and $f \in \mathfrak{m}^2_P$ a function which defines an isolated singularity. Then the degree of f over $Q := \mathbf{C}\langle\langle \partial f/\partial X_1, \ldots, \partial f/\partial X_n\rangle\rangle$ equals the rank of P over Q and has hence the maximal value.*

Proof. Let $\partial_i := \partial/\partial X_i = (dX_i)^*$ be the partial derivations and A the normalization of the hypersurface algebra

$$Q\langle\langle f\rangle\rangle = Q[f] = \mathbf{C}\langle\langle \partial_1 f, \ldots, \partial_n f, f\rangle\rangle.$$

We have to show that $P = A$. From

$$\omega_i := \mathrm{Sp}^B_A(dX_i) = d\,\mathrm{Sp}^B_A(X_i) \in \tilde{\Omega}^1_A = \Omega^1_A/\mathrm{t}\Omega^1_A \subseteq M^1_A$$

and the equality $df = \sum_i \partial_i f\, dX_i \in \tilde{\Omega}^1_A \subseteq \Omega^1_P$ we get with the trace mapping

$$r df = \sum_i (\partial_i f)\, \omega_i \in \tilde{\Omega}^1_A \subseteq \Omega^1_P,$$

where $r \geqslant 1$ is the rank of P over A. Applying the linear forms ∂_j we get

$$r\partial_j f = \sum_i (\partial_i f)(\partial_j \omega_i).$$

Since $\partial_1 f, \ldots, \partial_n f$ generate minimally the P-ideal $\sum_j P\partial_j f$ the matrix $(1/r)(\partial_j \omega_i)$ concides modulo $\mathfrak{m}_p$ with the identity matrix. In particular $(\partial_j \omega_i)$ is invertible and $\omega_1, \ldots, \omega_n$ is a P-base of Ω_P^1. Therefore P is an unramified finite extension of A, and this means $P = A$.

We remark that in the proof of Proposition (3) besides the finiteness of P over A we only used the facts that A is normal and contains the function f together with its partial derivatives $\partial_1 f, \ldots, \partial_n f$ and that $\partial_1 f, \ldots, \partial_n f$ generate minimally a P-ideal. The last condition means (by Zariski's lemma, see [14]), that there is no coordinate system $\tilde{X}_1, \ldots, \tilde{X}_n$ for P, such that f is independent of one of the functions $\tilde{X}_i$.

In the situation of Proposition (3) the rank of P over Q, i.e. the degree of f over Q is equal to the Milnor number μ of the singularity P/Pf. The Jacobian of the extension $Q \to P$ is the Hesse determinant $\operatorname{Hesse}(f) = \operatorname{Det}(\partial^2 f/\partial X_i \partial X_j)$. From a representation $\partial f/\partial X_j = \sum_i a_{ij} X_i$ with $a_{ij} \in P$ one gets

$$\operatorname{Hesse}(f) = \mu \operatorname{Det}(a_{ij}) \bmod (\partial f/\partial X_1, \ldots, \partial f/\partial X_n),$$

cf. [24]. Furthermore we mention that the minimal equation of the function f over $\mathbf{C}\langle\langle \partial_1 f, \ldots, \partial_n f\rangle\rangle$ is also an equation of integral dependence of f over the ideal $\sum_j (\partial_j f) P$ generated by its partial derivatives, cf. [22].

(4) Regular and extendable differential forms. In general the trace Sp_A^B: $M_B^\cdot \to M_A^\cdot$ doesn't map $\tilde{\Omega}_B^\cdot = \Omega_B^\cdot/t\Omega_B^\cdot$ into $\tilde{\Omega}_A^\cdot = \Omega_A^\cdot/t\Omega_A^\cdot$. But the complexes of regular resp. extendable differential forms which we shall define now are invariant under the trace mapping. ($\mathrm{Sp}(\tilde{\Omega}_B^\cdot) \subseteq \tilde{\Omega}_A^\cdot$ holds in the special case, that $A \to B$ is a finite free extension, cf. for example [7], § 4.)

