COMPUTATIONAL MATHEMATICS
BANACH CENTER PUBLICATIONS, VOLUME 13
PWN – POLISH SCIENTIFIC PUBLISHERS
WARSAW 1984

A METHOD OF APPROXIMATION OF SPACES OF GENERALIZED FUNCTIONS

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1. Introduction

The aim of this paper is the construction of the approximation of generalized functions, which will be useful in studying the approximate solutions of differential equations permitting only generalized solutions.

Let us start from the definition of the approximation.

Let U be a linear topological space with the topology generated by the family of seminorms $\{\|\cdot\|_a\}_{a\in A}$ (A may consist of one element a, then $\|\cdot\|_a$ is a norm), let $\dot{H}$ be a metric space of parameters with an accumulation point denoted by 0, and let $H = \dot{H}\setminus\{0\}$. Together with the space $(U, \{\|\cdot\|_a\}_{a\in A})$ let us also consider a space $(V, \{\|\cdot\|_a^V\}_{a\in B})$ where

$$(1.1)\qquad U\subset V,\ A\subset B,\ \forall F\in U\ \forall a\in A\quad \|F\|_a = \|F\|_a^V,$$

and a family $\{(V, \{\|\cdot\|_{h,a}^V\}_{a\in B})\}_{h\in H}$ where $\|\cdot\|_{h,a}$ are seminorms with the properties

$$(1.2)\qquad \begin{aligned}&\forall a\in B\ \forall F\in V\ \forall h\in H\quad \|F\|_{h,a}^V\leqslant \|F\|_a^V,\\ &\forall a\in B\ \forall F\in V\quad \lim_{h\to 0}\|F\|_{h,a}^V = \|F\|_a^V.\end{aligned}$$

As an example, let us consider $V = W_1^1(0,1)$ — the space of absolutely continuous functions on the interval $[0,1]$ with the norm

$$\|F\|^V = \int_0^1 |F(X)|\,dX + \int_0^1 |F'(X)|\,dX,$$

and $U = \mathring{W}_1^1(0,1)$ – the subspace of V consisting of all functions vanishing at the endpoints, with the norm $\|F\| = \|F\|^V$. Further, let us define for each $h \in (0, 1/2)$ a seminorm

$$\|F\|_h^V = \int_h^{1-h} |F(X)|\,dX + \int_h^{1-h} |F'(X)|\,dX;$$

these seminorms satisfy condition (1.2).

The following definition is a generalization of that given by Aubin ([1], Introduction).

(2) DEFINITION. The *approximation of U* is a family of triples $\{(u_h, \mathrm{rs}_h, \mathrm{PR}_h)\}_{h\in H}$, where

1) for each $h \in H$ u_h is a linear topological space with the family of seminorms $\{\|\cdot\|_{h,c}\}_{c\in C_h}$,

2) rs_h: $U \to u_h$ (restriction) is a linear operator and

$$\exists K > 0\ \forall h \in H\ \forall a \in A\ \exists c(a,h) \in C_h\ \forall F \in U \quad \|\mathrm{rs}_h F\|_{h,c(a,h)} \leqslant K\|F\|_a,$$

3) PR_h: $u_h \to V$ (prolongation) is a linear operator and

$$\exists K > 0\ \forall h \in H\ \forall c \in C_h\ \exists a(c,h) \in B\ \forall f_h \in u_h \quad \|\mathrm{PR}_h f_h\|^V_{h,a(c,h)} \leqslant K\|f_h\|_{h,c},$$

$$\forall a \in A\ \forall F \in U \quad \lim_{h\to 0}\|F - \mathrm{PR}_h \mathrm{rs}_h F\|^V_{h,a} = 0.$$

If $U = V$ then the approximation is called *internal*, in the opposite case it is called *external*. Further, an internal approximation is called *reflexive* iff

4) $\forall h \in H\ \forall f_h \in u_h \quad \mathrm{rs}_h \mathrm{PR}_h f_h = f_h$.

The present paper, which is a continuation of [2], is devoted to the construction of the approximation of the spaces $W_q^p(B)$ and $W_q^{-p}(B)$ where $B \subset \mathbf{R}^m$. The definition of these spaces and of some operators acting in them are given in Section 2.

The approximation is based on the so-called double partition of unity which is constructed on the mesh introduced on $\mathbf{R}^m$. The mesh and the spaces of mesh functions are defined in Section 3.

The definition of the double partition of unity is introduced in Section 4 (Definition (18)). Theorem (23) gives examples of internal, external and reflexive approximations of $L_q(B)$ when q is finite. Theorem (24) allows us to build an approximation of $L_q(B)^{\mathrm{loc}}$. The estimations of the convergence of the approximation are given.

Section 5 is devoted to constructing an approximation of other spaces of functions. The operators are defined which allow us to obtain a double partition of unity of higher as well as lower regularity. With the aid of these operators the external and internal approximations of

$W_q^p(B)$ (q finite in Theorem (36), q infinite in Corollary (38.1)), $W_q^{-p}(B)$ (Theorem (37) and Corollary (38.2)), $W_q^p(B)^{\mathrm{loc}}$ and $W_q^{-p}(B)^{\mathrm{loc}}$ (Corollary (38.3)) are constructed. A reflexive approximation is built in Corollary (39.4).

2. Definitions and notation

In our considerations U will be a space of functions or distributions defined on a subset of the m-dimensional real space $\boldsymbol{R}^m$. The following notation will be used.

(3.1) If σ is a logical sentence then

$$\delta(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is true,} \\ 0 & \text{if } \sigma \text{ is false;} \end{cases}$$

I — the set of integers, $\overline{m} = \{i \in I\colon 1 \leqslant i \leqslant m\}$;
if $X,\ Y \in \boldsymbol{R}^m$, then

$$X \leqslant Y \Leftrightarrow \forall i \in \overline{m} \quad X_i \leqslant Y_i,$$

$$X < Y \Leftrightarrow X \leqslant Y \wedge X \neq Y, \quad X \circ Y = (X_1 Y_1, \ldots, X_m Y_m),$$

$$|X| = \sum_{i=1}^{m} |X_i|;$$

if $X,\ Y \in \boldsymbol{R}^m$, $\forall i\ X_i > 0$, then

$$X^Y = \prod_{i=1}^{m} X_i^{Y_i};$$

if $A, B \subset \boldsymbol{R}^m$, then

$$A+B = \{X \in \boldsymbol{R}^m\colon X = Y+Z,\ Y \in A,\ Z \in B\},$$

$$-A = \{X \in \boldsymbol{R}^m\colon -X \in A\}, \quad A-B = A+(-B).$$

Let us now consider a set $K \subset \overline{m}$ such that the number of its elements, $\mathrm{ne}K$, equals k. Let $\setminus K = \overline{m} \setminus K$. We will use the following notation.

(3.2) If $X \in \boldsymbol{R}^m$ and the elements of K are arranged in the increasing order $i_1 < i_2 < \ldots < i_k$, then $X_K = (X_{i_1}, \ldots, X_{i_k})$;
if $X \in \boldsymbol{R}^k$, $Y \in \boldsymbol{R}^{m-k}$, then $Z = X \oplus_K Y$ is the vector from $\boldsymbol{R}^m$ with the coordinates $Z_j = X_{a_j}\delta(j \in K) + Y_{j-a_j}\delta(j \notin K)$, $a_j = \mathrm{ne}\{i \in K\colon i \leqslant j\}$;
if $X \in \boldsymbol{R}^k$, then $X'_K = X \oplus_K 0$;
if $A \subset \boldsymbol{R}^k$, $B \subset \boldsymbol{R}^{m-k}$, then $A \oplus_K B = \{X \oplus_K Y\colon X \in A,\ Y \in B\}$;
if $A \subset \boldsymbol{R}^m$, $X \in \boldsymbol{R}^{m-k}$, then the X-section of A with respect to Kth variables is the set $\mathrm{SC}_K^X A = \{Y \in \boldsymbol{R}^k\colon Y \oplus_K X \in A\}$.

Further,

(3.3) let $e = (1, 1, \ldots, 1) \in \boldsymbol{R}^m$, hence following (3.2),

$$e'_K = \big(\delta(1 \in K),\ \delta(2 \in K),\ \ldots,\ \delta(m \in K)\big);$$

let for $i \in \overline{m}$

$$e'_i = e'_{\{i\}} \quad \text{and} \quad I^m_K = \{j \in I^m \colon\ j_{\setminus K} = 0\}.$$

Let us also introduce the lower and upper bounds of a subset of $\boldsymbol{R}^m$.

(3.4) If $B \subset \boldsymbol{R}^m$ is a bounded non-empty set, then

$$\mathrm{U}B(A) = \sum_{i \in \overline{m}} e'_i \cdot \sup\{X_i \colon\ X \in A\}, \qquad \mathrm{L}B(A) = \sum_{i \in \overline{m}} e'_i \cdot \inf\{X_i \colon\ X \in A\}.$$

Consider now the Lebesgue measure μ on $\boldsymbol{R}^m$ and define the relations in the set of all measurable subsets of $\boldsymbol{R}^m$:

$$A \underset{\mu}{\subset} B \Leftrightarrow \mu(A \setminus B) = 0, \qquad A \underset{\mu}{=} B \Leftrightarrow A \underset{\mu}{\subset} B \wedge B \underset{\mu}{\subset} A.$$

Introduce the following equivalence relation in the space $\mathrm{MS}(A)$ of all measurable real-valued functions on A:

$$F \underset{\mu}{\sim} G \Leftrightarrow \{X \in A \colon\ F(X) \neq G(X)\} \underset{\mu}{=} \emptyset,$$

and denote by $M(A)$ the space $\mathrm{MS}(A)/N_\mu(A)$, where $N_\mu(A) = \big\{F \in \mathrm{MS}(A) \colon\ \{X \in A \colon\ F(X) = 0\} \underset{\mu}{=} \emptyset\big\}$. An element G of $M(A)$, that is, the class of equivalent functions $F_x \in \mathrm{MS}(A)$, is, however, identified with any of the functions F_x and therefore a relation written for $G = \{F_x\} \in M(A)$ is understood as follows:

$$\sigma(G) \Leftrightarrow \exists x\ \ \sigma(F_x).$$

For example, we will write $F \geqslant 0$ if and only if there exists $G \in \mathrm{MS}(A)$ such that $G \underset{\mu}{\sim} F$ and at each point $X \in A$, $G(X) \geqslant 0$.

From now on we assume that A is a regular subset of $\boldsymbol{R}^m$, that is, $A \underset{\mu}{\neq} \emptyset$ and $\operatorname{int} A \underset{\mu}{=} \operatorname{cl} A$.

(4.1) By $L_q(A)$ where $1 \leqslant q \leqslant \infty$ we denote the space consisting of all functions F from $M(A)$ such that

$$\|F\|_q = \begin{cases} \Big(\int\limits_A |F(X)|^q\, dX\Big)^{1/q} & \text{if} \quad q < \infty, \\ \operatorname{ess\,sup}\{|F(X)| \colon\ X \in A\} & \text{if} \quad q = \infty \end{cases}$$

is finite; $\|\cdot\|_q$ is the norm in $L_q(A)$.

The definitions of a projection of a measurable set $A \subset \boldsymbol{R}^m$ onto the $X_{\setminus K}$-hyperplane where $K \subset \overline{m}$,

(4.2) $\mathrm{PJ}_K A = \{Y \in \boldsymbol{R}^{m - \mathrm{no}\,K} \colon\ \mathrm{SC}^X_K A \underset{\mu_K}{\neq} \emptyset\}$ (μ_K is the $\mathrm{no}\,K$-dimensional measure);

and of the function with reducted number of variables,

(4.3) if $F\colon A\to \boldsymbol{R}$ then $\mathrm{RV}_K^Y F(X) = F(X\oplus_K Y)$ for $X\in \mathrm{SC}_K^Y A$, $Y\in \mathrm{PJ}_K A$,

allow us to define the set of the functions which are integrable with respect to the Kth variables.

(4.4) If $1\leqslant q\leqslant \infty$, $K\subset \overline{m}$, then $L_q(A;K)$ is the set of all functions $F\in M(A)$ such that for almost every $Y\in \mathrm{PJ}_K A$ the function $\mathrm{RV}_K^Y F$ belongs to $L_q(\mathrm{SC}_K^Y A)$.

(4.5) If $1\leqslant qx\leqslant \infty$, $1\leqslant qi\leqslant \infty$, $K\subset \overline{m}$, then the space $L_{(qx,qi,K)}(A)$ is the subset of $L_{qi}(A;K)$ consisting of all functions F such that the function G defined by $G(Y) = \|\mathrm{RV}_K^Y F\|_{qi}$ belongs to $L_{qx}(\mathrm{PJ}_K A)$; the norm is given by

$$\|F\|_{(qx,qi,K)} = \|G\|_{qx}$$

(qx, qi are called the *external* and *internal index* of the norm, respectively, and (qx, qi, K) is called *finite* if $qx<\infty$ and $qi<\infty$).

Further, we introduce operators of restricting and extending the domain of definition of measurable functions.

(5.1) If $A\subset B$ then $\mathrm{EX}_B\colon M(A)\to M(B)$ and $\mathrm{RD}_A\colon M(B)\to M(A)$ are defined by the formulas

$$(\mathrm{EX}_B F)(X) = \begin{cases} F(X) & \text{if } X\in A,\\ 0 & \text{if } X\in B\setminus A,\end{cases}\quad (\mathrm{RD}_A F)(X) = F(X) \text{ if } X\in A.$$

If $B=\boldsymbol{R}^m$, we shall write EX_m instead of $\mathrm{EX}_{\boldsymbol{R}^m}$.

Now, let us define the following scalar products.

(5.2) Let A, $B\subset \boldsymbol{R}^m$, $C=A\cup B$, $F\in M(A)$, $G\in M(B)$, $K\subset \overline{m}$.

If $H = \mathrm{EX}_C F\cdot \mathrm{EX}_C G\in L_1(C)$ then $\langle F, G\rangle = \int\limits_C H(X)dX$,

if $H = \mathrm{EX}_C F\cdot \mathrm{EX}_C G\in L_1(C;K)$ then $\langle F, G\rangle_K(Y) = \int\limits_{\mathrm{SC}_K^Y C} \mathrm{RV}_K^Y H(X)dX$ for almost every $Y\in \mathrm{PJ}_K C$.

If $F\in M(A)$, we can define its derivatives.

(6.1) A function $G\in M(A)$ is called the *k-th derivative of F* ($k\in I^m$, $k\geqslant 0$), iff for every $U\in C_0^\infty(\mathrm{cl}A)$ (that is, the space of infinitely many times differentiable functions with compact support contained in $\mathrm{cl}A$) the following equality holds

$$\langle G, U\rangle = \langle F, D^{*k}U\rangle,$$

where $D^k U = \dfrac{\partial^{|k|} U}{\partial X_1^{k_1} \ldots \partial X_m^{k_m}}$, $D^{*k} = (-1)^{|k|} D^k$. In this case we write $G = D^k F$. If $K \subset \overline{m}$, the derivative $D^{e'_K} F$ will also be denoted by $D_K F$.

Note that if $F \in M(A)$ has the kth derivative $D^k F$, its extension $\mathrm{EX}_B F$ belonging to $M(B)$ need not possess any derivative; an example is the function $F(X) = 1$ for $X \in A$ if $A \underset{\mu}{\neq} B$.

Before introducing the spaces of differentiable functions let us make the following definition.

(6.2) The class SL^m is the class of star-like sets of nonnegative vectors, that is, the set of all non-empty subsets p of I^m satisfying
if $k \in p$ then $k \geqslant 0$,
if $k \in p$, $l \in I^m$, $0 \leqslant l \leqslant k$, then $l \in p$;
the class SL_K^m (where $K \subset \overline{m}$) consists of all subsets of I_K^m belonging to SL^m.

An example of an element of the class SL_K^m in the case $m = 3$, $K = \{1, 2\}$, is the set $p = \{k \in I^3\colon k_3 = 0 \wedge |k| \leqslant 1\} = \{(0,0,0), (0,1,0), (1,0,0)\}$. Further, let us introduce the following notation.

(6.3) If $p \in \mathrm{SL}^m$, $l \in I^m$, $l \geqslant 0$, then $p \oplus l$ is the minimal set in SL^m containing $p + \{l\}$; if $l \in p$ then $p \ominus l = (p - \{l\}) \cap \{k \in I^m\colon k \geqslant 0\}$; the set $\{0\} \oplus l = \{k\colon 0 \leqslant k \leqslant l\}$ will be denoted by l''; if $l \in \boldsymbol{R}$ then the set $\{k\colon k \geqslant 0 \wedge |k| \leqslant l\}$ will be often denoted by l; if $p \in \mathrm{SL}_K^m$ then $p_K = \{i_K\colon i \in p\}$; the upper boundary of p is defined as follows: $p^+ = \{k \in p\colon \forall l \text{ if } l > k \text{ then } l \notin p\}$.

