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A NEW SCENARIO IN DISSIPATIVE SYSTEMS: SEQUENTIAL EXTINCTION OF STRANGE ATTRACTORS

GÉZA GYÖRGYI*

Department of Physics, Stevens Institute of Technology
Hoboken, NJ, U.S.A

This paper is a review of a work done in collaboration with C. Chen and G. Schmidt [1]. Strange attractors in two-dimensional mappings are revisited. It is demonstrated that as the Jacobian tends to unity, strange attractors of periods 1, 2, 4... are extinguished one by one at critical Jacobian values. The scenario is attributed to a sequence of heteroclinic crises between invariant manifolds of the period doubling set. Scaling properties of crises are accounted for by renormalization theory. It is shown that there is a single relevant dissipative eigendirection of the period doubling operator emanating from the Hamiltonian fixed point map, having the eigenvalue 2.

Introduction

Since the discovery of universality in the period doubling scenario by Feigenbaum [2], the period doubling route to chaos has been one of the most studied nonlinear phenomena. Universality implies that scaling properties of the period doubling sequence are quantitatively the same for large classes of systems. One important universality class is represented by one-dimensional maps with a single quadratic maximum and negative Schwarzian derivative (for a review see [3]). An example of such systems is the quadratic map $x_{n+1} = 1-Kx_n^2$, where K is sometimes called the nonlinearity parameter. By our increasing the parameter K, an originally stable fixed point becomes unstable and bifurcates into a stable period-2 cycle. Then this cycle destabilizes through bifurcation, and so on, the system undergoes an infinite sequence of period doubling bifurcations. Beyond the limiting state where a period-2^∞ orbit is born, chaotic regions can be observed. These are interrupted by periodic windows, i.e. regions where stable periodic orbits live. The chaotic attractor consists, as a coarse grained feature, of 2^n pieces, which are visited by the iteration in a periodic manner [4]. While the nonlinearity

* On leave from the Institute for Theoretical Physics Eötvös University, Budapest, P.O. Box 327, 1088, Hungary

parameter is further increased, the atractor grows in size and its pieces merge sequentially. The pieces of the period-2^n attractor merge pairwise, resulting in a 2^{n-1}-piece attractor. When this happens, the 2^n-th image of the maximum of the map falls onto the unstable period-2^{n-1} orbit which was originally born via period doubling bifurcation. (We mention that the existence of an absolutely continuous invariant measure in such cases follows from a more general theorem [3].) At the end of the inverse bifurcation sequence of chaotic attractors, there is a one-piece attractor. It disintegrates at a critical parameter value K_c and the iteration is driven to another attractor, which is at infinity in the case of the quadratic map. The bifurcation parameter values scale in a universal way. No matter which is the particular form of the map with a quadratic maximum and exhibiting the above scenario, if the bifurcation $2^{n-1} \to 2^n$ and the inverse bifurcation $2^n \to 2^{n-1}$ occurs at $\bar{K}_n$ and K_n, respectively, then the bifurcation parameters obey $(\bar{K}_{n-1} - \bar{K}_n)/(\bar{K}_n - \bar{K}_{n+1}) \to \delta_F$ and $(K_{n-1} - K_n)/(K_n - K_{n+1}) \to \delta_F$. Here $\delta_F = 4.669\ldots$ is a universal number. Both $\bar{K}_n$ and K_n accumulate at the same point, $\bar{K}_\infty = K_\infty$.

Besides one-dimensional maps, also a large family of area-contracting two-dimensional maps exhibit the period doubling scenario [6]. The asymptotic scaling properties of such systems coincide with those of one-dimensional maps. Some nonlinear dissipative differential equations as the Rössler- [7] and Lorenz-equations [8] also seem to belong to the same universality class in particular parameter ranges.

Hamiltonian flows with three phase space dimensions correspond to area-preserving maps of the plane, that is, maps with Jacobian one (see [9] for a review). Such maps, depending on the initial condition, exhibit periodic, quasiperiodic, or chaotic orbits. A periodic orbit can be stable or unstable. It often occurs that a stable orbit becomes unstable due to change in the nonlinearity parameter and also gives birth to a stable orbit with twice the periodicity. In case of an infinite sequence of period doubling bifurcations the limiting state exhibits chaotic behavior in the same neighborhood of the plane, or, the iteration is driven away to infinity. The parameters $\bar{K}_n(1)$ where the period doubling bifurcations occur – the argument 1 refers to the unit Jacobian of the map – accumulate at $K_\infty(1)$. In case of analytic maps, which we restrict our attention to, the convergence is geometric with the universal rate $\delta_H = 8.72\ldots$ [10]–[15].