Let A be pure-d-dimensional and reduced and

$$P \to A, \ P := \mathbf{C}\langle\langle X_1, \ldots, X_d\rangle\rangle,$$

a noetherian normalization of A. Then we get a subcomplex $\Delta_A^\cdot = (\Delta_A^\cdot, d)$ of $(M_A^\cdot, d)$ by setting

$$\Delta_A^i := \{\omega \in M_A^i \colon \mathrm{Sp}_P^A(\omega \wedge \tilde{\Omega}_A^{d-i}) \subseteq \Omega_P^d\} \cong \operatorname{Hom}_P(\Omega_A^{d-i}, \Omega_P^d),$$

$i \in \mathbf{Z}$. Obviously we have $\Delta_A^i = \{\omega \in M_A^i \colon \omega \wedge \tilde{\Omega}_A^{d-i} \subseteq \Delta_A^d\} \cong \operatorname{Hom}_A(\Omega_A^{d-i}, \Delta_A^d)$. The independence of Δ_A^i, $i \in \mathbf{Z}$, from the chosen normalization $P \to A$ is a consequence of this independence for the case $i = d$, which is proved e.g. in [12]. There is a canonical inclusion $\tilde{\Omega}_A^\cdot \subseteq \Delta_A^\cdot$. We call $\Delta_A^\cdot$ the *complex of regular differential forms* of A. This complex can be constructed (with considerably more effort) for arbitrary (not necessarily reduced) analytic algebras, see [8]. For normal A we have simply $\Delta_A^\cdot \cong (\Omega_A^\cdot)^{**}$ because in this case both complexes are A-reflexive and coincide in codimension $\leqslant 1$.

In the convergent case the *complex $\Gamma^{\bullet}_A$ of extendable differential forms* may be defined as $(\pi_* \Omega^{\bullet}_{X'})_x$, where $\pi\colon X' \to X$ is a desingularization of the complex space germ $X = (X, x)$ associated to A. Γ^p_A is also the module of locally square integrable meromorphic differential forms in degree p, $p \in \mathbf{Z}$ (with the normalization of A as Γ^0_A).

In the general case one has to replace X' by a desingularization of Spec A. In the convergent case both definitions give the same complex. Obviously $\Gamma^{\bullet}_A$ can be identified with a subcomplex of $\Delta^{\bullet}_A$ and is even an algebra (whereas in general $\Delta^{\bullet}_A$ isn't). Moreover $\Gamma^{\bullet}_A = \Gamma^{\bullet}_{A'}$, where A' is the normalization of A. For details see [11]. cf. also [4].

(5) PROPOSITION. *Let $A \to B$ be a finite extension of pure-d-dimensional analytic algebras with a well defined rank. Then $\mathrm{Sp}^B_A(\Delta^{\bullet}_B) \subseteq \Delta^{\bullet}_A$ and $\mathrm{Sp}^B_A(\Gamma^{\bullet}_B) = \Gamma^{\bullet}_A$. So there is a canonical diagram*

$$\begin{array}{ccccc} \Gamma^{\bullet}_A & \to & \Gamma^{\bullet}_B & \xrightarrow{\mathrm{Sp}^B_A} & \Gamma^{\bullet}_A \\ & & \downarrow & & \downarrow \\ & & \Delta^{\bullet}_B & \xrightarrow{\mathrm{Sp}^B_A} & \Delta^{\bullet}_A, \end{array}$$

where the composition of the mappings in the top row is the multiplication with $r := \mathrm{rank}_A B$. If A is normal the inclusion $\tilde{\Omega}^{\bullet}_A \to \tilde{\Omega}^{\bullet}_B$ extends to an inclusion $\Delta^{\bullet}_A \to \Delta^{\bullet}_B$.

Proof. First we consider regular d-forms. By definition we have the equalities $\Delta^d_B = C^B_P\, dt_1 \wedge \ldots \wedge dt_d$ and $\Delta^d_A = C^A_P\, dt_1 \wedge \ldots \wedge dt_d$ with a system of parameters $t_1, \ldots, t_d$ in A and the Dedekind complementary modules C^B_P resp. C^A_P associated to the finite extensions $P := \mathbf{C}\langle\langle t_1, \ldots, t_d \rangle\rangle \to B$ resp. $P \to A$.

By the transitivity of the trace C^B_P is mapped by the trace into C^A_P and hence Δ^d_B into Δ^d_A. For regular p-forms, $p \in \mathbf{Z}$, the assertion follows from the commutative diagram

$$\begin{array}{ccccc} \Delta^p_B & & \xrightarrow{\qquad\qquad\mathrm{Sp}^B_A\qquad\qquad} & & \Delta^p_A \\ \| & & & & \| \\ \mathrm{Hom}_B(\Omega^{d-p}_B, \Delta^d_B) & \xrightarrow{\mathrm{Hom}(\Omega^{d-p}_B, \mathrm{Sp}^B_A)} & \mathrm{Hom}_A(\Omega^{d-p}_B, \Delta^d_A) & \to & \mathrm{Hom}_A(\Omega^{d-p}_A, \Delta^d_A), \end{array}$$

which is obtained from the case $p = d$ just proved. The inclusion $\Delta^{\bullet}_A \to \Delta^{\bullet}_B$ is obvious if A and B are both normal. But in general $\Delta^{\bullet}_{B'} \subseteq \Delta^{\bullet}_B$, where B' is the normalization of B.