(6.4) If $A \subset \boldsymbol{R}^m$, $p \in \mathrm{SL}^m$, $1 \leqslant q \leqslant \infty$, then the space $W_q^p(A)$ is defined as the subspace of $L_q(A)$ consisting of all functions F such that $D^k F \in L_q(A)$ for every $k \in p$; the norm is given by

$$\|F\|_{p,q} = \begin{cases} \Big(\sum\limits_{k \in p} \|D^k F\|_q^q\Big)^{1/q} & \text{if} \quad q < \infty, \\ \max\{\|D^k F\|_\infty\colon k \in p\} & \text{if} \quad q = \infty; \end{cases}$$

if $K \subset \overline{m}$, $p \in \mathrm{SL}_K^m$, $1 \leqslant qx \leqslant \infty$, $1 \leqslant qi \leqslant \infty$, then $W^p_{(qx,qi,K)}(A)$ is the set $\{F\colon \forall l \in p \ D^l F \in L_{(qx,qi,K)}(A)\}$ normed by

$$\|F\|_{p,(qx,qi,K)} = \begin{cases} \Big(\int\limits_{\mathrm{PJ}_K A} \|\mathrm{RV}_K^Y F\|_{p_K,qi}^{qx} dY\Big)^{1/qx} & \text{if} \quad qx < \infty, \\ \operatorname{ess\,sup}\{\|\mathrm{RV}_K^Y F\|_{p_K,qi}\colon Y \in \mathrm{PJ}_K A\} & \text{if} \quad qx = \infty; \end{cases}$$

$\mathring{W}_q^p(A)$ ($\mathring{W}^p_{(qx,qi,K)}(A)$) is the subspace of $W_q^p(A)$ ($W^p_{(qx,qi,K)}(A)$, respectively) consisting of all functions F such that for every $k \in p$, $D^k(\mathrm{EX}_m F) = \mathrm{EX}_m(D^k F)$.

If A is unbounded, the following space with the local topology will be considered.

(6.5) $W_q^p(A)^{\mathrm{loc}}$ — the space of all functions $F \in M(A)$ such that for every bounded regular set $B \subset A$ $\mathrm{RD}_B F \in W_q^p(B)$, with the topology given by the family of seminorms

$$\|F\|_{p,q}^{(B)} = \|\mathrm{RD}_B F\|_{p,q};$$

$W_q^0(A)^{\mathrm{loc}}$ will be denoted by $L_q(A)^{\mathrm{loc}}$.

Let us also introduce the index q' conjugate to q by the formula

(6.6) $1/q+1/q' = 1$ if $1 \leqslant q \leqslant \infty$,
$q' = (qx', qi', K)$, $1/qx+1/qx' = 1$, $1/qi+1/qi' = 1$, if
$q = (qx, qi, K)$.

The space $\overset{\circ}{W}{}_q^p(A)$ may be considered as a subspace of the Cartesian product $\big(L_q(A)\big)^{\mathrm{ne}p}$ consisting of all vectors $\vec{G}_p = (G_k)_{k\in p}$ such that $G_k = D^k G_0$ (cf. [3], Ch. I, § 12). Therefore any linear continuous mapping U from $\overset{\circ}{W}{}_q^p(A)$ into $\boldsymbol{R}$ can be prolongated onto the whole space $\big(L_q(A)\big)^{\mathrm{ne}p}$ according to the Hahn–Banach theorem, and hence, if $1 < q < \infty$ or $q = (qx, qi, K)$ where $1 < qx < \infty$, $1 < qi < \infty$, then there exist functions $U_k \in L_{q'}(A)$ $(k \in p)$ such that

$$\langle F, U\rangle = \sum_{k\in p} \langle D^k F, U_k\rangle$$

($\langle F, U\rangle$ is the value of U for the function $F \in \overset{\circ}{W}{}_q^p(A)$, and on the right-hand side of the equation we have the scalar product of D^kF and U_k — but it may also be considered as the value of the functional U_k for the function D^kF). Therefore we obtain by applying Hölder's inequality

(7.1) $$|\langle F, U\rangle| \leqslant \|F\|_{p,q}\Big(\sum_{k\in p} \|U_k\|_{q'}^{q'}\Big)^{1/q'} \qquad \text{if } 1 < q < \infty,$$

$$|\langle F, U\rangle| \leqslant \|F\|_{p,q}\Big(\int_{\mathrm{PJ}_K A}\Big(\sum_{k\in p} \|\mathrm{RV}_K^Y U_k\|_{qi'}^{qi'}\Big)^{qx'/qi'} dY\Big)^{1/qx'}$$
$$\text{if } q = (qx, qi, K).$$

On the other hand, every vector $\vec{U}_p \in \big(L_{q'}(A)\big)^{\mathrm{ne}p}$ may be considered as a linear functional on $\big(L_q(A)\big)^{\mathrm{ne}p}$.

Let us thus consider the space $\big(L_q(A)\big)^{\mathrm{ne}p}$ ($1 \leqslant q \leqslant \infty$ or $q = (qx, qi, K)$) and define in it the following equivalence relation

(7.2) $$\mathrm{EQ}_q^p(\vec{U}_p, \vec{Y}_p) \Leftrightarrow \forall F \in \overset{\circ}{W}{}_{q'}^p(A) \quad \sum_{k\in p} \langle D^kF, U_k\rangle = \sum_{k\in p} \langle D^kF, Y_k\rangle.$$

Definition (7.2) will be used for defining distributions on A.

(7.3) The equivalence classes in $(L_q(A))^{\mathrm{ne}\,p}$ with respect to the relation EQ_q^p will be called *distributions of order* (p, q) *on* A; if U is the equivalence class of EQ_q^p generated by $\vec{U}_p$ (we will write $U = \mathrm{EC}_q^p(\vec{U}_p)$) then we define

$$\langle F, U\rangle = \sum_{k\in p} \langle D^k F, U_k\rangle.$$

(7.4) The set of distributions of order (p, q) on A is a linear space. Introducing the norm by the formula

$$\|U\|_{-p,q} = \sup\{\langle F, U\rangle : F \in \mathring{W}_{q'}^p(A),\ \|F\|_{p,q'} = 1\}$$

we obtain the space $W_q^{-p}(A)$.

Let us also consider the following equivalence relation in $(L_q(A)^{\mathrm{loc}})^{\mathrm{ne}\,p}$:

(7.5) $\mathrm{EQ}_q^{p,\mathrm{loc}}(\vec{U}_p, \vec{Y}_p) \Leftrightarrow$ for each bounded regular $B \subset A$ and for every $F \in \mathring{W}_{q'}^p(B)$, $\sum_{k\in p} \langle D^k F, U_k\rangle = \sum_{k\in p} \langle D^k F, Y_k\rangle$.

It can easily be shown that

$\mathrm{EQ}_q^{p,\mathrm{loc}}(\vec{U}_p, \vec{Y}_p) \Leftrightarrow$ for each bounded regular $B \subset A$ $\mathrm{EQ}_q^p(\overrightarrow{\mathrm{RD}_B U_p}, \overrightarrow{\mathrm{RD}_B Y_p})$ (if S is an operator on $L_q(A)$, $\vec{U}_p \in (L_q(A))^{\mathrm{ne}(p)}$, then $\vec{S U_p} = (S U_k)_{k\in p}$).

Therefore we introduce the following definitions.

(7.6) The equivalence classes in $(L_q(A)^{\mathrm{loc}})^{\mathrm{ne}\,p}$ with respect to the relation $\mathrm{EQ}_q^{p,\mathrm{loc}}$ are called the *local distributions of order* (p, q) *on* A. If $U = \mathrm{EC}_q^{p,\mathrm{loc}}(\vec{U}_p)$ is a local distribution on A and $B \subset A$, then we define $\mathrm{RD}_B U = \mathrm{EC}_q^{p,\mathrm{loc}}(\overrightarrow{\mathrm{RD}_B U_p})$.

(7.7) The space $W_q^{-p}(A)^{\mathrm{loc}}$ is the space of local distributions of order (p, q) with the topology inducted by the seminorms

$$\|U\|_{-p,q}^{(B)} = \|\mathrm{RD}_B U\|_{-p,q},$$

where B runs over the set of all bounded regular subsets of A. ⊣

Let us also define the support of functions and distributions.

(8.1) The support of the function $F \in L_q(A)$ is contained in the set $B \subset A$ ($\mathrm{SUPP}(F) \subset B$) iff $\{X \in A : F(X) \neq 0\} \underset{\mu}{\subset} B$; if F is a distribution from $W_q^{-p}(A)$ then $\mathrm{SUPP}(F) \subset B$ iff for every $G \in \mathring{W}_{q'}^p(A)$ such that $\mathrm{RD}_B G = 0$ we have $\langle G, F\rangle = 0$.

Now we define some operators on the spaces $W_q^p(A)$ and $W_q^{-p}(A)$. First, let us extend the definition of the derivative onto the space of distributions. We prove the following lemma.

(9.1) LEMMA. *If* $F \in W_q^p(A)$, $G \in \mathring{W}_{q'}^p(A)$, $k \in p$, *then*

$$\langle F, D^k G\rangle = \langle D^{*k} F, G\rangle.$$

Proof. Let $q > 1$. In this case there exists a sequence $\{G_n\}_{n=1}^{\infty} \subset C_0^{\infty}(A)$ such that $\|G-G_n\|_{p,q'} \to 0$ as $n\to\infty$. Definition (6.1) yields for every n $\langle F, D^k G_n\rangle = \langle D^{*k}F, G_n\rangle$. Since $|\langle F, D^kG_n\rangle - \langle F, D^kG\rangle| \leqslant \|F\|_q \cdot \|D^kG_n - -D^kG\|_{q'} \leqslant \|F\|_{p,q}\|G_n-G\|_{p,q'}$ and similarly $|\langle D^{*k}F, G_n\rangle - \langle D^{*k}F, G\rangle| \leqslant \|F\|_{p,q}\|G_n-G\|_{p,q'}$, we obtain $\langle F, D^kG\rangle = \langle D^{*k}F, G\rangle$.

Let now $q = 1$ and let for every natural n

$$B_n = \{X \in A\colon\ |X| \geqslant n\}, \qquad C_n = \{X \in A\setminus B_n\colon\ \exists k \in p\ \ |D^kF(X)| \geqslant n\},$$
$$A_n = A\setminus(B_n \cup C_n).$$

Since $\mathrm{RD}_{A\setminus B_n} G \in L_r(A\setminus B_n)$ for each $r < \infty$, we can choose a function $G_n \in C^{\infty}(A\setminus B_n)$ such that for fixed r $\|G-G_n\|_{p,r}^{(A\setminus B_n)} \leqslant \varepsilon_n$, where ε_n is an arbitrary positive number, and moreover, G_n can be prolongated onto A in such a way that $\|G_n\|_{p,\infty}^{(A)} \leqslant C\|G\|_{p,\infty}$, C is a constant independent of n. Hence we have the estimation

$$|\langle F, D^kG_n\rangle - \langle F, D^kG\rangle|$$
$$\leqslant \|F\|_{r'}^{(A_n)}\|D^kG_n - D^kG\|_r^{(A_n)} + \|F\|_1^{(B_n\cup C_n)}\|D^kG_n - D^kG\|_{\infty}^{(B_n\cup C_n)}.$$

But $\|F\|_{r'}^{(A_n)} \leqslant n^{1/r}\|F\|_1$, $\|D^kG_n - D^kG\|_r^{(A_n)} \leqslant \varepsilon_n$, $\|F\|_1^{(B_n\cup C_n)} \leqslant \|F\|_{p,1}^{(B_n\cup C_n)} \to 0$ as $n\to\infty$, $\|D^kG_n - D^kG\|^{(B_n\cup C_n)} \leqslant (C+1)\|G\|_{p,\infty}$. Taking ε_n such that $\varepsilon_n n^{1/r} \to 0$ as $n\to\infty$, we obtain $|\langle F, D^kG_n\rangle - \langle F, D^kG\rangle| \to 0$ as $n\to\infty$. In the similar way we prove that $|\langle D^{*k}F, G_n\rangle - \langle D^{*k}F, G\rangle| \to 0$ as $n\to\infty$, hence $\langle D^{*k}F, G\rangle = \langle F, D^kG\rangle$ if $1 \leqslant q \leqslant \infty$. The proof in the case $q = (qx, qi, K)$ is based on the Fubini theorem. ∎

Therefore we define the derivative of the distribution as follows.

(9.2) DEFINITION. If $U \in W_q^{-p}(A)$ then $D^kU \in W_q^{-(p\oplus k)}(A)$ is the distribution satisfying

$$\langle F, D^kU\rangle = \langle D^{*k}F, U\rangle \qquad \forall F \in \mathring{W}_{q'}^{p\oplus k}(A);$$

if $U \in W_q^{-p}(A)^{\mathrm{loc}}$ then $D^kU \in W_q^{-(p\oplus k)}(A)^{\mathrm{loc}}$ is the local distribution satisfying

$$\langle F, D^kU\rangle = \langle D^{*k}F, U\rangle \qquad \forall F \in \mathring{W}_{q'}^{p\oplus k}(B),$$

where $B \subset A$ is a bounded regular set.

Hence if $U = \mathrm{EC}_q^p(\vec{U}_p)$ or $U = \mathrm{EC}_q^{p,\mathrm{loc}}(\vec{U}_p)$ then

$$\langle F, D^k U\rangle = (-1)^{|k|}\langle D^k F, U\rangle = (-1)^{|k|}\sum_{l\in p}\langle D^{k+l}F, U_l\rangle,$$

thus $D^k U = \mathrm{EC}_q^{p\oplus k}(\vec{Y}_{p\oplus k})$ or $D^k U = \mathrm{EC}_q^{p\oplus k,\mathrm{loc}}(\vec{Y}_{p\oplus k})$, respectively, where

$$Y_l = \delta(l \geqslant k)(-1)^{|k|}U_{l-k}.$$

The extension and restriction operators $\mathrm{EX}_B\colon W_q^{-p}(A)\to W_q^{-p}(B)$, $\mathrm{RD}_A\colon W_q^{-p}(B)\to W_q^{-p}(A)$, where $A \subset B$, are defined as follows.

(9.3) If $U = \mathrm{EC}_q^p(\vec{U}_p) \in W_q^{-p}(A)$ then $\mathrm{EX}_B U = \mathrm{EC}_q^p(\overrightarrow{\mathrm{EX}_B U_p})$,
if $U = \mathrm{EC}_q^p(\vec{U}_p) \in W_q^{-p}(B)$ then $\mathrm{RD}_A U = \mathrm{EC}_q^p(\overrightarrow{\mathrm{RD}_A U_p})$.

The next operator considered here is the translation operator. If $X, Y \in \boldsymbol{R}^m$, $A \subset \boldsymbol{R}^m$, then we define $\mathrm{TS}^Y X = X+Y$, $\mathrm{TS}^Y A = \{X \in \boldsymbol{R}^m\colon X = \mathrm{TS}^Y Z,\ Z \in A\}$. The *translation operator* $\mathrm{TS}^Y\colon M(A)\to M(\mathrm{TS}^{-Y}A)$ is given by the formula

(9.4) $\quad (\mathrm{TS}^Y F)(X) = F(\mathrm{TS}^Y X) \quad \forall X \in \mathrm{TS}^{-Y}A.$

If $U = \mathrm{EC}_q^{p,\mathrm{loc}}(\vec{U}_p)$ is a (local) distribution of order (p, q) on A, we define the (local) distribution $\mathrm{TS}^Y U$ on $\mathrm{TS}^{-Y}A$ by the formula

(9.5) $\quad \mathrm{TS}^Y U = \mathrm{EC}_q^{p,\mathrm{loc}}(\overrightarrow{\mathrm{TS}^Y U_p}).$

It is easy to show that

(9.6) $\quad \langle \mathrm{TS}^Y F, U\rangle = \langle F, \mathrm{TS}^{-Y}U\rangle \quad$ if $\quad F \in \dot{W}_q^p(A),\ U \in W_{q'}^{-p}(\mathrm{TS}^{-Y}A).$

The next operators defined here are integration operators. Let us first define auxiliary functions

(9.7) *constant function* $\mathrm{FC}(X) = 1\ \forall X \in \boldsymbol{R}^m$;
step functions

$\mathrm{FS}_{K,X}^-(Z) = \delta(Z_K < X)$, $\mathrm{FS}_{K,X}^+(Z) = \delta(Z_K > X)$ for $K \subset \overline{m}$, $X \in \boldsymbol{R}^{\mathrm{ne}\,K}$, $Z \in \boldsymbol{R}^m$;

$\mathrm{FS}_K^{X,Y} = \mathrm{FS}_{K,Y}^- - \mathrm{FS}_{K,X}^- = \mathrm{FS}_{K,X}^+ - \mathrm{FS}_{K,Y}^+$ for $K \subset \overline{m}$, $X, Y \in \boldsymbol{R}^{\mathrm{ne}\,K}$, $X \leqslant Y$.