A major difference between the one-dimensional and the Hamiltonian scenario is that the former exhibits an inverse bifurcation sequence in the chaotic region, whereas the inverse sequence is absent in Hamiltonian systems. We address the question, what is the fate of strange attractors in a dissipative two-dimensional map as the Hamiltonian limit is approached. It turns out that for a large family of maps the region in parameter space where strange attractors live shrinks gradually to zero. Numerical results are presented first, which then are explained by means of renormalization theory.

Disappearance of strange attractors

For the sake of simplicity we first consider maps with constant Jacobian determinant and shall return to the question of nonuniform Jacobians later on. With the Jacobian becoming larger an increasing number of basins coexists in different parts of the plane. We restrict, therefore, our attention to the continuation of the basin of attraction of the originally stable fixed point, the source of the bifurcation sequence we study.

Recent numerical investigation of the dissipative standard map by Schmidt and Wang [16] showed that when the Jacobian determinant of the map was increased, the inverse bifurcation sequence became less and less complete. In other words, strange attractors with low periodicity could not be observed any more for large Jacobians, while strange attractors with higher periodicities were stilll present. The disappearance of the period-1 strange attractor has also been demonstrated by Tél [17] on the Lozi map for large negative Jacobians. We studied numerically iterations of the type

$$x' = f(K, x) - Jy, \qquad y' = x - f(K, x'), \tag{1}$$

the Jacobian of which is J. We considered maps with $f(K, x) = -Kx-(1+K)x^2$ and $f(K, x) = K\sin x$; the following description virtually fits both cases. The only difference is that in the case of the sine map the iteration does not go out to infinity, whereas it may in the quadratic mapping for K large. The regions in the K-J plane where strange attractors born out of the period doubling sequence can be found are shown in Fig. 1. The K axis is

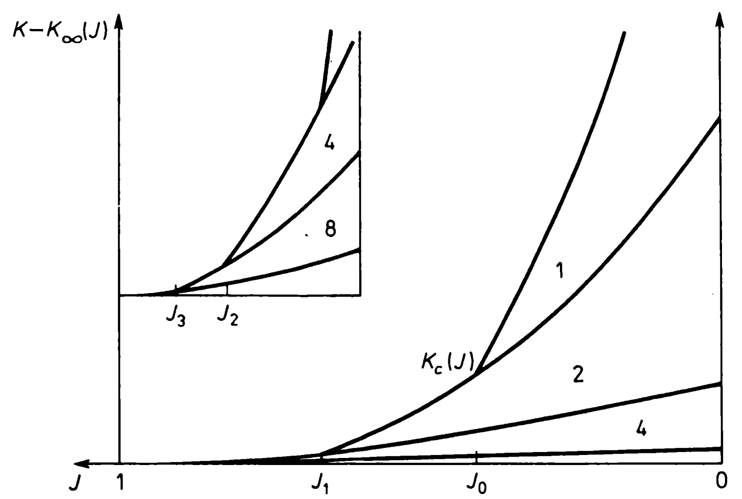

Fig. 1. Regions where strange attractors of periods 1, 2, 4, ... were observed. Inset displays a magnification of part of the diagram. For $J > J_n$ no period-2^n chaos is present within the original basin of attraction. The basin is destroyed beyond the $K_c(J)$ curve

rescaled in the way that $K_\infty(J)$ is a straight line, representing the accumulation of the bifurcations. In the triangular shaped region labeled by p a strange attractor of period $p = 2^n$ can be seen, as a gross feature. A finer resolution would reveal the existence of stable orbits and periodic chaos of periods larger than p, as e.g. period $3p$, in small regions within the triangle. Beyond the upper envelope, $K_c(J)$, of the triangles the attractor born out of the period doubling set is destroyed; in the case of the quadratic map the iteration is driven, in general, towards infinity. A triangle labeled by 2^n extends to the one-dimensional limit, $J \doteq 0$, the edge there forming the interval (K_n, K_{n-1}) where the 2^n-piece chaotic band lives (for $n = 0$ the interval is $(K_0, K_c(0))$. The triangle, however, does not stretch to $J = 1$, but it terminates at a $J_n < 1$. In other words, J_n is the highest possible Jacobian where the period-2^n strange attractor can exist. We find that the series of $J_n - s$ increases monotonically with n and seems to converge towards $J_\infty = 1$. Furthermore, the approximate relation $(J_n)^2 \cong J_{n-1}$ is found, thus the effective Jacobian $J_{\text{eff}} \equiv (J_n)^p$, $p = 2^n$, is virtually independent of n.