Now we consider extendable differential forms. For functions the assertion is clear, because the trace of entire elements is again entire.

In the convergent case the homomorphism $A \to B$ is associated to a finite covering $Y \to X$ of complex spaces. To prove that for any locally square integrable differential form ω on Y the trace form $\mathrm{Sp}(\omega)$ is again locally square integrable on X it is sufficient to consider d-forms. But outside a suitable exceptional set $S \subseteq X$ (depending on ω) the covering $Y \to X$ is an unramified covering $Y' \to X'$, $X' := X - S$, of complex manifolds such that ω is holomorphic on Y' and square integrable. Then it is trivial that the form $\mathrm{Sp}(\omega)$ on X' is square integrable too. In the general case let $X := \mathrm{Spec}\, A$, $Y := \mathrm{Spec}\, B$ and π: $\tilde{X} \to X$ a desingularization. We consider the commutative diagram

$$\begin{array}{ccc} (\tilde{X} \times_X Y)_{\mathrm{red}} & \to & Y \\ \downarrow & & \downarrow \\ \tilde{X} & \xrightarrow{\pi} & X. \end{array}$$

An extendable differential form ω on Y has an extendable and therefore regular pull-back $\tilde{\omega}$ on $(\tilde{X} \times_X Y)_{\mathrm{red}}$. Hence $\mathrm{Sp}(\tilde{\omega}) = \pi^* \mathrm{Sp}(\omega)$ is again regular on $\tilde{X}$ and therefore holomorphic on $\tilde{X}$.

(6) COROLLARY. *In the situation of Proposition* (5) *for* $p \in \mathbf{Z}$ *the* A*-module* Γ_A^p *is a direct summand of* Γ_B^p, *and the vector space* $H^p(\Gamma_A^\cdot)$ *is a direct summand of* $H^p(\Gamma_B^\cdot)$. *If* A *is normal* Δ_A^p *is a direct* A*-summand of* Δ_B^p *and* $H^p(\Gamma_A^\cdot)$ *a direct summand of* $H^p(\Delta_B^\cdot)$.

Proof. The assertions follow from the fact that the composition of the complex homomorphism $\Gamma_A^\cdot \to \Gamma_B^\cdot$ and Sp: $\Gamma_B^\cdot \to \Gamma_A^\cdot$ resp. (in the normal case) the composition of $\Delta_A^\cdot \to \Delta_B^\cdot$ and Sp: $\Delta_B^\cdot \to \Delta_A^\cdot$ is the multiplication with $\mathrm{rank}_A B$.

As a special case of Corollary (6) let A be normal and B smooth. Then $\Delta_A^p = (\Omega_A^p)^{**}$ is a Macaulay module for every $p \in \mathbf{Z}$ because it is a direct summand of $\Delta_B^p \cong B^{\binom{d}{p}}$. Furthermore $B^* = \mathrm{Hom}_A(B, A)$ is isomorphic to B as a reflexive B-module of rank 1. Therefore the direct A-summand $(\Omega_A^p)^* = (\Omega_A^p)^{***}$ of $B^{*\binom{d}{p}} \cong B^{\binom{d}{p}}$ is a Macaulay module too. As a consequence one gets for example $\mathrm{Ext}_A^i(\Omega_A^p, A) = 0$ for all $i = 1, \ldots, s-2$, $s :=$ codim(sing A) (and especially the rigidity of A if $s \geqslant 3$), cf. [1], 16.E and [17].

(7) COROLLARY. *Let* $A \to B$ *be a nondegenerate extension of analytic algebras (i.e.* $\dim B = \dim A + \dim B/\mathfrak{m}_A B$*) with* A *normal and* B *reduced and pure dimensional. If* $\Gamma_B^p = \Delta_B^p$ *for some* $p \in \mathbf{Z}$ *then also* $\Gamma_A^p = \Delta_A^p$.

Proof. If the elements $f_1, \ldots, f_m \in B$ form a system of parameters in the fibre $B/\mathfrak{m}_A B$ the algebra B is a finite extension of $A' := A\langle\langle f_1, \ldots, f_m\rangle\rangle$. By (5) we have $\Gamma_{A'}^p = \mathrm{Sp}_{A'}^B(\Gamma_B^p) = \mathrm{Sp}_{A'}^B(\Delta_B^p) = \Delta_{A'}^p$. Because of the formula

$$\Delta_{A'}^p = \Delta_A^p\langle\langle f_1, \ldots, f_m\rangle\rangle \oplus \sum_i df_i \wedge \Delta_{A'}^{p-1}$$

and the analogous one for $\Gamma_{A'}^p$, we get $\Gamma_A^p = \Delta_A^p$.