Operators IN_K, $\mathrm{IN}_K^*\colon L_1(\boldsymbol{R}^m; K)\to M(\boldsymbol{R}^m)$ are given by the formulas

(9.8) $(\mathrm{IN}_K U)(X) = \langle U, \mathrm{FS}_{K,X_K}^-\rangle_K(X_{\setminus K})$,
$(\mathrm{IN}_K^* U)(X) = \langle U, \mathrm{FS}_{K,X_K}^+\rangle_K(X_{\setminus K})$ for almost every $X \in \boldsymbol{R}^m$.

It can easily be shown that

(9.9) if $U, F \in L_1(\boldsymbol{R}^m; K)$ then $\langle \mathrm{IN}_K U, F\rangle_K = \langle U, \mathrm{IN}_K^* F\rangle_K$.

The relation between the derivative and the operator IN_K is described by the following lemma.

(9.10) LEMMA. *If* $F \in L_1(\boldsymbol{R}^m)$ *then* $D_K \mathrm{IN}_K F = D_K^* \mathrm{IN}_K^* F = F$, *if, moreover,* $F \in W_1^{e'K''}(\boldsymbol{R}^m)$, *then* $\mathrm{IN}_K D_K F = \mathrm{IN}_K^* D_K^* F = F$.

Proof. Take an arbitrary $U \in C_0^\infty(\boldsymbol{R}^m)$. Since $\mathrm{IN}_K^* D_K^* U = U$, by applying Fubini's theorem and formula (9.9) we obtain

$$\begin{aligned}\langle F, U\rangle &= \langle\langle F, U\rangle_K, \mathrm{FC}\rangle = \langle\langle F, \mathrm{IN}_K^* D_K^* U\rangle_K, \mathrm{FC}\rangle\\ &= \langle\langle \mathrm{IN}_K F, D_K^* U\rangle_K, \mathrm{FC}\rangle = \langle \mathrm{IN}_K F, D_K^* U\rangle\end{aligned}$$

(the external integration is carried out over the $(m-\mathrm{ne}K)$-dimensional space), and hence, by definition (6.1), $D_K \mathrm{IN}_K F = F$.

Now, let $Q \subset \boldsymbol{R}^m$ be a bounded regular set and let $U \in L_\infty(Q)$. Then $\mathrm{IN}_K(\mathrm{EX}_m U) \in L_\infty(\boldsymbol{R}^m)$ and by using (9.9) and Fubini's theorem we obtain

$$\langle D_K^* F, \mathrm{IN}_K \mathrm{EX}_m U\rangle = \langle \mathrm{IN}_K^* D_K^* F, \mathrm{EX}_m U\rangle,$$

the applying of Lemma (9.1) and of the first part of our lemma yields

$$\langle D_K^* F, \mathrm{IN}_K \mathrm{EX}_m U\rangle = \langle F, D_K \mathrm{IN}_K \mathrm{EX}_m U\rangle = \langle F, \mathrm{EX}_m U\rangle.$$

Since Q and U are arbitrary, $\mathrm{IN}_K^* D_K^* F = F$. The proof of the remaining equalities is similar. ∎

The following equalities can easily be obtained from the definitions.

(9.11) $\mathrm{TS}^Y(\mathrm{IN}_K U) = \mathrm{IN}_K(\mathrm{TS}^Y U)$, $\mathrm{TS}^Y(\mathrm{IN}_K^* U) = \mathrm{IN}_K^*(\mathrm{TS}^Y U)$, if $U \in L_1(\boldsymbol{R}^m; K)$.

At last, we introduce the modulus of continuity of functions and distributions. Let us first define an operator of *finite difference*

$$\mathrm{DF}_Z^k \colon W_q^r(A) \to W_q^r\Big(\bigcap_{0 \leqslant l \leqslant k} \mathrm{TS}^{-l\circ Z} A\Big),$$

where r denotes p or $-p$, $k \in I^m$, $k \geqslant 0$, and $Z \in \boldsymbol{R}^m$, by the recurrent formula

(10.1) $\mathrm{DF}_Z^0 U = U$;

if $U \in W_q^r(A)$, $i \in \overline{m}$, then $\mathrm{DF}_Z^{e_i'} U = \mathrm{TS}^{Z\circ e_i'} U - U$;

if $U \in W_q^r(A)$, $k \in I^m$, $k = l + e_i'$, $l > 0$, then

$$\mathrm{DF}_Z^k U = \mathrm{DF}_Z^{e_i'}(\mathrm{DF}_Z^l U).$$

The *k-th axial modulus of continuity* is defined as follows.

(10.2) $\mathrm{MC}_{r,q}(U; C, k) = \sup\{\|\mathrm{DF}_Z^k U\|_{r,q}\colon\ 0 \leqslant Z \leqslant C\}$
for $U \in W_q^r(A)$, $k \in I^m$, $k \geqslant 0$, $C \in \boldsymbol{R}^m$, $C \geqslant 0$;
$\mathrm{MC}_q(U; C, k) = \mathrm{MC}_{0,q}(U; C, k)$;

the *global modulus of first order* is defined by

(10.3) $\mathrm{MC}_{r,q}(U; C) = \sup\{\|\mathrm{TS}^Z U - U\|_{r,q}\colon\ -C \leqslant Z \leqslant C\}$ for $U \in W_q^r(A)$, $C \in \boldsymbol{R}^m$, $C \geqslant 0$.

As can easily be checked,

(10.4) $\mathrm{MC}_{r,q}(U; C) \leqslant \sum_{i\in\overline{m}} \mathrm{MC}_{r,q}(U; C, e_i')$.

It can be proved that if $U \in W_q^r(A)$ with q finite then

(10.5) $\lim_{C\to 0} \mathrm{MC}_{r,q}(U; C, k) = 0$ for every $k > 0$.

Moreover, the following lemma is true.

(10.6) LEMMA. *If* $U \in W_q^p(A)$, $p \in \mathrm{SL}^m$, $k \in p$, *then* $\|\mathrm{DF}_C^k U\|_q \leqslant |C^k| \cdot \|D^k U\|_q$ *for every* $C \in \boldsymbol{R}^m$, *hence* $\mathrm{MC}_q(U; C, k) \leqslant C^k \|D^k U\|_q$ *for* $C \geqslant 0$.

Proof. The proof will be carried out by induction with respect to k. Let first $k = e_i' \in p$.

Let us take an arbitrary bounded regular set $B \subset A$ and define $\tilde{U} = \mathrm{RD}_B U$. Then $\tilde{U} \in W_1^p(B)$, what follows from Hölder's inequality. According to lemma (9.10), at almost every $X \in B$ and for $Z \geqslant 0$

$$\begin{aligned}\mathrm{DF}_Z^{e_i'}\tilde{U}(X) &= \mathrm{IN}_i \mathrm{DF}_Z^{e_i'} D_i\tilde{U}(X) = \langle \mathrm{TS}^{Z\circ e_i'} D_i\tilde{U} - D_i\tilde{U}, \mathrm{FS}_{i,X_i}^{-}\rangle_i (X_{\setminus i})\\ &= \langle D_i\tilde{U}, \mathrm{TS}^{-Z\circ e_i'}\mathrm{FS}_{i,X_i}^{-} - \mathrm{FS}_{i,X_i}^{-}\rangle_i(X_{\setminus i}) = \langle D_i\tilde{U}, \mathrm{FS}_i^{X_i, X_i+Z_i}\rangle_i(X_{\setminus i}),\end{aligned}$$

what follows from formula (9.6) and definition (9.7). Thus, at almost each $X \in A$

$$\mathrm{DF}_Z^{e_i'} U(X) = \langle D_i U, \mathrm{FS}_i^{X_i, X_i+Z_i}\rangle_i (X_{\setminus i}),$$

since B has been taken arbitrarily. Hence for $1 \leqslant q < \infty$

$$\begin{aligned}\|\mathrm{DF}_Z^{e_i'} U\|_q^q &= \int_{\boldsymbol{R}^m} \Big| \int_{\boldsymbol{R}} D_i U(Y \oplus_i X_{\setminus i})\, \delta(X_i \leqslant Y \leqslant X_i + Z_i)\, dY \Big|^q dX\\ &\leqslant \int_{\boldsymbol{R}^m} \Big\{ \int_{\boldsymbol{R}} |D_i U(Y \oplus_i X_{\setminus i})|^q\, \delta(X_i \leqslant Y \leqslant X_i + Z_i)\, dY \Big\} Z_i^{q-1}\, dX,\end{aligned}$$

what follows from Hölder's inequality. By introducing new variable $W = X_i - Y$ instead of X_i and applying Fubini's theorem we obtain thus

$$\|\mathrm{DF}_Z^{e_i'} U\|_q^q \leqslant Z_i^{q-1} \int_{\boldsymbol{R}^m} \Big\{ \int_{\boldsymbol{R}} |D_i U(Y \oplus_i X_{\setminus i})|^q \delta(-Z_i \leqslant W \leqslant 0)\, dY \Big\}\, d(W \oplus_i X_{\setminus i})$$

$$= Z_i^{q-1} \int_{-Z_i}^{0} \Big\{ \int_{\boldsymbol{R}^m} |D_i U(Y \oplus_i X_{\setminus i})|^q d(Y \oplus_i X_{\setminus i}) \Big\}\, dW = Z_i^q \|D_i U\|_q^q.$$

Hence $\|\mathrm{DF}_Z^{e_i'} U\|_q \leqslant Z_i \|D_i U\|_q$ and for each $k \in p$ $\|\mathrm{DF}_Z^k U\|_q \leqslant Z^k \|D^k U\|_q$, what was to be shown. The proof in the case when Z is an arbitrary vector is based on the formula $\mathrm{DF}_{-Z}^{e_i'} U = -\mathrm{TS}^{-Z \circ e_i'} \mathrm{DF}_Z^{e_i'} U$, which can be obtained from definition (10.1). This completes the proof of the lemma. ∎

If q is not finite, formula (10.5) is not true for every F. We can, however, define the following subspaces of $W_q^r(A)$.

(10.7) $C^r(A) = W_*^r(A) = \{F \in W_\infty^r(A)\colon \lim_{C \to 0} \mathrm{MC}_{r,\infty}(F; C, e_i') = 0 \ \ \forall\, i \in \overline{m}\}$

(in the case $r = 0$ $C^r(A)$ is the space of uniformly continuous functions);

$W_{(qx,*,K)}^r(A) = \{F \in W_{(qx,\infty,K)}^r(A)\colon \lim_{C \to 0} \mathrm{MC}_{r,(qx,\infty,K)}(F; C, e_i') = 0 \ \ \forall\, i \in K\}$;

$W_{(*,qi,K)}^r(A) = \{F \in W_{(\infty,qi,K)}^r(A)\colon \lim_{C \to 0} \mathrm{MC}_{r,(\infty,qi,K)}(F; C, e_i') = 0 \ \ \forall\, i \in \setminus K\}$.

The modulus of continuity of local functions and distributions can be defined in a similar way, using the seminorms.

3. Spaces of mesh functions

An approximation of the spaces introduced in the previous section will be based on a mesh defined on $\boldsymbol{R}^m$.

Let the family H of parameters be a bounded subset of $(0, \infty)^m$ with zero as an accumulation point. Consider the family $r^m = \{r_h^m\}_{h \in H}$ of the sets defined as follows:

(11.1) $$r_h^m = \{X \in \boldsymbol{R}^m\colon X = k \circ h,\ k \in I^m\}.$$

The set defined by (11.1) will be called a *mesh* on $\boldsymbol{R}^m$.

Let us also consider a fixed (independent of h) vector $\theta \in [0, 1]^m$ and state the following one-to-one correspondence between the mesh points and subsets of $\boldsymbol{R}^m$.

(11.2) If $x_h \in r_h^m$ then $\mathrm{CE}\, x_h = \{X \in \boldsymbol{R}^m\colon \forall i\ (x_h + (\theta - e) \circ h)_i \leqslant X_i < (x_h + \theta \circ h)_i\}$ is the cell corresponding to x_h;

if $X \in \boldsymbol{R}^m$ then the relation $\mathrm{pt}_h X = x_h \Leftrightarrow X \in \mathrm{CE}\, x_h$ defines the *mesh point* corresponding to X.

For simplifying the notation we will omit the subscript h in the symbols of mesh points, mesh functions, etc. Strictly speaking, if for every $h \in H$ x_h is a point of r_h^m then x will be treated as a family $x = \{x_h\}_{h \in H}$; if ϱ is a relation, $\varrho(x)$ is a shortened symbol for: $\forall h \in H$ $\varrho(x_h)$. To avoid misunderstanding, the points, functions, etc. on the mesh will be denoted by small letters, when the points of $\boldsymbol{R}^m$, functions on $\boldsymbol{R}^m$, etc. — by capital letters, as has been done in definition (11.2). The constants or constant vectors denoted by capital letters do not depend on the mesh width even if they are connected with mesh sets or functions; by small letters we will denote not only the constants depending on the mesh width, but also, for instance, the index of the space W_q^p or the power of differentiating D^k. Since in many cases the mesh width h and the volume of the elementary cell h^e occur in the formulas, we introduce the symbols

(11.3) $$\mathrm{mw}_h = h, \qquad \mathrm{cv} = \mathrm{mw}^e,$$

which allow us to write, for instance, $|f(x)| < |\mathrm{mw}|$, instead of $\forall h \in H$ $|f_h(x_h)| < |h|$.

Now, let us define mesh neighbourhoods of points in r^m.

(12.1) Let ST be a finite non-empty subset of I^m independent of the mesh width. The set

$$\mathrm{nb}(x, \mathrm{ST}) = \{y \in r^m \colon\ y = x + j \circ \mathrm{mw},\ j \in \mathrm{ST}\}$$

is called the ST-*neighbourhood of the point* $x \in r^m$, ST is the stencil of $\mathrm{nb}(x, \mathrm{ST})$; if $b \subset r^m$ then we define

$$\mathrm{nb}(b, \mathrm{ST}) = \bigcup_{x \in b} \mathrm{nb}(x,\ \mathrm{ST}).$$

If $b \subset r^m$, the following mesh set will be also defined using the stencil ST.

(12.2) $$\mathrm{sb}(b, \mathrm{ST}) = \{y \in r^m \colon\ \mathrm{nb}(y, \mathrm{ST}) \subset b\}.$$

It can easily be shown that

(12.3) $\mathrm{sb}(b, \{0\}) = b = \mathrm{nb}(b, \{0\})$;
for any two stencils ST, ST′,
$\mathrm{sb}(b, \mathrm{ST} \cup \mathrm{ST}') = \mathrm{sb}(b, \mathrm{ST}) \cap \mathrm{sb}(b, \mathrm{ST}')$,
$\mathrm{nb}(b, \mathrm{ST} \cup \mathrm{ST}') = \mathrm{nb}(b, \mathrm{ST}) \cup \mathrm{nb}(b, \mathrm{ST}')$,
$\mathrm{sb}(b, \mathrm{ST} + \mathrm{ST}') = \mathrm{sb}(\mathrm{sb}(b, \mathrm{ST}), \mathrm{ST}')$,
$\mathrm{nb}(b, \mathrm{ST} + \mathrm{ST}') = \mathrm{nb}(\mathrm{nb}(b, \mathrm{ST}), \mathrm{ST}')$;
for every stencil ST,
$\mathrm{nb}(\mathrm{sb}(b, \mathrm{ST}), \mathrm{ST}) \subset b \subset \mathrm{sb}(\mathrm{nb}(b, \mathrm{ST}), \mathrm{ST})$.

Therefore, if $0 \in \mathrm{ST}$, $b \subset r^m$, then $\mathrm{sb}(b, \mathrm{ST}) \subset b \subset \mathrm{nb}(b, \mathrm{ST})$ and we define the mesh boundary of b

(12.4) $$\mathrm{bd}(b, \mathrm{ST}) = b \setminus \mathrm{sb}(b, \mathrm{ST}).$$

Further, using the notion of the stencil we can state the correspondence between subsets of $\boldsymbol{R}^m$ and mesh sets.