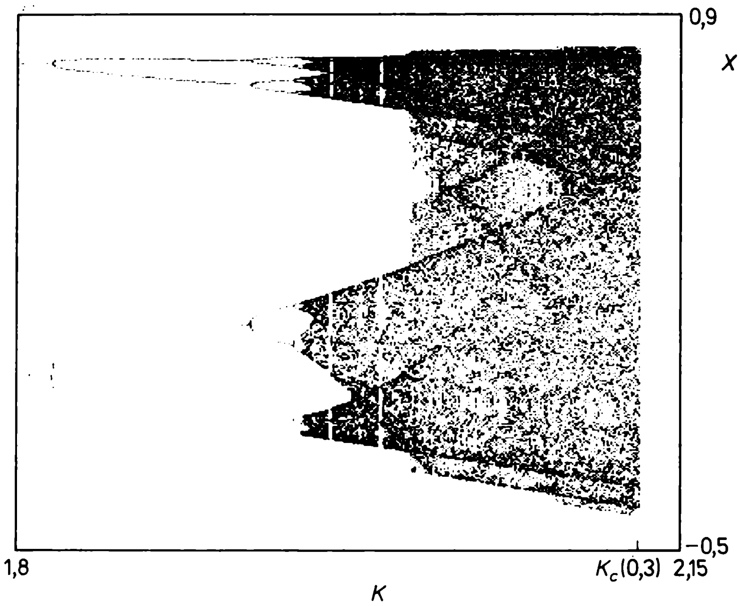

Fig. 2. The projection of the attractor of the Hénon map to the x axis as a function of the nonlinearity parameter (a) $J = 0.3$: there is one-piece chaotic attractor; (b) $J = \sqrt{0.3}$: no one-piece chaos, instead, an escaping iteration can be seen at $K_c(\sqrt{0.3})$; (c) $J = \sqrt{0.3}$: the lower piece of (b) enlarged

We illustrate the disappearance of the period-1 strange attractor on the example of the Hénon map ($x' = 1 - Kx^2 - y$, $y' = Jx$). On Fig. 2a the projection of the attractor to the x axis at $\tilde{J} = 0.3$ is displayed. The parameter K scans first through the region of bifurcations, then strange attractors are explored. The period-1 chaotic attractor is also present. If the parameter is increased beyond a critical value, $K_c(\tilde{J})$, the attractor is destroyed and the iteration is driven away from the neighborhood. Figure 2b shows the attractor for $\sqrt{\bar{J}}$ and varying K. It is apparent that the chaotic attractor is destroyed at $K_c(\sqrt{\bar{J}})$ while still having two pieces, and the one-piece attractor is absent. An enlargement and reflection of the lower piece of the attractor resulted in Fig. 2c. Its similarity to Fig. 2a is manifest. At some K values there are stepwise changes in the size of the chaotic attractor, which are due to crises [18]. Also, lighter and darker spots can be distinguished