We remark that the equality $\Gamma_B^p = \Delta_B^p$ always holds if $p \leqslant$ codim(sing B)-2, cf. [3].

(8) COROLLARY. *Let* $A \to B$ *be a nondegenerate extension of normal analytic algebras with* $\Gamma_B^1 = \Delta_B^1$. *If* $\Delta_A^1 (= (\Omega_A^1)^{**})$ *is free then* A *is smooth.*

Proof. By (7) $\Gamma_A^1 = \Delta_A^1$ is free. The result now follows from the following criterion for smoothness: *If A is reduced and pure dimensional and if Γ_A^1 is free as an A-module then A is smooth*, cf. [10], [11].

Corollary (8) is a partial generalization of a result in [19], where Ω_B^1 is assumed to be reflexive.

(9) Corollary. *For a quasihomogeneous normal analytic algebra B the canonical homomorphism*

$$\Omega_B^1/\mathfrak{m}_B\Omega_B^1 \to \Delta_B^1/\mathfrak{m}_B\Delta_B^1$$

is injective.

Proof. Let $f_1, \ldots, f_d$ be a homogeneous system of parameters in B and A the algebra $\mathbf{C}\langle\langle f_1, \ldots, f_d\rangle\rangle \subseteq B$. By δ we denote the Euler derivations and the corresponding linear forms on the modules of differentials too. The diagram

$$\begin{array}{ccccccc} \Omega_A^1 & \to & \Omega_B^1 & \to & \Delta_B^1 & \xrightarrow{\mathrm{Sp}} & \Omega_A^1 \\ {\scriptstyle\delta}\downarrow & & {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\ A & \to & B & = & B & \xrightarrow{\mathrm{Sp}} & A \end{array}$$

is commutative with $\delta(\Omega_B^1) = \mathfrak{m}_B$. So δ induces an isomorphism $\bar{\delta}$: $\Omega_B^1/\mathfrak{m}_B\Omega_B^1 \to \mathfrak{m}_B/\mathfrak{m}_B^2$. The B-linear form δ: $\Delta_B^1 \to B$ isn't surjective, otherwise $\mathrm{Sp}\circ\delta = \delta\circ\mathrm{Sp}$ and hence δ: $\Omega_A^1 \to A$ would be surjective too. Therefore δ induces a homomorphism $\bar{\delta}$: $\Delta_A^1/\mathfrak{m}_B\Delta_B^1 \to \mathfrak{m}_B/\mathfrak{m}_B^2$. The commutative diagram

$$\begin{tikzcd}[column sep=small] \Omega_B^1/\mathfrak{m}_B\Omega_B^1 \arrow[rr] \arrow[dr, "\bar{\delta}"'] & & \Delta_B^1/\mathfrak{m}_B\Delta_B^1 \arrow[dl, "\bar{\delta}"] \\ & \mathfrak{m}_B/\mathfrak{m}_B^2 & \end{tikzcd}$$

gives now the assertion.

(10) Corollary (Hochster [6]). *Let B be a quasi-homogeneous normal analytic algebra. If Δ_B^1 $(= (\Omega_B^1)^*)$ is free then B is smooth.*

Proof. By (9) the minimal number of generators of the B-module Ω_B^1 is at most the minimal number of generators of Δ_B^1. By assumption this is $d := \dim B$. But Ω_B^1 is a module of rank d, so it is free.

Corollary (9) answers a question raised by E. Platte in [20], Rem. 2.5. A further proof of (10) is given in [16].

(11) Nonramification. Related to the considerations above is the question whether for a finite extension $A \to B$ the property of being unramified can be characterized with the help of regular or extendable differentials instead by using Kähler differentials. More precisely: Let $A \to B$ be a finite extension of normal analytic algebras. By a classical result (which we have used before) the equality $A = B$ holds if and only if the canonical homomorphism