(12.5) If $b \subset r^m$, ST is a stencil, then

$$\mathrm{NB}(b, \mathrm{ST}) = \bigcup_{x \in \mathrm{nb}(b, \mathrm{ST})} \mathrm{CE}x,$$

$$\mathrm{SB}(b, \mathrm{ST}) = \bigcup_{x \in \mathrm{sb}(b, \mathrm{ST})} \mathrm{CE}x$$

are the ST-*neighbourhood* and ST-*subset* corresponding to b; if $B \subset \boldsymbol{R}^m$, ST is a stencil, then

$$\mathrm{nb}(B, \mathrm{ST}) = \mathrm{nb}(\{x\colon \mathrm{CE}x \cap B \underset{\mu}{\neq} \emptyset\},\ \mathrm{ST}),$$

$$\mathrm{sb}(B, \mathrm{ST}) = \mathrm{sb}(\{x\colon \mathrm{CE}x \underset{\mu}{\subset} B\},\ \mathrm{ST})$$

are the *mesh sets* associated to B.

Now, let us introduce some notation, analogous to that for the functions on $\boldsymbol{R}^m$.

(13.1) If A is a linear space, $b \subset r^m$, then the symbol $(b \to A)$ will denote the space of all functions defined on b with the values in A.

(13.2) If $a \subset b \subset r^m$ then the *extension* and *restriction* operators on the mesh, $\mathrm{ex}_b\colon (a \to A) \to (b \to A)$, $\mathrm{rd}_a\colon (b \to A) \to (a \to A)$, are defined by the formulas

$$(\mathrm{ex}_b f)(x) = \begin{cases} f(x) & \text{if} \quad x \in a \\ 0 & \text{if} \quad x \in b \setminus a, \end{cases} \quad \mathrm{ex}_m = \mathrm{ex}_{r^m},$$

$$(\mathrm{rd}_a f)(x) = f(x) \quad \text{if} \quad x \in a.$$

(13.3) The *support* of a function $f \in (b \to A)$ is defined by

$$\mathrm{supp}(f) = \{x \in b\colon f(x) \neq 0\}.$$

Similarly as in the second paragraph, if $K \subset \overline{m}$, $\mathrm{ne}K = k$, then we define

(13.4) r^k_K – the family of meshes on $\boldsymbol{R}^k$ with the mesh width mw_K,

$$\mathrm{cv}_K = \mathrm{mw}^{e'_K};$$

$$\mathrm{sc}^x_K a = \{y \in r^k_K\colon y \oplus_K x \in a\} \quad \text{if} \quad a \subset r^m,\ x \in r^{m-k}_{\setminus K};$$

$$\mathrm{pj}_K a = \{y \in r^{m-k}_{\setminus K}\colon \mathrm{sc}^y_K a \neq \emptyset\} \quad \text{if} \quad a \subset r^m;$$

$$(\mathrm{rv}^y_K f)(x) = f(x \oplus_K y) \quad \text{if} \quad f \in (a \to B),\ y \in \mathrm{pj}_K a,\ x \in \mathrm{sc}^y_K a.$$

(13.5) The following scalar products of mesh functions are defined: if $u \in (b \to \boldsymbol{R})$, $f \in (o \to A)$, A is a linear topological space with the topology inducted by the family of seminorms $\{|\cdot|_z\}_{z \in Z}$, and the series $\sum_{x_h \in r_h^m} |\mathrm{ex}_m u_h(x_h)|\,|\mathrm{ex}_m f_h(x_h)|_z$ is convergent for every $h \in H$ and $z \in Z$, then

$$[u, f] = [f, u] = \mathrm{cv} \sum_{x \in r^m} \mathrm{ex}_m u(x) \mathrm{ex}_m f(x);$$

if $K \subset \overline{m}$ and for every $x \in r_{\setminus K}^{m-\mathrm{ne}K}$ the $\mathrm{ne}K$-dimensional scalar product $[\mathrm{rv}_K^x u, \mathrm{rv}_K^x f]$ is well defined, then

$$[u, f]_K(x) = [\mathrm{rv}_K^x u, \mathrm{rv}_K^x f] \quad \forall\, x \in r_{\setminus K}^{m-\mathrm{ne}K}.$$

Now, we define mesh translations and finite differences as the discrete analogue of translations and derivatives.

(13.6) If $x \in r^m$, $k \in I^m$, then $\mathrm{ts}^k x = x + k \circ \mathrm{mw}$,
if $b \subset r^m$, $k \in I^m$, then $\mathrm{ts}^k b = \{x \in r^m \colon x = \mathrm{ts}^k y,\ y \in b\}$.

The operator $\mathrm{ts}^k\colon (b \to A) \to (\mathrm{ts}^{-k} b \to A)$ is given by

(13.7) $(\mathrm{ts}^k f)(x) = f(\mathrm{ts}^k x)\ \forall\, x \in \mathrm{ts}^{-k} b$;
if $K \subset \overline{m}$ then $\mathrm{ts}_K f = \mathrm{ts}^{e'_K} f$, $\mathrm{ts}_K^- f = \mathrm{ts}^{-e'_K} f$.

The finite differences are defined by the recurrent formulas.

(13.8) $\Delta^0 f = \Delta^{*0} f = f$; $\Delta^{e'_i} f = \mathrm{ts}_i f - f$, $\Delta^{*e'_i} f = \mathrm{ts}_i^- f - f$ for $i \in \overline{m}$;
$\Delta^{k+e'_i} f = \Delta^{e'_i}(\Delta^k f)$, $\Delta^{*k+e'_i} f = \Delta^{*e'_i}(\Delta^{*k} f)$ for $k \in I^m$, $k \geqslant 0$;
$\partial^k f = \mathrm{mw}^{-k} \Delta^k f$, $\partial^{*k} f = \mathrm{mw}^{-k} \Delta^{*k} f$ for $k \in I^m$, $k \geqslant 0$;
$\partial_K f = \partial^{e'_K} f$, $\partial_K^* f = \partial^{*e'_K} f$ for $K \subset \overline{m}$.

It follows from the definitions that

(13.9) $\partial^k f(x)$ is the linear combination of the values of f at the points from $\mathrm{nb}(x, k'')$, that is,

$$\partial^k\colon (b \to A) \to \big(\mathrm{sb}(b, k'') \to A\big),$$

and similarly,

$$\partial^{*k}\colon (b \to A) \to \big(\mathrm{sb}(b, -k'') \to A\big)$$

(the symbol k'' was introduced in (6.3)).

Further, let us define auxiliary functions

(13.10) $\mathrm{fc}(x) = 1 \ \forall x \in r^m$; $\mathrm{fs}_K^{x,y} = \mathrm{fs}_{K,y}^- - \mathrm{fs}_{K,x}^-$ if $x < y$;
$\mathrm{fs}_{K,x}^-(z) = \delta(\forall i \in K \ z_i < x_i)$, $\mathrm{fs}_{K,x}^+(z) = \delta(\forall i \in K \ z_i > x_i)$
$\forall x, y \in r_K^{\mathrm{ne}\,K}$, $z \in r^m$.

Let A be a linear space with the family of seminorms $\{|\cdot|_z\}_{z\in Z}$ (in the special case Z may consist of one element, then A is a normed space). We define the spaces analogous to $L_q(B)$, $W_q^p(B)$, $W_q^{-p}(B)$.

(14.1) By $l_q(b; A)$ $(1 \leqslant q \leqslant \infty)$ we will understand the subspace of $(b \to A)$ consisting of all functions f such that

$$\|f\|_{q,z} = \begin{cases} [|f|_z^q, \mathrm{fc}]^{1/q} & \text{if} \quad 1 \leqslant q < \infty, \\ \sup\{|f(x)|_z \colon x \in b\} & \text{if} \quad q = \infty \end{cases}$$

is finite for every $z \in Z$. $\{\|\cdot\|_{q,z}\}_{z\in Z}$ is the family of seminorms in $l_q(b; A)$.

(14.2) If $K \subset \overline{m}$, then $l_q(b, K; A)$ is the set consisting of all functions $f \in (b \to A)$ such that $\mathrm{rv}_K^y f \in l_q(\mathrm{sc}_K^y b; A)$ for every $y \in \mathrm{pj}_K b$; the space $l_{(qx,qi,K)}(b; A)$ is the subset of $l_{qi}(b, K; A)$ consisting of all functions f such that the functions g_z defined by $g_z(y) = \|\mathrm{rv}_K^y f\|_{qi,z}$ belong to $l_{qx}(\mathrm{pj}_K b; \boldsymbol{R})$ for every $z \in Z$; the seminorms are given by $\|f\|_{(qx,qi,K),z} = \|g_z\|_{qx}$. If $b = a \oplus_K c$ then $l_{(qx,qi,K)}(b; A)$ is isomorphic with $l_{qx}(c; l_{qi}(a; A))$.

(14.3) The symbol $w_q^p(b; A)$ will denote the subspace of $(b \to A)$ consisting of all functions f such that $\partial^k f \in l_q(\mathrm{sb}(b, k''); A)$ for every $k \in p$;

$$\|f\|_{p,q,z} = \begin{cases} \Big(\sum\limits_{k\in p} \|\partial^k f\|_{q,z}^q\Big)^{1/q} & \text{if} \quad 1 \leqslant q < \infty, \\ \max\{\|\partial^k f\|_{\infty,z} \colon k \in p\} & \text{if} \quad q = \infty, \\ \|g\|_{qx} \quad \text{if} \quad q = (qx, qi, K),\ p \in \mathrm{SL}_K^m, \\ \qquad g(y) = \|\mathrm{rv}_K^y f\|_{p_K,qi,z} \quad \text{for } y \in \mathrm{pj}_K b \end{cases}$$

form the family of seminorms.

(14.4) The space $\mathring{w}_q^p(b; A)$ is defined as $w_q^p(b; A) \cap \mathrm{vn}_p(b; A)$, where vn_p is the set of the functions vanishing together with their pth finite differences on the boundary of b:

$$\mathrm{vn}_p(b; A) = \{f \in (b \to A) \colon \mathrm{ex}_m(\partial^k f) = \partial^k(\mathrm{ex}_m f) \text{ for every } k \in p\},$$

or, in the equivalent form,

$$\mathrm{vn}_p(b; A) = \{f \in (b \to A) \colon f(x) = 0 \text{ if } x \in \mathrm{bd}(b, p \cup -p)\}.$$

If $A = \boldsymbol{R}$, we will write $l_q(b)$, $w_q^p(b)$, $\mathring{w}_q^p(b)$ instead of $l_q(b; \boldsymbol{R})$, $w_q^p(b; \boldsymbol{R})$, $\mathring{w}_q^p(b; \boldsymbol{R})$.

We prove now some properties of the functions and operators defined before. First, let us note that if $u \in l_q(b; A)$ then $\partial^k u \in l_q(\mathrm{sb}(b, k''); A)$, since ∂^k is the linear combination of the translation operators. Thus,

(15.1) if $u \in l_q(b; A)$ then $u \in w_q^p(b; A)$ for each $p \in \mathrm{SL}^m$.

Moreover, we have the following lemma.

(15.2) LEMMA. *If $f \in l_q(r^m; A)$, $u \in l_{q'}(r^m)$, q' is conjugate to q, then*

$$\forall k \geqslant 0 \qquad [\partial^k f, u] = [f, \partial^{*k} u].$$

Proof. The fact that $[\partial^k f, u]$ is well defined follows from Hölder's inequality. To prove the lemma let us take first $k = e_i'$. We have

$$[\partial_i f, u] = \mathrm{mw}_i^{-1}[\mathrm{ts}_i f - f, u].$$

Since $[\mathrm{ts}^k f, u] = [f, \mathrm{ts}^{-k} u]$, we obtain

$$[\partial_i f, u] = \mathrm{mw}_i^{-1}[f, \mathrm{ts}_i^{-} u - u] = [f, \partial_i^* u],$$

and the lemma can be proved by induction. ■

(15.3) COROLLARY. *If $f \in \mathring{w}_q^p(b; A)$, $u \in l_{q'}(b)$, then for every $k \in p$*

$$[\partial^k f, u] = [f, \partial^{*k} u].$$

Proof. Following the definition of $\mathring{w}_q^p$, for every $k \in p$ $\partial^k \mathrm{ex}_m f = \mathrm{ex}_m \partial^k f$. Since $\mathrm{ex}_m \partial^k f \in l_q(r^m; A)$, $\mathrm{ex}_m u \in l_{q'}(r^m)$, we obtain by applying the previous lemma

$$[\partial^k \mathrm{ex}_m f, \mathrm{ex}_m u] = [\mathrm{ex}_m f, \partial^{*k} \mathrm{ex}_m u].$$

Since $\mathrm{ex}_m f(x) = 0$ if $x \in \mathrm{bd}(b, p \cup -p)$ and $\partial^{*k} \mathrm{ex}_m u(x) = \mathrm{ex}_m \partial^{*k} u(x)$ if $x \in \mathrm{sb}(b, -k'') \supset \mathrm{sb}(b, -p)$, the corollary is proved. ■

Let us now consider the space $\mathring{w}_q^p(b)$. It follows from definition (14.4) that if $k \in p$ then $\partial^k f$ vanishes outside the set $b_{kp} = \mathrm{sb}\big(b, (p \ominus k) \cup -(p \ominus k)\big)$. Thus, the space $\mathring{w}_q^p(b)$ may be isometrically embedded into the Cartesian product $\prod_{k \in p} l_q(b_{kp})$; similarly to the case considered in Section 2, we take the embedding $f \to (\mathrm{rd}_{b_{pk}} \partial^k f)_{k \in p}$.

Therefore, each linear continuous functional on $\mathring{w}_q^p(b)$ can be represented by a vector $\vec{u}_p \in \prod_{k \in p} l_{q'}(b_{kp})$. Let us define the distributions of order (p, q) on b as the equivalence classes in $\prod_{k\in p} l_q(b_{kp})$ with respect to the relation

$$(16.1)\qquad \mathrm{eq}_q^p(\vec{u}_p, \vec{y}_p) \Leftrightarrow \forall f \in \mathring{w}_{q'}^p(b) \ \sum_{k\in p} [\partial^k f, u_k] = \sum_{k\in p} [\partial^k f, y_k].$$

If $u = \mathrm{ec}_q^p(\vec{u}_p)$, where ec_q^p is the equivalence class with respect to the relation eq_q^p, we will write

$$[f, u] = \sum_{k\in p} [\partial^k f, u_k].$$

For every $k \in p$, Corollary (15.3) yields that $[\partial^k f, \mathrm{ex}_m u_k] = [f, \partial^{*k}\mathrm{ex}_m u_k]$, but since $\partial^{*k}\mathrm{ex}_m u_k(x) = \partial^{*k} u_k(x)$ if $x \in \mathrm{supp}(f) \subset b_{0p}$, we obtain

$$[\partial^k f, u_k] = [f, \partial^{*k} u_k].$$

Hence

$$[f, u] = \sum_{k\in p} [\partial^k f, u_k] = \Big[f, \sum_{k\in p} \mathrm{ex}_b \partial^{*k} u_k\Big],$$

what means that every distribution of order (p, q) on the mesh can be represented by a mesh function belonging to $l_q(b)$.

(16.2) The space $w_q^{-p}(b)$ is defined as the space of all distributions of order (p, q) on b, normed by

$$\|u\|_{-p,q} = \sup\{[f, u]\colon f \in \mathring{w}_{q'}^p(b),\ \|f\|_{p,q'} = 1\}.$$

If b is a bounded subset of r^m then it is finite, therefore $l_q(b)$, $w_q^p(b)$, $w_q^{-p}(b)$ contain all real-valued functions defined on b. Hence the definition of local spaces is the following:

(16.3) The local spaces $l_q(b)^{\mathrm{loc}}$, $w_q^p(b)^{\mathrm{loc}}$, $w_q^{-p}(b)^{\mathrm{loc}}$, are the sets of all real-valued functions defined on b, with the topology inducted by the families of seminorms:

$$\|f\|_q^{(c)} = \|\mathrm{rd}_c f\|_q, \qquad \|f\|_{p,q}^{(c)} = \|\mathrm{rd}_c f\|_{p,q}, \qquad \|f\|_{-p,q}^{(c)} = \|\mathrm{rd}_c f\|_{-p,q},$$

where c runs over the family of all bounded subsets of b.

We define now the operators in_K, $\mathrm{in}_K^*\colon l_1(r^m, K; A) \to (r^m \to A)$ which are discrete analogues of the operators IN_K, IN_K^*. If $K \subset \overline{m}$, $f \in l_1(r^m, K; A)$, then

$$(17.1)\qquad (\mathrm{in}_K f)(x) = [f, \mathrm{fs}_{K,x_K}^-]_K(x_{\setminus K}), \qquad (\mathrm{in}_K^* f)(x) = [f, \mathrm{fs}_{K,x_K}^+]_K(x_{\setminus K})$$

(the functions fs are defined by (13.10)).