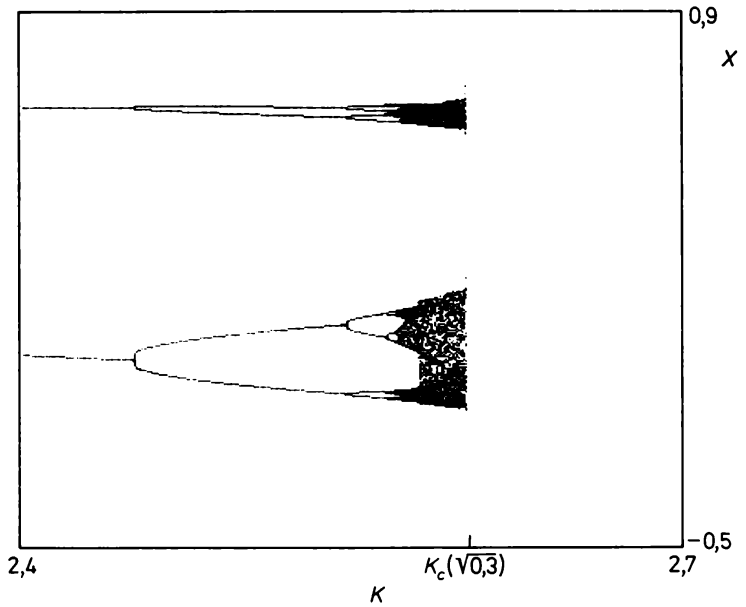

Fig. 2b

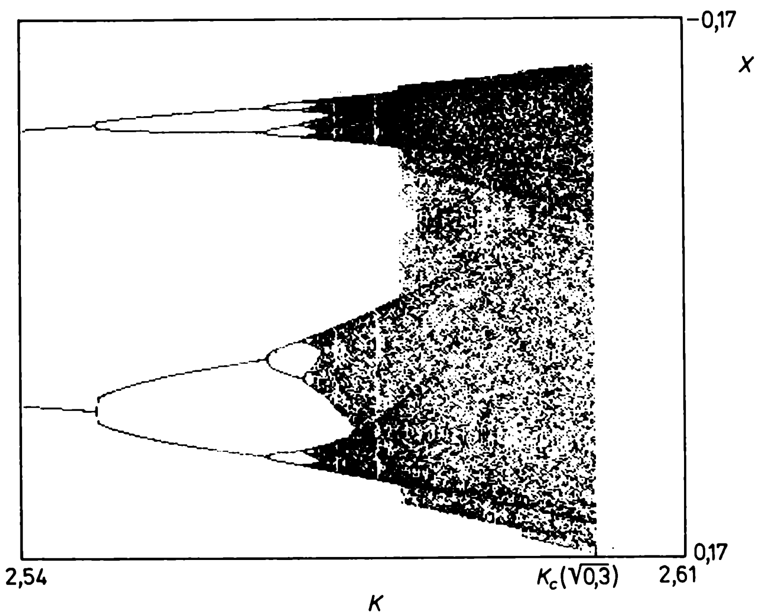

Fig. 2c

within the attractor, indicating regions visited with lower and higher probability, respectively. All these features seem to appear repeatedly for $\tilde{J}$ and $\sqrt{\tilde{J}}$.

Scaling of crises

The sequential disappearance of strange attractors is a consequence of heteroclinic crises as described in the followings. Beyond the line $\bar{K}_n(J)$, the period-2^n orbit of the main period doubling sequence becomes unstable and possesses a stable and unstable manifold. A manifold can intersect other manifolds of opposite stability. Depending on whether the intersecting manifolds belong to the same or a different periodic orbit, the intersection is called *homoclinic* or *heteroclinic*, respectively. Let us assume that the orbit of period 2^{n-1} is unstable. As shown by Holmes and Whitley [19], the unstable manifold of period 2^{n-1} always intersects the stable manifold of its daughter orbit of period 2^n. If, however, the period-2^n orbit becomes unstable, its unstable manifold does not necessarily cross the stable manifold of period