$B \otimes_A \Omega_A^1 \to \Omega_B^1$ is surjective. Is it possible to replace (Ω_A^1, Ω_B^1) by (Δ_A^1, Δ_B^1) or (Γ_A^1, Γ_B^1)? In general this seems to be a difficult problem. The surjectivity of $B \otimes_A \Delta_A^1 \to \Delta_B^1$ or $B \otimes_A \Gamma_A^1 \to \Gamma_B^1$ implies that the extension $A \to B$ is unramified in codimension 1, because the modules Δ_A^1, Γ_A^1 and Ω_A^1 resp. Δ_A^1, Γ_B^1 and Ω_B^1 coincide for the nonsingular locus of A resp. B. Let us assume that Δ_A^1 is a free A-module. Then conversely the mapping $B \otimes_A \Delta_A^1 \to \Delta_B^1$ is surjective if the extension $A \to B$ is unramified in codimension 1 because then the two reflexive B-modules $B \otimes_A \Delta_A^1$ and Δ_B^1 coincide in codimension 1. A positive answer to our question would imply that A has no strict extensions unramified in codimension 1 (if Δ_A^1 is free). So in the case $\dim A = 2$ the algebra A would be pure and hence smooth using a result of Flenner [2]. This would solve the Zariski–Lipman problem in dimension 2.

Here we prove:

(12) *Let $A \to B$ be a finite extension of normal analytic algebras. Then we have*:

(i) *If $B \otimes_A \Delta_A^1 \to \Delta_B^1$ (resp. $B \otimes_A \Gamma_A^1 \to \Gamma_B^1$) is surjective and if*

$$\Omega_B^1/\mathfrak{m}_B \Omega_B^1 \to \Delta_B^1/\mathfrak{m}_B \Delta_B^1 \qquad (\textit{resp. } \Omega_B^1/\mathfrak{m}_B \Omega_B^1 \to \Gamma_B^1/\mathfrak{m}_B \Gamma_B^1)$$

is injective (which by (9) *holds if B is quasihomogeneous) then $A = B$.*

(ii) *If $B \otimes_B \Delta_A^1 \to \Delta_B^1$ (resp. $B \otimes_A \Gamma_A^1 \to \Gamma_B^1$) is surjective and if A is quasihomogeneous then $A = B$.*

Proof. We treat the case of regular differential forms. The proof for extendable forms runs along the same lines. From the surjectivity of $B \otimes \Delta_A^1 \to \Delta_B^1$ one deduces the surjectivity of $\Delta_A^1/\mathfrak{m}_A \Delta_A^1 \to \Delta_B^1/\mathfrak{m}_B \Delta_B^1$ and the equality $\mathfrak{m}_B \Delta_B^1 = \mathfrak{m}_B \Delta_A^1$. Because of $\mathrm{Sp}(\mathfrak{m}_B) = \mathfrak{m}_A$ the trace induces by Theorem (5) a mapping $\Delta_B^1/\mathfrak{m}_B \Delta_B^1 \to \Delta_A^1/\mathfrak{m}_A \Delta_A^1$ which is up to the factor $\operatorname{rank}_A B$ left inverse to the first one and therefore invertible.

To prove (i) we consider the commutative diagram

$$\begin{array}{ccc}
\mathfrak{m}_A & \xrightarrow{\qquad\qquad\qquad\qquad} & \Delta_A^1/\mathfrak{m}_A \Delta_A^1 \\
\downarrow & & \downarrow \\
\mathfrak{m}_B & \to \mathfrak{m}_B/\mathfrak{m}_B^2 = \Omega_B^1/\mathfrak{m}_B \Omega_B^1 \to & \Delta_B^1/\mathfrak{m}_B \Delta_B^1 \\
{\scriptstyle \mathrm{Sp}}\downarrow & & \downarrow{\scriptstyle \overline{\mathrm{Sp}}} \\
\mathfrak{m}_A & \xrightarrow{\qquad\qquad\qquad\qquad} & \Delta_A^1/\mathfrak{m}_A \Delta_A^1
\end{array}$$

and get $\operatorname{kern} \mathrm{Sp} \subseteq \mathfrak{m}_B^2$. It follows $\mathfrak{m}_B = \mathfrak{m}_A + \operatorname{kern} \mathrm{Sp} = \mathfrak{m}_A B + \mathfrak{m}_B^2$ and $\mathfrak{m}_B = \mathfrak{m}_A B$, i.e. $A = B$.

The case (ii) can be reduced to the case (i): Without loss of generality we may assume that A and B are complete. The Euler derivation δ of A can be extended to a derivation of B because the A-linear form $\Delta_A^1 \to A$

corresponding to δ defines a B-linear form on $\Delta_B^1 = (B \otimes_A \Delta_A^1)/\mathrm{t}(B \otimes_A \Delta_A^1)$. (The extendability of δ follows already from the property that the extension $A \to B$ is unramified in codimension 1.) The result (2.14) in [25] implies that B is quasihomogeneous, and part (i) can be applied.

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