The following properties can easily be obtained from the definition of scalar products.

(17.2) If $f \in l_1(r^m, K; A)$, $g \in l_1(r^m, K)$, then $[\mathrm{in}_K f, g]_K = [f, \mathrm{in}_K^* g]_K$,

(17.3) if $f \in l_1(r^m, K; A)$, then $\mathrm{ts}^k \mathrm{in}_K f = \mathrm{in}_K \mathrm{ts}^k f$ for every $k \in I^m$.

The following relation between ∂_K and in_K holds.

(17.4) LEMMA. *If* $f \in l_1(r^1; A)$ *then*

$$\partial_1 \mathrm{in}_1 f = \partial_1^* \mathrm{in}_1^* f = \mathrm{in}_1 \partial_1 f = \mathrm{in}_1^* \partial_1^* f = f.$$

Proof. Using the definitions of ∂_K and in_K we obtain

$$\partial_1 \mathrm{in}_1 f(x) = \mathrm{mw}^{-1}\Big\{\mathrm{mw}\sum_{y \in r^1} f(y)\,\delta(y < \mathrm{ts}^1 x) - \mathrm{mw}\sum_{y \in r^1} f(y)\,\delta(y < x)\Big\} = f(x),$$

in the same way we show that $\partial_1^* \mathrm{in}_1^* f = f$. Take now an arbitrary $g \in l_1(r^1)$. Applying (17.2), Lemma (15.2) and the equality just proven we obtain

$$[\mathrm{in}_1 \partial_1 f, g] = [\partial_1 f, \mathrm{in}_1^* g] = [f, \partial_1^* \mathrm{in}_1^* g] = [g, f].$$

Since g has been taken arbitrarily, the lemma is proved. ∎

(17.5) COROLLARY. *If* $f \in l_1(r^m, K; A)$, $L \subset K$, *then*

$$\partial_L \mathrm{in}_K f = \mathrm{in}_K \partial_L f = \mathrm{in}_{K \setminus L} f, \qquad \partial_L^* \mathrm{in}_K^* f = \mathrm{in}_K^* \partial_L^* f = \mathrm{in}_{K \setminus L}^* f$$

(the notation $\mathrm{in}_\emptyset f = \mathrm{in}_\emptyset^* f = f$ is here used).

The proof is based on Lemma (17.4) and Fubini's theorem. ∎

4. Approximation of $L_q(B)$

In order to construct an approximation of $L_q(B)$ we introduce a set of double partitions of unity, that is, a set pu_l^m of functions of two variables — $x \in r^m$ and $X \in \boldsymbol{R}^m$, which sum up to 1 with respect to each variable.

Let us consider functions $\hat{v} \in \big(r^m \to L_\infty(\boldsymbol{R}^m)^{\mathrm{loc}}\big)$, that is, functions defined on the mesh with values in $L_\infty(\boldsymbol{R}^m)^{\mathrm{loc}}$. The following notation will be used:

$$\hat{v}(x, X) = \big(\hat{v}(x)\big)(X),$$

i.e. for every $h \in H$, $X \in \boldsymbol{R}^m$, $x_h \in r_h^m$,

$$\hat{v}_h(x_h, X) = \big(\hat{v}_h(x_h)\big)(X);$$

$\hat{v}(\cdot, X)$ is the mesh function defined for almost every $X \in \boldsymbol{R}^m$;

if A is an operator acting from $B \subset L_\infty(\boldsymbol{R}^m)^{\mathrm{loc}}$ into B' then $A\hat{v}$ is the function from $(r^m \to B')$ given by $(A\hat{v})(x) = A\big(\hat{v}(x)\big)$ for each $x \in r^m$, for example, if $F \in L_\infty(\boldsymbol{R}^m)^{\mathrm{loc}}$ and for every $x \in r^m$ the scalar product $\langle \hat{v}(x), F\rangle$ exists, then $\langle \hat{v}, F\rangle$ is the mesh function defined by $\langle \hat{v}, F\rangle(x) = \langle \hat{v}(x), F\rangle$.

Now, we can introduce the definition.

(18) DEFINITION. The *largest set of double partitions of unity*, pu_l^m (denoted also, if needed, by $\mathrm{pu}_l^m[\mathrm{mw}]$), is the set of all functions $\hat{v} \in (r^m \to L_\infty(\boldsymbol{R}^m)^{\mathrm{loc}})$ satisfying:

1) $\forall x \in r^m \;\; \hat{v}(x) \geqslant 0$;

2) $[\hat{v}, \mathrm{fc}] = \mathrm{FC}$;

3) $\langle \hat{v}, \mathrm{FC}\rangle = \mathrm{fc}$;

4) there exists a stencil $\mathrm{ST}\,\hat{v}$ such that for each $x \in r^m$

$$\mathrm{SUPP}\,\hat{v}(x) \subset \mathrm{NB}(x, \mathrm{ST}\,\hat{v});$$

pu_t^m is the subset of pu_l^m consisting of all functions $\hat{v}$ invariant with respect to the translation operator, that is, satisfying

5) $\forall k \in I^m \;\; \hat{v} = \mathrm{TS}^{k\circ\mathrm{mw}}(\mathrm{ts}^k \hat{v})$;

pu_0^m is the subset of pu_t^m consisting of all functions $\hat{v}$ which are products of one-dimensional functions, that is,

6) $\hat{v}(x, X) = \prod_{i=1}^{m} \hat{v}_i(x_i, X_i)$, where $\hat{v}_i \in \mathrm{pu}_t^1[\mathrm{mw}_i]$ for $i = 1, 2, \ldots, m$.

The scalar product occurring in (18.2) is well defined since as is required in definition (13.5) the sum $\sum_{x \in r^m} \mathrm{fc}(x) \|\hat{v}(x)\|_\infty^{(A)}$ is convergent for every bounded set $A \subset \boldsymbol{R}^m$, because only finite number of the components of the sum does not vanish on A what follows from assumption (18.4). Moreover, at almost every $X \in \boldsymbol{R}^m$, $[\hat{v}, \mathrm{fc}](X) = [\hat{v}(\cdot, X), \mathrm{fc}]$.

The set pu_0^m is non-empty; for instance, the following characteristic function $\hat{c}$ is often used

$$\hat{c}(y, Y) = \mathrm{mw}^{-e}\,\delta(Y \in \mathrm{CE}\,y). \tag{18.7}$$

It can easily be checked that $\hat{c} \in \mathrm{pu}_0^m$, as the stencil we can take $\mathrm{ST}\,\hat{c} = \{0\}$.

For each function $\hat{v} \in \mathrm{pu}_l^m$ let us also introduce the vectors

$$\mathrm{UP}\,\hat{v} = \mathrm{UB}(\mathrm{ST}\,\hat{v}) + \theta, \qquad \mathrm{LW}\,\hat{v} = \mathrm{LB}(\mathrm{ST}\,\hat{v}) + \theta - e, \tag{18.8}$$

which are called *upper* and *lower bounding vectors* for $\mathrm{SUPP}\,\hat{v}$ since

$$\mathrm{NB}(x, \mathrm{ST}\hat{v}) \subset \{X \in \boldsymbol{R}^m\colon \mathrm{LW}\,\hat{v} \circ \mathrm{mw} \leqslant X - x \leqslant \mathrm{UP}\,\hat{v} \circ \mathrm{mw}\}, \tag{18.9}$$

what follows from definitions (3.4) and (11.2).

Together with the set pu_l^m we will also consider the sets pc_l^m, pc_t^m, pc_0^m of combinations of partitions of unity.

(19.1) DEFINITION. The set pc_a^m (where a is an arbitrary index from $\{l, t, 0\}$) consists of all functions $\hat{v}$ for which there exist real numbers $s_1, \ldots, s_k$ and functions $\hat{v}^1, \ldots, \hat{v}^k$ from pu_a^m, such that $\sum_{i=1}^{k} s_i = 1$ and $\sum_{i=1}^{k} s_i\hat{v}^i = \hat{v}$.

The numbers $(s_1, \ldots, s_k)$ and the functions $(\hat{v}^1, \ldots, \hat{v}^k)$ occurring in Definition (19.1) are not defined uniquely. The following representation of a function $\hat{v} \in \mathrm{pc}_a^m$ will also be used.

$$\hat{v} = \sum_{i=1}^{kp} t_i\hat{w}^i + \sum_{i=1}^{kd} u_i(\hat{y}^i - \hat{z}^i); \quad t_i, u_i > 0, \quad \sum_{i=1}^{kp} t_i = 1; \quad \hat{w}^i, \hat{y}^i, \hat{z}^i \in \{\hat{v}^j\colon 1 \leqslant j \leqslant k\}. \tag{19.2}$$

For instance,

$$0.5\hat{v}^1 + 0.8\hat{v}^2 - 0.3\hat{v}^3 = \{0.4\hat{v}^1 + 0.6\hat{v}^2\} + \{0.1(\hat{v}^1 - \hat{v}^3) + 0.2(\hat{v}^2 - \hat{v}^3)\}.$$

It follows directly from the definition that if $\hat{v} \in \mathrm{pc}_l^m$ (pc_t^m) then $\hat{v}$ satisfies conditions (18.2)–(18.4) ((18.2)–(18.5)), respectively).

Let us also define a modulus of $\hat{v} \in \mathrm{pc}_a^m$ by the formula

$$\mathrm{md}_a(\hat{v}) = \inf\Big\{\sum |s_i|\colon \hat{v} = \sum s_i\hat{v}^i, \ \sum s_i = 1, \ \hat{v}^i \in \mathrm{pu}_a^m\Big\}. \tag{19.3}$$

It can be checked that

$$\inf\Big\{\sum_{i=1}^{kd} u_i\colon \hat{v} \text{ is of the form (19.2)}\Big\} = \big(\mathrm{md}_a(\hat{v}) - 1\big)/2. \tag{19.4}$$

Let us now prove several lemmas.

(20.1) LEMMA. *If* $u \in (r^m \to \boldsymbol{R})$, $1 \leqslant q < \infty$, $\hat{v} \in \mathrm{pu}_l^m$, *then for almost every* $X \in \boldsymbol{R}^m$

$$|[u, \hat{v}](X)|^q \leqslant [|u|^q, \hat{v}](X).$$

Proof. Let q' be the index conjugate to q. We apply (18.1) to obtain that $|[u, \hat{v}](X)|^q = |[u\hat{v}^{1/q}, \hat{v}^{1/q'}](X)|^q$ for almost every $X \in \boldsymbol{R}^m$. It follows from Hölder's inequality and (18.2) that

$$\begin{aligned} |[u\hat{v}^{1/q}, \hat{v}^{1/q'}](X)|^q &\leqslant \|u\hat{v}(\cdot, X)^{1/q}\|_q^q \cdot \|\hat{v}(\cdot, X)^{1/q'}\|_{q'}^q \\ &= [|u|^q, \hat{v}](X)[\hat{v}, \mathrm{fc}](X)^{q/q'} = [|u|^q, \hat{v}](X). \ \blacksquare \end{aligned}$$

(20.2) LEMMA. *If* $U \in L_q(\boldsymbol{R}^m)^{\mathrm{loc}}$, $1 \leqslant q < \infty$, $\hat{v} \in \mathrm{pu}_l^m$, *then*

$$\forall x \in r^m \quad |\langle U, \hat{v}(x)\rangle|^q \leqslant \langle |U|^q, \hat{v}(x)\rangle.$$

Proof. Let $x \in r^m$. As in the proof of the previous lemma, we use first (18.1), then Hölder's inequality and at last (18.3) to obtain

$$|\langle U, \hat{v}(x)\rangle|^q = |\langle U\hat{v}(x)^{1/q}, \hat{v}(x)^{1/q'}\rangle|^q \leqslant \|U\hat{v}(x)^{1/q}\|_q^q \|\hat{v}(x)^{1/q'}\|_{q'}^q$$
$$= \langle |U|^q, \hat{v}(x)\rangle \langle \hat{v}(x), \mathrm{FC}\rangle^{q/q'} = \langle |U|^q, \hat{v}(x)\rangle. \blacksquare$$

(20.3) LEMMA. *If* $u \in l_q(r^m)$ *and* $\hat{v} \in \mathrm{pu}^m_{a(q)}$, *where*

$$a(q) = \begin{cases} l & \text{if} \quad 1 \leqslant q \leqslant \infty, \\ 0 & \text{if} \quad q = (qx, qi, K), \end{cases}$$

then $[u, \hat{v}] \in L_q(\boldsymbol{R}^m)$ *and* $\|[u, \hat{v}]\|_q \leqslant \|u\|_q$.

Proof. Let, at first, $1 \leqslant q < \infty$. We have to prove that $[u, \hat{v}] \in L_q(\boldsymbol{R}^m)$. Since we obtain from Lemma (20.1) that $|[u, \hat{v}]| \leqslant [|u|^q, \hat{v}]^{1/q}$, it is sufficient to prove that $[|u|^q, \hat{v}]^{1/q} \in L_q(\boldsymbol{R}^m)$. The definition of the norm yields

$$\|[|u|^q, \hat{v}]^{1/q}\|_q^q = \langle [|u|^q, \hat{v}], \mathrm{FC}\rangle = \int_{\boldsymbol{R}^m} \mathrm{cv} \sum_{x \in r^m} |u(x)|^q \hat{v}(x, X)\, dX.$$

The integral of the infinite sum of nonnegative functions is equal to the sum of the integrals of these functions, therefore by applying (18.3) we obtain

$$\int_{\boldsymbol{R}^m} \mathrm{cv} \sum_{x \in r^m} |u(x)|^q \hat{v}(x, X)\, dX = \mathrm{cv} \sum_{x \in r^m} |u(x)|^q \langle \hat{v}(x), \mathrm{FC}\rangle = \|u\|_q^q.$$

Hence $\|[u, \hat{v}]\|_q \leqslant \|u\|_q$ if $1 \leqslant q < \infty$. If $q = \infty$ then following Hölder's inequality and (18.2) we have

$$\|[u, \hat{v}]\|_\infty = \operatorname{ess\,sup}\{|[u, \hat{v}](X)|: X \in \boldsymbol{R}^m\}$$
$$\leqslant \operatorname{ess\,sup}\{\|u\|_\infty \|\hat{v}(\cdot, X)\|_1: X \in \boldsymbol{R}^m\} = \|u\|_\infty.$$

If $q = (qx, qi, K)$ then following the definitions of the norm we have

$$\|[u, \hat{v}]\|_q = \|G\|_{qx}, \quad \text{where} \quad G(Y) = \|\mathrm{RV}_K^Y [u, \hat{v}]\|_{qi};$$
$$\|u\|_q = \|g\|_{qx}, \quad \text{where} \quad g(y) = \|\mathrm{rv}_K^y u\|_{qi}.$$

Since $\hat{v} \in \mathrm{pu}_0^m$, it can be represented as a product

$$\hat{v}(x, X) = \hat{v}_K(x_K, X_K)\hat{v}_{\setminus K}(x_{\setminus K}, X_{\setminus K}) = \prod_{i \in K} \hat{v}_i(x_i, X_i) \prod_{i \notin K} \hat{v}_i(x_i, X_i),$$

and following Fubini's theorem we have for $X \in \boldsymbol{R}^k$, $Y \in \boldsymbol{R}^{m-k}$ ($k = \mathrm{ne}\, K$)

$$[u, \hat{v}](X \oplus_K Y) = \mathrm{cv}_{\setminus K} \sum_{y \in r^{m-k}_{\setminus K}} [\mathrm{rv}_K^y u, \hat{v}_K](X)\hat{v}_{\setminus K}(y, Y).$$

Therefore

$$G(Y) = \Big\| \mathrm{cv}_{\setminus K} \sum_{\nu \in r^{m-K}_{\setminus K}} [\mathrm{rv}^{y}_{K} u,\ \hat{v}_K] \hat{v}_{\setminus K}(y, Y) \Big\|_{qi}$$

$$\leqslant \mathrm{cv}_{\setminus K} \sum_{\nu} \|[\mathrm{rv}^{y}_{K} u,\ \hat{v}_K]\|_{qi} \hat{v}_{\setminus K}(y, Y).$$

By applying the first part of the present lemma we obtain

$$\|[\mathrm{rv}^{y}_{K} u,\ \hat{v}_K]\|_{qi} \leqslant \|\mathrm{rv}^{y}_{K} u\|_{qi},$$

hence combining the last two inequalities we get

$$G(Y) \leqslant [g, \hat{v}_{\setminus K}](Y),$$

and by applying the first part of our lemma to the function g we finally obtain

$$\|G\|_{qx} \leqslant \|g\|_{qx}, \quad \text{that is,} \quad \|[u, \hat{v}]\|_q \leqslant \|u\|_q,$$

hence the lemma is proved. ∎

Let us note that if we choose as the function $\hat{v}$ occurring in Lemma (20.3) the function $\hat{c}$ defined in (18.7), then we obtain the following result, which can be proved by the direct computation of the norms.