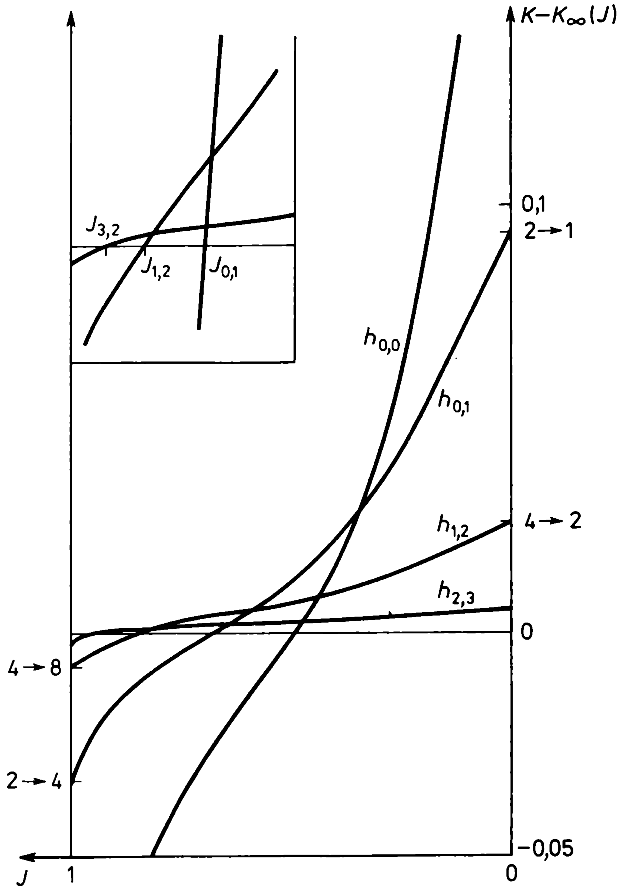

Fig. 3

2^{n-1}. This kind of heteroclinic intersection develops if a line is crossed in the parameter plane K-J, as displayed in Fig. 3. Beyond the curve $h_{n-1,n}$, the unstable manifold of period 2^n intersects the stable one of period 2^{n-1}. On the curve $h_{n-1,n}$ heteroclinic tangency, i.e., crisis [18], can be observed, below it the manifolds do not contact. At $J = 0$ $h_{n-1,n}$ starts at K_n, where 2^n bands merge into 2^{n-1} bands, and for $J = 1$ it winds up in $\bar{K}_{n+1}(1)$, where the period-2^n elliptic orbit bifurcates into the period-2^{n+1} orbit. The curve $h_{0,0}$ marks the crisis between the unstable manifold of the fixed point which gave birth to the period doubling sequence, and the stable manifold of another unstable fixed point. (This stable manifold can be thought of as the boundary of the basin of attraction for small J.) A schematic representation of what happens if $h_{n-1,n}$ is crossed is given in Fig. 4.

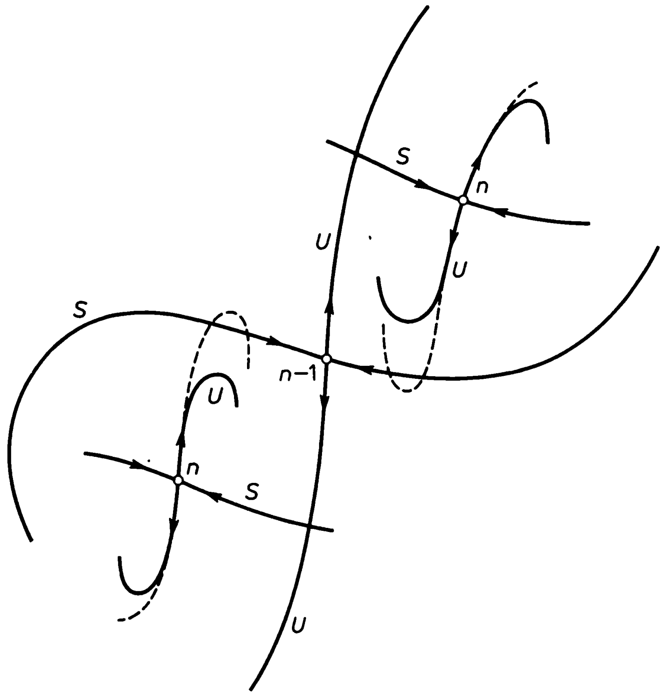

Fig. 4. Simplified scheme of stable (S) and unstable (U) manifolds of periodic points belonging to period-2^{n-1} and -2^n unstable orbits. Below $h_{n-1,n}$, the unstable manifold of period-2^n does not cross the stable one of period 2^{n-1}, whereas beyond $h_{n-1,n}$ the intersection takes place (see dashed line)

Heteroclinic intersections strongly influence the dynamics. An unstable manifold of a periodic orbit acts as attractor, and the iteration is driven towards it along the stable directions. If, however, the unstable manifold intersects a stable manifold of another unstable periodic orbit, the iteration will be driven towards the unstable manifold of the latter. If there are heteroclinic contacts the other way about, too, that is, the second unstable manifold intersects the first stable one, then this provides a mechanism for reinjection back to the first unstable manifold. Consider, for instance, a map in the triangle between $h_{0,1}$ and $h_{1,2}$ with an edge on the $J = 0$ axis. Then the iteration can be driven from the period-2^n unstable manifold to the period-2^{n+1} for any n, whereas reinjection is provided for every n but $n = 0$. As a consequence, if the iteration starts near the unstable fixed point, it will go towards the connected set of manifolds of the orbits with higher periodicity. Among these periods the lowest is 2, thus the attractor will have a periodicity 2 as a coarse grained feature. Along these lines it can be argued for that the (right-hand side) triangle between $h_{n-1,n}$ and $h_{n,n+1}$ is the region where the period-2^n strange attractor exists. These regions are just the triangles represented on Fig. 1, showing a self-similar structure. Self-similarity carries over to all properties, as for instance to the appearance of stable periodic orbits [1].