(20.4) *Remark.* If $u \in l_q(r^m)$, then $[u, \hat{c}] \in L_q(\boldsymbol{R}^m)$ and $\|[u, \hat{c}]\|_q = \|u\|_q$.

We also have the following analogue of Lemma (20.3).

(20.5) Lemma. *If* $U \in L_q(\boldsymbol{R}^m)$ *and* $\hat{v} \in \mathrm{pu}^m_{a(q)}$, *then*

$$\langle U, \hat{v}\rangle \in l_q(r^m) \quad \text{and} \quad \|\langle U, \hat{v}\rangle\|_q \leqslant \|U\|_q.$$

The proof is similar to that of Lemma (20.3).

(20.6) Lemma. *If* $F \in L_q(\boldsymbol{R}^m)$, $f \in l_{q'}(r^m)$, $\hat{v} \in \mathrm{pu}^m_{a(q)}$, *then*

$$[f, \langle \hat{v}, F\rangle] = \langle [f, \hat{v}], F\rangle.$$

Proof. According to Lemmas (20.3), (20.5) $\langle \hat{v}, F\rangle$, $\langle \hat{v}, |F|\rangle \in l_q(r^m)$ and $[f, \hat{v}]$, $[|f|, \hat{v}] \in L_{q'}(\boldsymbol{R}^m)$, hence $[f, \langle \hat{v}, F\rangle]$ and $\langle [f, \hat{v}], F\rangle$ are well defined. Moreover, for every finite subset b of r^m and almost every $X \in \boldsymbol{R}^m$ we have the estimation

$$\Big|\mathrm{cv} \sum_{x \in b} f(x) \hat{v}(x, X) F(X)\Big| \leqslant [|f|, \hat{v}](X) |F(X)|,$$

therefore by applying (for every fixed $h \in H$) the majorized convergence theorem (see, e.g. [4]) we obtain the required result. ∎

(21) Lemma. *Let us define for* $\hat{v}$, $\hat{w} \in \mathrm{pu}^m_l$ *the maximal vector of the pair* $\{\hat{v}, \hat{w}\}$ *as*

$$\mathrm{MX}(\hat{v}, \hat{w}) = \mathrm{UB}(\{\mathrm{UP}\,\hat{v} - \mathrm{LW}\,\hat{w},\ \mathrm{UP}\,\hat{w} - \mathrm{LW}\,\hat{v}\}),$$

and the volume of $\{\hat{v}, \hat{w}\}$ *as the number*

$$\mathrm{VM}(\hat{v}, \hat{w}) = (\mathrm{UP}\,\hat{v} - \mathrm{LW}\,\hat{v} + \mathrm{UP}\,\hat{w} - \mathrm{LW}\,\hat{w})^{e}.$$

Assume that $U \in L_q(\boldsymbol{R}^m)$ *and* $\hat{v}, \hat{w} \in \mathrm{pu}^m_{a(q)}$. *Then* $\langle \hat{w}, U\rangle \in l_q(r^m)$, $[\langle \hat{w}, U\rangle, \hat{v}] \in L_q(\boldsymbol{R}^m)$, *and the following inequality holds:*

$$\|U - [\langle \hat{w}, U\rangle, \hat{v}]\|_q \leqslant \mathrm{CT}(q, \hat{v}, \hat{w})\mathrm{MC}_q\big(U;\ \mathrm{MX}(\hat{v}, \hat{w})\circ \mathrm{mw}\big),$$

where

$$\mathrm{CT}(q, \hat{v}, \hat{w}) = \begin{cases} \mathrm{VM}(\hat{v}, \hat{w})^{1/q} & \text{if } 1 \leqslant q \leqslant \infty, \\ \big(\mathrm{VM}(\hat{v}_K, \hat{w}_K)^{1/qi} + \mathrm{VM}(\hat{v}_{\setminus K}, \hat{w}_{\setminus K})^{1/qx}\big) & \text{if } q = (qx, qi, K), \end{cases}$$

and

$$\hat{v}_K(x_K, X_K) = \prod_{i\in K} \hat{v}_i(x_i, X_i), \qquad \hat{v}_{\setminus K}(x_{\setminus K}, X_{\setminus K}) = \prod_{i\notin K} \hat{v}_i(x_i, X_i).$$

Proof. Equality (18.2) yields for almost every $X \in \boldsymbol{R}^m$

$$U(X) = U(X)[\hat{v}(\cdot, X), \mathrm{fc}] = [\hat{v}(\cdot, X),\ U(X)\mathrm{fc}].$$

Further, from (18.3) we obtain

$$U(X)\mathrm{fc} = \langle \hat{w}, \mathrm{FC}\rangle U(X) = \langle \hat{w}, U(X)\mathrm{FC}\rangle.$$

Hence

(21.1) $$\begin{aligned} &U(X) - [\langle \hat{w}, U\rangle, \hat{v}](X) \\ &= [\langle \hat{w}, U(X)\mathrm{FC}\rangle,\ \hat{v}(\cdot, X)] - [\langle \hat{w}, U\rangle,\ \hat{v}(\cdot, X)] \\ &= [\langle \hat{w}, U(X)\mathrm{FC} - U\rangle,\ \hat{v}(\cdot, X)]. \end{aligned}$$

Let now $1 \leqslant q < \infty$ and let $d = \|U - [\langle \hat{w}, U\rangle, \hat{v}]\|_q^q$. Applying equality (21.1) and Lemmas (20.1), (20.2) we obtain

$$\begin{aligned} d &= \int_{\boldsymbol{R}^m} |[\langle \hat{w}, U(X)\mathrm{FC} - U\rangle,\ \hat{v}(\cdot, X)]|^q dX \\ &\leqslant \int_{\boldsymbol{R}^m} [|\langle \hat{w}, U(X)\mathrm{FC} - U\rangle|^q,\ \hat{v}(\cdot, X)]\, dX \\ &\leqslant \int_{\boldsymbol{R}^m} [\langle \hat{w},\ |U(X)\mathrm{FC} - U|^q\rangle,\ \hat{v}(\cdot, X)]\, dX, \end{aligned}$$

that is,

$$\begin{aligned} d &\leqslant \int_{\boldsymbol{R}^m} \mathrm{cv} \sum_{x\in r^m} \int_{\boldsymbol{R}^m} \hat{w}(x, Y)\,|U(X) - U(Y)|^q dY \hat{v}(x, X)\, dX \\ &= \mathrm{cv} \sum_{x\in r^m} \int_{\boldsymbol{R}^m} \int_{\boldsymbol{R}^m} \hat{w}(x, Y)\,|U(X) - U(Y)|^q \hat{v}(x, X)\, dY\, dX. \end{aligned}$$

It follows from (18.4) that the function under the integral vanishes out of the set $\{(X, Y)\colon X \in \mathrm{NB}(x, \mathrm{ST}\hat{v}),\ Y \in \mathrm{NB}(x, \mathrm{ST}\hat{w})\}$. Let us introduce new integration variable $Z = Y - X$. It suffices to consider Z belonging to the set $\mathrm{NB}(x, \mathrm{ST}\hat{w}) - \mathrm{NB}(x, \mathrm{ST}\hat{v})$, which is contained in the set

$$\mathrm{ND}[\mathrm{mw}] = \{Z \in \boldsymbol{R}^m\colon (\mathrm{LW}\hat{w} - \mathrm{UP}\hat{v}) \circ \mathrm{mw} \leqslant Z \leqslant (\mathrm{UP}\hat{w} - \mathrm{LW}\hat{v}) \circ \mathrm{mw}\},$$

because of (18.9). It follows from the definition of MX and VM that

$$(21.2)\quad \mathrm{ND}[\mathrm{mw}] \subset \{Z \in \boldsymbol{R}^m\colon -\mathrm{MX}(\hat{v}, \hat{w}) \circ \mathrm{mw} \leqslant Z \leqslant \mathrm{MX}(\hat{v}, \hat{w}) \circ \mathrm{mw}\},$$

(21.3) the volume of $\mathrm{ND}[\mathrm{mw}]$ is equal to $\mathrm{VM}(\hat{v}, \hat{w})\mathrm{cv}$.

Hence we obtain

$$d \leqslant \mathrm{cv} \sum_{x \in r^m} \int_{\boldsymbol{R}^m} \Big\{ \int_{\mathrm{ND}[\mathrm{mw}]} \hat{w}(x, X+Z)\,|U(X) - U(X+Z)|^q \hat{v}(x, X)\,dZ \Big\} dX$$

$$= \int_{\mathrm{ND}[\mathrm{mw}]} \Big\{ \int_{\boldsymbol{R}^m} \mathrm{cv} \sum_{x \in r^m} \hat{v}(x, X)\,|U(X) - U(X+Z)|^q \hat{w}(x, X+Z)\,dX \Big\} dZ.$$

From (18.1,2) we have $0 \leqslant \hat{w}(x, X+Z) \leqslant \mathrm{cv}^{-1}$ and $\mathrm{cv}\sum \hat{v}(x, X) = 1$, therefore

$$d \leqslant \mathrm{cv}^{-1} \int_{\mathrm{ND}[\mathrm{mw}]} \int_{\boldsymbol{R}^m} |U(X) - U(X+Z)|^q\,dX\,dZ.$$

Taking into account (21.2,3), we obtain

$$d \leqslant \mathrm{cv}^{-1} \int_{\mathrm{ND}[\mathrm{mw}]} \mathrm{MC}_q\big(U; \mathrm{MX}(\hat{v}, \hat{w}) \circ \mathrm{mw}\big)^q dZ$$

$$\leqslant \mathrm{VM}(\hat{v}, \hat{w})\mathrm{MC}_q\big(U; \mathrm{MX}(\hat{v}, \hat{w}) \circ \mathrm{mw}\big)^q,$$

what concludes the proof in the case $q < \infty$.

If $q = \infty$ then (21.1) and (18) yield for almost every $X \in \boldsymbol{R}^m$

$$|U(X) - [\langle \hat{w}, U\rangle, \hat{v}](X)| = |[\langle \hat{w}, U(X)\mathrm{FC} - U\rangle, \hat{v}(\cdot, X)]|$$

$$\leqslant [\langle \hat{w}, |U(X)\mathrm{FC} - U|\rangle, \hat{v}(\cdot, X)]$$

$$= \mathrm{cv} \sum_{x \in r^m} \int_{\mathrm{NB}(x, \mathrm{ST}\hat{w})} \hat{w}(x, Y)\,|U(X) - U(Y)|\,dY\,\hat{v}(x, X)$$

$$\leqslant \mathrm{cv} \sum_{x \in r^m} \int_{\mathrm{NB}(x, \mathrm{ST}\hat{w})} \hat{w}(x, Y)\mathrm{MC}_\infty\big(U;\ \mathrm{MX}(\hat{v}, \hat{w}) \circ \mathrm{mw}\big)\,dY\,\hat{v}(x, X)$$

$$= \mathrm{MC}_\infty\big(U;\ \mathrm{MX}(\hat{v}, \hat{w}) \circ \mathrm{mw}\big).$$

Let us now consider the case $q = (qx, qi, K)$. According to the definition of the norm, we have to estimate the function

$$G(Y) = \|\mathrm{RV}_K^Y(U - [\langle\hat{w}, U\rangle, \hat{v}])\|_{qi}.$$

By using Fubini's theorem, similarly as in (20.3), (20.5), we obtain

$$(21.4)\quad G(Y) = \Big\|\Big[\int \hat{w}_{\setminus K}(\cdot, Z)\{\mathrm{RV}_K^Y U - [\langle\hat{w}_K, \mathrm{RV}_K^Z U\rangle, \hat{v}_K]\}dZ, \hat{v}_{\setminus K}\Big](Y)\Big\|_{qi}$$

$$\leqslant \Big[\int \hat{w}_{\setminus K}(\cdot, Z)\|\mathrm{RV}_K^Y U - \mathrm{RV}_K^Z U\|_{qi} dZ, \hat{v}_{\setminus K}\Big](Y) +$$

$$+\Big[\int \hat{w}_{\setminus K}(\cdot, Z)\|\mathrm{RV}_K^Z U - [\langle\hat{w}_K, \mathrm{RV}_K^Z U\rangle, \hat{v}_K]\|_{qi} dZ, \hat{v}_{\setminus K}\Big](Y)$$

$$\leqslant \Big[\int \hat{w}_{\setminus K}(\cdot, Z)\|\mathrm{RV}_K^Y U - \mathrm{RV}_K^Z U\|_{qi} dZ, \hat{v}_{\setminus K}\Big](Y) +$$

$$+[\langle\hat{w}_{\setminus K}, G_M\rangle, \hat{v}_{\setminus K}](Y),$$

where $G_M(Z) = \mathrm{VM}(\hat{v}_K, \hat{w}_K)^{1/qi}\mathrm{MC}_{qi}\big(\mathrm{RV}_K^Z U; \mathrm{MX}(\hat{v}_K, \hat{w}_K)\circ \mathrm{mw}_K\big)$ for $Z \in \boldsymbol{R}^{m-\mathrm{ne}K}$; the last inequality can be obtained by applying the first part of our lemma to the function $\mathrm{RV}_K^Z U$.

Let us now estimate $\|G\|_{qx}$. The norm of the first component of the right-hand side of (21.4) can be estimated as in the first part of the proof of the present lemma, and the second part — by applying Lemmas (20.3), (20.5). Finally, we obtain the inequality

$$\|G\|_{qx} \leqslant \mathrm{VM}(\hat{v}_{\setminus K}, \hat{w}_{\setminus K})^{1/qx}\mathrm{MC}_q\big(U; \mathrm{mw}\circ\big(\mathrm{MX}(\hat{v}_{\setminus K}, \hat{w}_{\setminus K})\oplus_{\setminus K} 0\big)\big) +$$
$$+\mathrm{VM}(\hat{v}_K, \hat{w}_K)^{1/qi}\mathrm{MC}_q\big(U; \mathrm{mw}\circ\big(\mathrm{MX}(\hat{v}_K, \hat{w}_K)\oplus_K 0\big)\big),$$

from which we obtain the required result. ∎

(22.1) COROLLARY. *If* $B \subset \boldsymbol{R}^m$, $F \in L_q(B)$, $\hat{v}, \hat{w} \in \mathrm{pu}^m_{a(q)}$, $b = \mathrm{sb}(B, \mathrm{ST}\hat{w})$, $B^-[\mathrm{mw}] = \mathrm{SB}(b, \mathrm{ST}\hat{v})$, *then*

$$\langle\hat{w}, F\rangle \in l_q(r^m), \quad [\langle\hat{w}, F\rangle, \hat{v}] \in L_q(\boldsymbol{R}^m)$$

and

$$\|F - [\langle\hat{w}, F\rangle, \hat{v}]\|_q^{(B^-[\mathrm{mw}])} \leqslant \mathrm{CT}(q, \hat{v}, \hat{w})\mathrm{MC}_q\big(F; \mathrm{MX}(\hat{v}, \hat{w})\circ\mathrm{mw}\big),$$

where the notation is the same as in Lemma (21).

Proof. According to the definition of the scalar product, $\langle\hat{w}, F\rangle = \langle\hat{w}, \mathrm{EX}_m F\rangle$. Hence we can apply Lemma (21) with U replaced by $\mathrm{EX}_m F$. Further, following definitions (12.2,5), if $x \in b$ then $\mathrm{SUPP}\hat{w}(x) \subset B$, if $X \in B^-[\mathrm{mw}]$ then $\mathrm{supp}\big(\hat{v}(\cdot, X)\big) \subset b$. Hence the required inequalities can be obtained similarly as those from Lemma (21). ∎

Let us now consider a function $F \in L_q(\boldsymbol{R}^m)$ and $\hat{v} \in \mathrm{pc}^m_{a(q)}$. Following formula (19.2) and Lemma (20.5), $\langle F, \hat{v}\rangle \in l_q(r^m)$ and

$$(22.2)\qquad \|\langle F, \hat{v}\rangle\|_q \leqslant \sum_{i=1}^{kp} t_i\|\langle F, \hat{w}^i\rangle\|_q + \sum_{i=1}^{kd} u_i\|\langle F, \hat{y}^i - \hat{z}^i\rangle\|_q.$$

Remark (20.4) yields that

$$(22.3)\qquad \|\langle F, \hat{y}^i - \hat{z}^i\rangle\|_q = \|[\langle F, \hat{y}^i - \hat{z}^i\rangle, \hat{c}\,]\|_q \leqslant \|[\langle F, \hat{y}^i\rangle, \hat{c}\,] - F\|_q + \|F - [\langle F, \hat{z}^i\rangle, \hat{c}\,]\|_q.$$

If we apply Lemma (20.5) to the first sum on the right-hand side of (22.2), Lemma (21) to the right-hand side of formula (22.3), and substitute (22.3) into (22.2), we obtain the inequality

$$\|\langle F, \hat{v}\rangle\|_q \leqslant \|F\|_q + \sum_{i=1}^{kd} u_i\big\{\mathrm{CT}(q, \hat{c}, \hat{y}^i)\mathrm{MC}_q\big(F; \mathrm{MX}(\hat{c}, \hat{y}^i)\circ \mathrm{mw}\big) + \mathrm{CT}(q, \hat{c}, \hat{z}^i)\mathrm{MC}_q\big(F; \mathrm{MX}(\hat{c}, \hat{z}^i)\circ \mathrm{mw}\big)\big\}.$$

Hence we have proved the following corollary to Lemma (20.5).

(22.4) COROLLARY. *If* $F \in L_q(\boldsymbol{R}^m)$, $\hat{v} \in \mathrm{pc}^m_{a(q)}$, *then* $\langle F, \hat{v}\rangle \in l_q(r^m)$ *and*

$$\|\langle F, \hat{v}\rangle\|_q \leqslant \|F\|_q + \big(\mathrm{md}_a(\hat{v}) - 1\big)\mathrm{CT}(q, \hat{v})\mathrm{MC}_q\big(F; \mathrm{MX}(\hat{v})\circ \mathrm{mw}\big),$$

where

$$\mathrm{CT}(q, \hat{v}) = \max\{\mathrm{CT}(q, \hat{c}, \hat{v}^i) \colon 1 \leqslant i \leqslant k\},$$
$$\mathrm{MX}(\hat{v}) = \mathrm{UB}\big(\{\mathrm{MX}(\hat{c}, \hat{v}^i) \colon 1 \leqslant i \leqslant k\}\big),$$

and $\hat{v}^i$ *are the components of* $\hat{v}$ *occurring in Definition* (19.1).

With the aid of the above lemmas we prove the theorem concerning the approximation of $L_q(B)$.

(23) THEOREM. *Assume that* $B \subset \boldsymbol{R}^m$ *and* q *is a finite index* (*that is, either* $1 \leqslant q < \infty$ *or* $q = (qx, qi, K)$ *and* $1 \leqslant qx < \infty$, $1 \leqslant qi < \infty$). *Let* $\hat{v}, \hat{w} \in \mathrm{pc}^m_{a(q)}$ *and let there exist a constant* MD (*independent of the mesh width*) *such that* $\mathrm{md}_{a(q)}(\hat{v}) \leqslant \mathrm{MD}$, $\mathrm{md}_{a(q)}(\hat{w}) \leqslant \mathrm{MD}$. *Let* $b = \mathrm{sb}(B, \mathrm{ST}\hat{w})$, $B^-[\mathrm{mw}] = \mathrm{SB}(b, \mathrm{ST}\hat{v})$. *If we define the operators* $\mathrm{rs}_e \colon L_q(B) \to l_q(r^m)$, $\mathrm{PR}_e \colon l_q(r^m) \to L_q(B)$, *and* $\mathrm{rs}_i \colon L_q(B) \to l_q(b)$, $\mathrm{PR}_i \colon l_q(b) \to L_q(B)$, *by the formulas*

$$\mathrm{rs}_e F = \langle \hat{w}, F\rangle, \qquad \mathrm{PR}_e f = \mathrm{RD}_B[\hat{v}, f],$$
$$\mathrm{rs}_i F = \mathrm{rd}_b\langle \hat{w}, F\rangle, \qquad \mathrm{PR}_i f = \mathrm{RD}_B[\hat{v}, f],$$

then the families of triples $\mathrm{Ap}_e(q, B, \hat{v}, \hat{w}) = \big\{\big(l_q(r^m)_h, \mathrm{rs}_{eh}, \mathrm{PR}_{eh}\big)\big\}_{h\in H}$ *and* $\mathrm{Ap}_i(q, B, \hat{v}, \hat{w}) = \big\{\big(l_q(b)_h, \mathrm{rs}_{ih}, \mathrm{PR}_{ih}\big)\big\}_{h\in H}$ *are internal approximations of* $L_q(B)$. *Moreover, we have the following estimates*

$$\|\mathrm{rs}_j F\|_q \leqslant \mathrm{MD}\,\|F\|_q, \qquad \|\mathrm{PR}_j f\|_q \leqslant \mathrm{MD}\,\|f\|_q \quad \textit{for} \quad j = e, i,$$
$$\|\mathrm{rs}_e F\|_q \leqslant \|F\|_q + (\mathrm{MD} - 1)\,\mathrm{CT}(q, \hat{w})\mathrm{MC}_q\big(\mathrm{EX}_m F; \mathrm{MX}(\hat{w})\circ \mathrm{mw}\big),$$
$$\|\mathrm{rs}_i F\|_q \leqslant \|F\|_q + (\mathrm{MD} - 1)\,\mathrm{CT}(q, \hat{w})\mathrm{MC}_q\big(F; \mathrm{MX}(\hat{w})\circ \mathrm{mw}\big),$$

$$\|F-\mathrm{PR}_e\mathrm{rs}_eF\|_q \leqslant \mathrm{MD}^2\mathrm{CT}(q, \hat{v}, \hat{w})\mathrm{MC}_q(\mathrm{EX}_mF; \mathrm{MX}(\hat{v}, \hat{w})\circ \mathrm{mw}),$$

$$\|F-\mathrm{PR}_i\mathrm{rs}_iF\|_q^{(B-[\mathrm{mw}])} \leqslant \mathrm{MD}^2\mathrm{CT}(q, \hat{v}, \hat{w})\mathrm{MC}_q(F; \mathrm{MX}(\hat{v}, \hat{w})\circ \mathrm{mw}).$$

The symbol $a(q)$ has been defined in Lemma (20.3), *the constants* CT — *in Lemma* (21) *and Corollary* (22.4). *Moreover,* $\mathrm{Ap}_i(q, B, \hat{c}, \hat{c})$ *is reflexive and* $\|\mathrm{PR}_if\|_q = \|f\|_q$ *for each* $f \in l_q(b)$.

Proof. According to Definition (13.5), $\langle\hat{w}, F\rangle = \langle\hat{w}, \mathrm{EX}_mF\rangle$, hence the fact that the operators introduced above are well defined and that the estimations are true follows from Lemmas (20.3,5), (21) and Corollaries (22.1,4). Therefore if we define for each $h \in H$ either

$$U = V = L_q(B), \quad \|F\|_h^V = \|F\|_q,$$

or

$$U = V = L_q(B), \quad \|F\|_h^V = \|\mathrm{RD}_{B-[h]}F\|_q,$$

then the conditions (2.1–3) in the definition of the approximation are fulfilled for corresponding triples Ap_e and Ap_i. The last statement of the theorem directly follows from Definition (18.7) and Remark (20.4). ∎

An immediate consequence of the theorem formulated above is the following theorem concerning the approximation of local spaces.

(24) THEOREM. *Suppose that the assumptions of Theorem* (23) *are satisfied. If we define the operators*

$$\mathrm{rs}_e\colon L_q(B)^{\mathrm{loc}}\to l_q(r^m)^{\mathrm{loc}}, \quad \mathrm{PR}_e\colon l_q(r^m)^{\mathrm{loc}}\to L_q(B)^{\mathrm{loc}},$$

$$\mathrm{rs}_i\colon L_q(B)^{\mathrm{loc}}\to l_q(b)^{\mathrm{loc}}, \quad \mathrm{PR}_i\colon l_q(b)^{\mathrm{loc}}\to L_q(B)^{\mathrm{loc}},$$

by the same formulas as in Theorem (23) *then*

$$\mathrm{Ap}_e^{\mathrm{loc}}(q, B, \hat{v}, \hat{w}) = \{(l_q(r^m)_h^{\mathrm{loc}}, \mathrm{rs}_{eh}, \mathrm{PR}_{eh})\}_{h\in H}$$

and

$$\mathrm{Ap}_i^{\mathrm{loc}}(q, B, \hat{v}, \hat{w}) = \{(l_q(b)_h^{\mathrm{loc}}, \mathrm{rs}_{ih}, \mathrm{PR}_{ih})\}_{h\in H}$$

are internal approximations of $L_q(B)^{\mathrm{loc}}$, *and the following estimations are valid:*

$$\|\mathrm{rs}_eF\|_q^{(a)} \leqslant \mathrm{MD}\,\|\mathrm{EX}_mF\|_q^{(\mathrm{NB}(a,\mathrm{ST}\hat{w}))}, \quad \|\mathrm{PR}_ef\|_q^{(A)} \leqslant \mathrm{MD}\,\|f\|_q^{(\mathrm{nb}(A,\mathrm{ST}\hat{v}))},$$

$$\|F-\mathrm{PR}_e\mathrm{rs}_eF\|_q^{(A)}$$

$$\leqslant \mathrm{MD}^2\mathrm{CT}(q, \hat{v}, \hat{w})\,\mathrm{MC}_q^{(\mathrm{NB}(\mathrm{nb}(A,\mathrm{ST}\hat{v}),\mathrm{ST}\hat{w}))}(\mathrm{EX}_mF; \mathrm{MX}(\hat{v}, \hat{w})\circ\mathrm{mw})$$

for every regular bounded $A \subset B$ *and every bounded* $a \subset r^m$;

$$\|\mathrm{rs}_iF\|_q^{(\mathrm{sb}(A,\mathrm{ST}\hat{w}))} \leqslant \mathrm{MD}\,\|F\|_q^{(A)}, \quad \|\mathrm{PR}_if\|_q^{(\mathrm{SB}(a,\mathrm{ST}\hat{v}))} \leqslant \mathrm{MD}\,\|f\|_q^{(a)},$$

$$\|F-\mathrm{PR}_i\mathrm{rs}_iF\|_q^{(\mathrm{SB}(\mathrm{sb}(A,\mathrm{ST}\hat{w})\mathrm{ST}\hat{v}))} \leqslant \mathrm{MD}^2\mathrm{CT}(q, \hat{v}, \hat{w})\mathrm{MC}_q^{(A)}(F; \mathrm{MX}(\hat{v}, \hat{w})\circ\mathrm{mw})$$

for every regular bounded $A \subset B$ *and every bounded* $a \subset b$.

Proof. We put in Definition (2) either (for Ap_e)

$$U = V = L_q(B)^{\mathrm{loc}}, \quad \|F\|_A = \|\mathrm{RD}_A F\|_q, \quad \forall h \in H \ \ \|F\|^V_{h,A} = \|F\|_A,$$

or (for Ap_i)

$$U \doteq V = L_q(B)^{\mathrm{loc}}, \quad \|F\|_A = \|\mathrm{RD}_A F\|_q,$$

$$\forall h \in H \quad \|F\|^V_{h,A} = \|\mathrm{RD}_{A \cap B - [h]} F\|_q,$$

where A runs over all regular bounded subsets of B. The applying of Theorem (23) allows us to prove Theorem (24). ∎

(24.1) *Remark.* Theorems (23) and (24) remain true if we replace $l_q(r^m)$ and $l_q(r^m)^{\mathrm{loc}}$ by $l_q(b^+)$ and $l_q(b^+)^{\mathrm{loc}}$, respectively, where

$$b^+ = \mathrm{nb}(B, \mathrm{ST}\hat{w}).$$

This follows from the fact that $\mathrm{supp}\langle \hat{w}, F\rangle \subset b^+$. ∎

The same partition of unity can also be used to approximate the space $C^0(B)$. To obtain an internal approximation, however, we have to assume according to Definition (2) that the function $\hat{v}$ used for building the prolongation operator is taken from $\mathrm{pc}^m_l \cap (r^m \to C^0(R^m))$, that is, every $\hat{v}(x)$ is continuous.

The more detail investigation of this problem will be performed in the next section.

5. Approximation of the spaces $W^p_q(B)$ and $W^{-p}_q(B)$

To approximate the spaces $W^p_q(B)$, $W^{-p}_q(B)$ we need functions of higher regularity than pu^m_l. Let us first define smoothing operators sm_K by the formula

(25.1) $$\text{if } \hat{v} \in \mathrm{pu}^m_l,\ K \subset \overline{m}, \text{ then } \mathrm{sm}_K \hat{v} = \partial^*_K \mathrm{IN}_K \hat{v}.$$

The following lemma can be proved.

(26) Lemma. *If* $\hat{v} \in \mathrm{pu}^m_l$, $K \subset \overline{m}$, *then* $\mathrm{sm}_K\hat{v} \in \big(r^m \to W^{\overset{\prime}{e}K''}(\boldsymbol{R}^m)^{\mathrm{loc}}\big)$ *and* $D_K \mathrm{sm}_K \hat{v} = \partial^*_K \hat{v}$. *Moreover,* $\mathrm{sm}_K\hat{v}$ *satisfies conditions* (18.1–5), *that is,* $\mathrm{sm}_K\hat{v} \in \mathrm{pu}^m_l$. *The stencil of* $\mathrm{sm}_K\hat{v}$ *is equal to* $\mathrm{ST}(\mathrm{sm}_K\hat{v}) = \mathrm{ST}\hat{v} - \overset{\prime}{e}^{\,\overset{\prime}{e}K''}$ (the symbol k'' was introduced in (6.3)). *If, moreover,* $\hat{v} \in \mathrm{pu}^m_0$, *then* $\mathrm{sm}_K\hat{v} \in \mathrm{pu}^m_0$.

Proof. Let us start from the case where K is a one-element set, for instance, $K = \{k\}$ (for simplifying we will write k instead of $\{k\}$). From the definition of sm_K and assumption (18.5) for $\hat{v}$ we obtain

$$\mathrm{sm}_k\hat{v} = \partial^*_k \mathrm{IN}_k\hat{v} = \mathrm{mw}_k^{-1}(\mathrm{ts}_k^{-1}\mathrm{IN}_k\hat{v} - \mathrm{IN}_k\hat{v}) = \mathrm{mw}_k^{-1}\mathrm{DF}^{\overset{\prime}{e}_k}_{\mathrm{mw}}\mathrm{IN}_k\hat{v}.$$

Similarly as in the proof of Lemma (10.6), we can show that

$$(26.1)\quad \mathrm{sm}_k\hat{v}(\cdot, X) = \mathrm{mw}_k^{-1}\langle \hat{v}, \mathrm{FS}_k^{X_k, X_k+\mathrm{mw}_k}\rangle_k (X_{\setminus k}) \quad \text{at almost every } X \in \boldsymbol{R}^m.$$

Hence $\mathrm{sm}_k\hat{v}$ satisfies (18.1). To prove (18.2) let us observe that

$$\begin{aligned}[\mathrm{sm}_k\hat{v}, \mathrm{fc}](X) &= \mathrm{mw}_k^{-1}\langle [\hat{v}, \mathrm{fc}], \mathrm{FS}_k^{X_k, X_k+\mathrm{mw}_k}\rangle_k (X_{\setminus k}) \\ &= \mathrm{mw}_k^{-1}\langle \mathrm{FC}, \mathrm{FS}_k^{X_k, X_k+\mathrm{mw}_k}\rangle_k (X_{\setminus k}) = 1,\end{aligned}$$

what follows from (26.1), assumption (18.2) for $\hat{v}$ and the theorem on the integral of a series of nonnegative functions. The proof of property (18.3) can be carried out similarly as the second part of the proof of Lemma (10.6) with $q = 1$, using formula (26.1) and properties (18.1,3). We obtain at each $x \in r^m$

$$\langle \mathrm{sm}_k\hat{v}(x), \mathrm{FC}\rangle = \|\mathrm{sm}_k\hat{v}(x)\|_1 = \mathrm{mw}_k\|\mathrm{mw}_k^{-1}\hat{v}(x)\|_1 = 1.$$

Now, if $X \notin \mathrm{NB}(x, \mathrm{ST}\hat{v} - e'_{k''})$ then for every s satisfying $0 \leqslant s \leqslant \mathrm{mw}_k$, the point $X + s\cdot e'_k$ does not belong to $\mathrm{NB}(x, \mathrm{ST}\hat{v})$. Hence by applying formula (26.1) and assumption (18.4) for $\hat{v}$, we obtain that $\mathrm{sm}_k\hat{v}(x, X) = 0$, what was to be shown. Property (18.5) of $\mathrm{sm}_k\hat{v}$ can be proved by applying (18.5) for $\hat{v}$ and property (9.11): for every $i \in I^m$

$$\mathrm{sm}_k\hat{v} = \partial_k^*\mathrm{IN}_k\hat{v} = \partial_k^*\mathrm{IN}_k(\mathrm{TS}^{i\circ\mathrm{mw}}\mathrm{ts}^i\hat{v}) = \mathrm{TS}^{i\circ\mathrm{mw}}\mathrm{ts}^i\partial_k\mathrm{IN}_k\hat{v} = \mathrm{TS}^{i\circ\mathrm{mw}}\mathrm{ts}^i\mathrm{sm}_k\hat{v}.$$

Therefore $\mathrm{sm}_k\hat{v} \in \mathrm{pu}_l^m$. By using Fubini's theorem it can be shown that

$$\mathrm{sm}_k(\mathrm{sm}_i\hat{v}) = \mathrm{sm}_i(\mathrm{sm}_k\hat{v}) = \mathrm{sm}_{\{i,k\}}\hat{v} \quad \text{for} \quad i, k \in \overline{m},\ i \neq k.$$

Hence $\mathrm{sm}_K\hat{v} \in \mathrm{pu}_l^m$ for every $K \subset \overline{m}$ and by applying Lemma (9.10) we obtain that $\mathrm{sm}_K\hat{v} \in \left(r^m \to W^{e'_{K''}}(\boldsymbol{R}^m)^{\mathrm{loc}}\right)$ and $D_K\mathrm{sm}_K\hat{v} = \partial_K^*\hat{v}$.