Table I

Jacobians J_n where the heteroclinic crisis lines $h_{n-1,n}$ and $h_{n,n+1}$ cross each other, for the map (1) with $f(K, x) = K \sin x$. The effective Jacobians $(J_n)^p$, $p = 2^n$, are also listed

n	J_n	$(J_n)^p$, $p = 2^n$
1	0.6623 ± 0.0001	0.4386 ± 0.0002
2	0.8137 ± 0.0002	0.4384 ± 0.0008
3	0.90183 ± 0.00002	0.4375 ± 0.0002
4	0.94971 ± 0.00002	0.4380 ± 0.0003
5	0.97446 ± 0.00001	0.4370 ± 0.0006

The crisis lines $h_{n,n+1}$ converge to the accumulation line of period doubling $K_\infty(J)$. At $J = 0$ and $J = 1$ the convergence is geometrical with the rate δ_F and δ_H, respectively, and inbetween, for J fixed, we expect that the asymptotic rate is δ_F. The scaling in J is governed by the square relation. Table I contains some computed values of the $J_n - s$, i.e. the leftmost tip of the triangles between $h_{n-1,n}$ and $h_{n+1,n}$, for the map (1) with $f(K, x) = K \sin x$. The square relation, mentioned already in the previous section, is apparent. As another example, denoting the intersection of $h_{n,n+1}$ and $K_\infty(J)$ by $J_{n,n+1}$, we found again a square rule $J_{n-1,n} \cong (J_{n,n+1})^2$.

Renormalization theory for weakly dissipative maps

Assuming that the approximate square relation between the Jacobians J_n becomes exact for large n, we are facing a geometrically converging series with the rate $(1-J_{n-1})/(1-J_n) \to 2$. The corresponding points in the parameter plane, the tips of the triangles of Fig. 1, converge to the accumulation point of the area-preserving period doubling sequence. This leads us to applying renormalization theory for weakly dissipative maps. Several authors studied approximate renormalization schemes for dissipative maps, motivated by scaling within the period doubling regime [20]–[22]. Since we were not able, however, to produce a convergence rate 2 using these schemes, we proceeded as described below.

The period doubling operator $\mathscr{R}$ for area-preserving maps $\underline{T}(x, y)$ is defined as $\mathscr{R}[\underline{T}] \equiv \underline{B} \circ \underline{T} \circ \underline{T} \circ \underline{B}^{-1}$, where $\underline{B}$ means an appropriate rescaling of the coordinates. The fixed point equation $\mathscr{R}[\underline{T}^*] = \underline{T}^*$ determines $\underline{T}^*$ and the scaling matrix $\underline{B} = \left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)$, if $\underline{T}^*$ is required to exhibit certain symmetries [12], [13], [15]. The factors $\alpha = -4.018\ldots$ and $\beta = 16.36\ldots$ scale the x and y axes, respectively. The eigenvalue problem of the linearized renormalization operator near $\underline{T}^*$ reads as

$$\mathscr{R}[\underline{T}^* + \varepsilon \underline{U}] = \underline{T}^* + \lambda \varepsilon \underline{U} + O(\varepsilon^2), \tag{2}$$

where λ is the eigenvalue and $\underline{U}$ the corresponding eigenfunction. It can be shown that there are two families of eigenvalues $\lambda = \alpha^{-s}\beta^{1-t}$ and $\lambda = \alpha^{1-s}\beta^{-t}$, $s, t = 0, 1, 2\ldots$, corresponding to infinitesimal, smooth, coordinate changes [12]. If we consider maps connected by coordinate transformations equivalent, we can disregard the above set of eigenvalues. In addition, within the class of area preserving perturbations, there is a single eigenvalue with a modulus larger than one, $\lambda = \delta_H = 8.72\ldots$ [12], [15]. All other eigendirections are stable, $|\lambda| < 1$, thus a map close to $\underline{T}^*$ is driven towards the single unstable manifold upon repeated application of the renormalization operator. As a consequence, the convergence rate for the bifurcation parameter values is δ_H for a large class of maps.

In order to explain the convergence rate 2 of the Jacobians J_n we allow area-contracting perturbations of $\underline{T}^*$, too. Let us assume first that there is an eigen-perturbation such that the map $\underline{T}^* + \varepsilon\underline{U}$ has a uniform Jacobian, $1 - \varepsilon j$. Then composing the Jacobian determinant of both sides of Eq. (2), and taking into account that a rescaling by $\underline{B}$ does not change the Jacobian, we are led to $\lambda = 2$ [23]. Thus eigen-perturbations with a uniform Jacobian must have the eigenvalue 2. This leaves the questions open, whether there are such perturbations at all, and whether there are relevant, i.e. $|\lambda| > 1$, eigenvalues corresponding to non-uniform Jacobians. We therefore solved (2) directly, using the formula manipulation program REDUCE. First we obtained a polynomial approximation of $\underline{T}^*(x, y)$ by iterating and rescaling an area-preserving map in its threshold state at $K_\infty(1)$. Then a polynomial Ansatz for the perturbation $\underline{U}(x, y)$ was taken with undetermined coefficients. Solving (2) for these coefficients and for the eigenvalues, we recovered the six known $|\lambda| \geqslant 1$ eigenvalues β, δ_H, β/α, α, β/α^2, and 1 within 0.5%. In addition, a nondegenerate $\lambda = 2.000$ showed up. All other eigendirections proved to be irrelevant, $|\lambda| < 1$, including possibly those having non-uniform Jacobians.

As a consequence, apart from coordinate transformations, there are two relevant eigendirections emanating out of $\underset{\sim}{T}^*$, with eigenvalues δ_H and 2, respectively. Hence, scaling of maps close to $\underset{\sim}{T}^*$ is determined by a universal two-parameter family. Repeated application of the renormalization transformation $\mathscr{R}$ on a map of this family leads the map away from $\underset{\sim}{T}^*$, while its K and J parameters are rescaled. In the linear region the envelope of a renormalization trajectory can be written as

$$K - K_\infty(1) = A(1-J)^\gamma, \tag{3}$$

where $\gamma = \log \delta_H/\log 2 = 3.12$. This can easily be seen by noting that Eq. (3) is invariant under simultaneous rescaling of $K - K_\infty(1)$ and $1-J$ by δ_H and 2, respectively. A curve like (3) appears also in the parameter diagram (Fig. 1) as the envelope of the critical line $K_c(J)$.

We expect that the universal two-parameter family can be extended from near $\underset{\sim}{T}^*$ to $J = 0$, supported also by recent results of Quispel [24]. This family exhibits the sequential disappearance of strange attractors at universal Jacobians J_n^u. For members of this family are transformed into each other by the period doubling operator, the relation $(J_n^u)^2 = J_{n-1}^u$ holds exactly. Furthermore, if a map is driven close to the universal two-parameter family in functional space, which happens if the renormalization operator is applied, its critcal Jacobians must approach the universal values. In other words, we expect that asymptotically the critical Jacobians are universal numbers for maps exhibiting the scenario, and so is the effective critical Jacobian $J_{\text{eff}} = (J_n)^p$, $p = 2^n$, too. The universal value of the effective Jacobian can be read off from Table I, $J_{\text{eff}}^u = J_0^u \cong 0.437$.

Whereas the above results predict universality only as an asymptotic property, the quadratic and sine maps produced critical Jacobians J_n close to the universal values even for low n. The typical effective Jacobian obtained by directly interating the map and searching for the disappearance of the 2^n-piece strange attactor was $\tilde{J}_{\text{eff}} \cong 0.45$. Note that $\tilde{J}_{\text{eff}} > J_{\text{eff}}$, the latter measured at the intersection of the curves of heteroclinic crisis, as shown in Table I. We attribute the discrepancy to long transients. Indeed, even after the attractor has been destroyed, a long time may elapse before the iteration actually escapes. Finally we mention that the sequential extinction of strange attractors has been observed in two examples of nonlinear differential equations, too, yielding an effective critical Jacobian $\cong 0.45$ again [25].

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