The last statement of the lemma can be proved by using Fubini's theorem. ■

Let us now define operators $\mathrm{sm}^k\colon \mathrm{pc}_l^m \to \mathrm{pc}_l^m$ in the following recurrent way.

(27.1) DEFINITION. If $\hat{v} \in \mathrm{pu}_l^m$ then $\mathrm{sm}^0\hat{v} = \hat{v}$;

if $\hat{v} \in \mathrm{pu}_l^m$, $k \in I^m$, $k \geqslant 0$, $K \subset \overline{m}$, then $\mathrm{sm}^{k+e'_K}\hat{v} = \mathrm{sm}_K(\mathrm{sm}^k\hat{v})$;

if $\hat{v} = \sum_{i=1}^{l} s_i\hat{v}^i$, $\hat{v}^i \in \mathrm{pu}_l^m$, $k \in I^m$, $k \geqslant 0$, then $\mathrm{sm}^k\hat{v} = \sum_{i=1}^{l} s_i\mathrm{sm}^k\hat{v}$.

The following lemma can be obtained from Lemma (26).

(27.2) LEMMA. *If* $\hat{v} \in \mathrm{pu}_l^m$, $k \in I^m$, $k \geqslant 0$, *then* $\mathrm{sm}^k\hat{v} \in \left(r^m \to W^{k''}(\boldsymbol{R}^m)^{\mathrm{loc}}\right)$ *and* $D^l\mathrm{sm}^k\hat{v} = \partial^{*l}\mathrm{sm}^{k-l}\hat{v}$ *for every* $l \in I^m$ *such that* $0 \leqslant l \leqslant k$; *moreover,* $\mathrm{sm}^k\hat{v}$ *satisfies conditions* (18.1–5) (*with* $\mathrm{ST}(\mathrm{sm}^k\hat{v}) = \mathrm{ST}\hat{v} - k''$), *that is,* $\mathrm{sm}^k\hat{v} \in \mathrm{pu}_l^m$. *If* $\hat{v} \in \mathrm{pc}_l^m$ *then* $\mathrm{sm}^k\hat{v} \in \mathrm{pc}_l^m$.

It follows from Lemmas (27.2) and (17.5) that

(27.3) $\mathrm{sm}^{k-e'_K}\hat{v} = D_K(\mathrm{in}_K^*\mathrm{sm}^k\hat{v})$ for every $\hat{v} \in \mathrm{pu}_l^m$, $K \subset \overline{m}$, $k \geqslant e'_K$.

Formula (27.3) will be used for constructing operators sm^k when k is not a nonnegative vector; in this case, however, sm^k is defined on a subset of pc_l^m.

Let us first introduce the following definition.

(27.4) DEFINITION. If $K \subset \overline{m}$, $\hat{v} \in \mathrm{pc}_0^m$, then the operator sm_K^- is defined by

$$\mathrm{sm}_K^-\hat{v} = D_K\mathrm{in}_K^*\hat{v}.$$

Let us investigate sm_K^-. The following identity is true.

$$\mathrm{sm}_K^-\hat{v} = \sum_{M\subset K}\sum_{L\subset M}(-1)^{\mathrm{ne}\,M+\mathrm{ne}\,L}\mathrm{sm}_L^-\hat{v}.$$

Moreover, Definition (27.4) together with formula (27.3) yield for $L \subset M$

$$\mathrm{sm}_L^-\hat{v} = D_L\mathrm{in}_L^*(D_{M\setminus L}\mathrm{in}_{M\setminus L}^*\mathrm{sm}_{M\setminus L}\hat{v}) = D_M\mathrm{in}_M^*\mathrm{sm}_{M\setminus L}\hat{v};$$

the derivative can be replaced with the summation because of assumption (18.4). Further, it follows from Definition (27.1) and assumption (18.6) that

$$\sum_{L\subset M}(-1)^{\mathrm{ne}M+\mathrm{ne}L}\mathrm{in}_M^*\mathrm{sm}_{M\setminus L}\hat{v} = \sum_{L\subset M}(-1)^{\mathrm{ne}\,M+\mathrm{ne}\,L}\mathrm{in}_M^*\Big\{\prod_{i\notin M}\hat{v}_i\prod_{i\in M}\mathrm{sm}^{\delta(i\notin L)}\hat{v}_i\Big\}$$

$$= \prod_{i\notin M}\hat{v}_i\prod_{i\in M}\mathrm{in}_1^*(\hat{v}_i-\mathrm{sm}^1\hat{v}_i).$$

Since according to Definition (18) at each point $x_i \in r_i^1$

$$\mathrm{SUPP}\hat{v}_i(x_i) \subset \mathrm{NB}(x_i, \mathrm{ST}\hat{v}_i) \quad \text{and} \quad \hat{v}_i(x_i) \leqslant \mathrm{mw}_i^{-1}\mathrm{FC},$$

and because of Lemma (26)

$$\mathrm{SUPP}\,\mathrm{sm}^1\hat{v}_i(x_i) \subset \mathrm{NB}(x_i, \mathrm{ST}\hat{v}_i - 1''),$$

we conclude taking into account Definition (17.1) of in_K^* and property (18.2) of $\hat{v}_i$ and $\mathrm{sm}^1\hat{v}_i$ that

$$\mathrm{SUPP}\,|\mathrm{in}_1^*(\hat{v}_i-\mathrm{sm}^1\hat{v}_i)(x_i)| \subset \mathrm{NB}(x_i, \mathrm{ST}\hat{v}_i)$$

and

$$|\mathrm{in}_1^*(\hat{v}_i-\mathrm{sm}^1\hat{v}_i)(x_i)| \leqslant \mathrm{FC}.$$

Hence we have proven the following lemma.

(27.5) LEMMA. *If* $K \subset \overline{m}$, $\hat{v} \in \mathrm{pc}_0^m$, *then* $\mathrm{sm}_K^-\hat{v} \in \big(r^m \to W^{-e'_K{}''}(\boldsymbol{R}^m)^{\mathrm{loc}}\big)$ *and*

$$\mathrm{sm}_K^-\hat{v} = \sum_{M\subset K} D_M\hat{y}_M,$$

where $\hat{y}_M = \prod_{i \notin M} \hat{v}_i \prod_{i \in M} \mathrm{in}_1^*(\hat{v}_i - \mathrm{sm}^1 \hat{v}_i)$. *Moreover, at each* $x \in r^m$

$$\mathrm{SUPP}\hat{y}_M(x) \subset \mathrm{NB}(x, \mathrm{ST}\hat{v}), \qquad |\hat{y}_M(x)| \leqslant \mathrm{cv}_{\setminus M}^{-1}\mathrm{FC}.$$

The following formulas can easily be obtained from (27.3,4) and (17.5).

$$\text{(27.6)} \quad \mathrm{sm}_K^-(\Delta^{*l}\mathrm{sm}^k\hat{v}) = \mathrm{cv}_K \Delta^{*l-e'_K} D_K \mathrm{sm}^k \hat{v} \quad \text{if} \quad l - e'_K \geqslant 0,\ k \geqslant 0;$$

$$\mathrm{sm}_K^-(\Delta^{*l}\mathrm{sm}^k\hat{v}) = \Delta^{*l}\mathrm{sm}^{k-e'_K}\hat{v} \quad \text{if} \quad k - e'_K \geqslant 0,\ l \geqslant 0.$$

Hence we introduce the following definitions.

(27.7) DEFINITION. If $s \in I^m$, $s \geqslant 0$, then the set $\mathrm{pu}_s^m \subset \mathrm{pu}_0^m$ ($\mathrm{pc}_s^m \subset \mathrm{pc}_0^m$) is defined as the set of all functions $\hat{v}$ of the form

$$\hat{v} = \mathrm{sm}^s \hat{v}^0 + \sum_{0<l\leqslant s} t_l \Delta^{*l}\mathrm{sm}^{s-l}\hat{v}^l,$$

where $\hat{v}^l \in \mathrm{pu}_0^m$ (pc_0^m, respectively), $t_l \in \boldsymbol{R}$ for each l; for $\hat{v} \in \mathrm{pc}_s^m$ we define similarly as in (19.3)

$$\mathrm{md}_s(\hat{v}) = \inf\Big\{\sum_{i=1}^k |s_i| \colon \sum_{i=1}^k s_i \hat{w}^i = \hat{v},\ \sum_{i=1}^k s_i = 1,\ \hat{w}^i \in \mathrm{pu}_s^m\Big\}.$$

(28.8) DEFINITION. The operator sm^{-l}: $\mathrm{pc}_s^m \to (r^m \to W^{-l^+\prime\prime}(\boldsymbol{R}^m)^{\mathrm{loc}})$, where $l \leqslant s + e$, $l^+ = \sum_{i=1}^m l_i e'_i \delta(l_i > 0)$, is defined in the recurrent way:
if $K \subset \overline{m}$ then $\mathrm{sm}^{-e'_K}\hat{v} = \mathrm{sm}_K^-\hat{v}$;
if $K \subset \overline{m}$, $0 \leqslant l \leqslant s + e'_{\setminus K}$, then $\mathrm{sm}^{-l-e'_K}\hat{v} = \mathrm{sm}_K^-(\mathrm{sm}^{-l}\hat{v})$;
if $i \geqslant 0$, $0 \leqslant l \leqslant s + e$, then $\mathrm{sm}^{i-l}\hat{v} = \mathrm{sm}^{-l}(\mathrm{sm}^i\hat{v})$
for each $\hat{v} \in \mathrm{pc}_s^m$.

Let us now investigate the distribution $\mathrm{sm}^{-l}\hat{v}$. First, let us represent it in another form. If $\hat{v} \in \mathrm{pc}_s^m$ has the form as in Definition (27.7), then for fixed $l \leqslant s + e$ it can also be represented as the sum

$$\text{(28.1)} \qquad \hat{v} = \sum_{0\leqslant n \leqslant (l-e)^+} b_{nl} \Delta^{*n}\mathrm{sm}^{(l-e)^+-(-l)^+-n}\hat{v}^{nl},$$

where

$$\hat{v}^{0l} = \mathrm{sm}^{s-(l-e)^+-(-l)^+}\hat{v}^0 + \sum_{0<i\leqslant s-(l-e)^+} t_i \Delta^{*i}\mathrm{sm}^{s-(l-e)^++(-l)^+-i}\hat{v}^i,$$

$$\hat{v}^{nl} = \Delta^{*s-(l-e)^+}\mathrm{sm}^{(-l)^+}\hat{v}^{s-(l-e)^++n} \quad \text{for} \quad 0 < n \leqslant (l-e)^+,$$

$$b_{0l} = 1, \qquad b_{nl} = t_{n+s-(l-e)^+} \quad \text{for} \quad 0 < n \leqslant (l-e)^+.$$

Hence by applying (27.6) we obtain the formula

$$\mathrm{sm}^{-l}\hat{v} = \sum_{0\leqslant n\leqslant (l-e)^+} b_{nl}\,\mathrm{mw}^n D^n \mathrm{sm}^{(l-e)^+-l^+}\hat{v}^{nl}, \tag{28.2}$$

which will be useful in the following considerations.

Formula (28.2) together with Lemma (27.5) yield that the support of $\mathrm{sm}^{-l}\hat{v}(x)$ is bounded and thus if $F \in W^p_q(\boldsymbol{R}^m)^{\mathrm{loc}}$, $l^+ \in p$, $f \in l_q(r^m)^{\mathrm{loc}}$, then the dual pairings $\langle F, \mathrm{sm}^{-l}\hat{v}\rangle$ and $[f, \mathrm{sm}^{-l}\hat{v}]$ are well defined. Let us also extend the definition of the dual pairing $\langle \mathrm{sm}^{-l}\hat{v}, F\rangle$ onto the case when $F \in W^p_q(B)$, $B \subset \boldsymbol{R}^m$. According to formula (28.2) it is sufficient to consider the distributions of the form $D^k \mathrm{sm}^-_K \hat{v}$. First, if $F \in \mathring{W}^p_q(B)$, then $\mathrm{EX}_m F \in W^p_q(\boldsymbol{R}^m)$ and we can define in the natural way

(28.3) if $k+e'_K \in p$ then $\langle D^k \mathrm{sm}^-_K \hat{v}, F\rangle = \langle D^k \mathrm{sm}^-_K \hat{v}, \mathrm{EX}_m F\rangle$
$\forall F \in \mathring{W}^p_q(B)$.

The above definition cannot be applied if $F \in W^p_q(B)$. In this case, however, we use Lemma (27.5), which yields that for every $G \in W^p_q(\boldsymbol{R}^m)$ the value of $\langle D^k \mathrm{sm}^-_K \hat{v}(x), G\rangle$ depends only on $\mathrm{RD}_{\mathrm{NB}(x,\mathrm{ST}\hat{v})} G$, therefore it is reasonable to define

(28.4) if $k+e'_K \in p$, $F \in W^p_q(B)$, $\mathrm{NB}(x, \mathrm{ST}\hat{v}) \subset B$, then

$$\langle D^k \mathrm{sm}^-_K \hat{v}(x), F\rangle = \sum_{M\subset K} \langle \hat{y}_M(x), D^{*k} D^*_M F\rangle,$$

where $\hat{y}_M$ are taken from Lemma (27.5).

The following two formulas can be obtained from Definitions (27.1) and (27.8).

(28.5) If $F \in W^p_q(\boldsymbol{R}^m)$, $\hat{v} \in \mathrm{pc}^m_s$, $0 \leqslant i \leqslant s+e$, $i \in p$, $l \geqslant 0$, $l+(i-l)^+ \in p$, then

$$\partial^{*l}\langle F, \mathrm{sm}^{-i}\hat{v}\rangle = \langle D^{*l}F, \mathrm{sm}^{l-i}\hat{v}\rangle.$$

(28.6) If $f \in w^p_q(r^m)$, $k \in I^m$, $\hat{v} \in \mathrm{pc}^m_s$, $0 \leqslant l \leqslant s+k+e$, then

$$D^l[\mathrm{sm}^k\hat{v}, f] = [\mathrm{sm}^{k-l}\hat{v}, \partial^l f].$$

Let us now estimate the norms of $\langle F, \mathrm{sm}^i\hat{v}\rangle$ and $[\mathrm{sm}^k\hat{v}, f]$.

(29) Lemma. *If $f \in w^p_q(r^m)$, $\hat{v} \in \mathrm{pu}^m_0$, $k \geqslant \mathrm{UB}(p)$, then $[\mathrm{sm}^k\hat{v}, f] \in W^p_q(\boldsymbol{R}^m)$ and*

$$\|[\mathrm{sm}^k\hat{v}, f]\|_{p,q} \leqslant \|f\|_{p,q}.$$

Proof. If $\hat{v} \in \mathrm{pu}_0^m$ then by applying formula (28.6), Lemmas (27.2) and (20.3) we obtain for every $l \in p$ the inequality

$$\|D^l[\mathrm{sm}^k\hat{v}, f]\|_q = \|[\mathrm{sm}^{k-l}\hat{v}, \partial^l f]\|_q \leqslant \|\partial_l f\|_q.$$

Hence the lemma is true.

(30) LEMMA. *If $F \in W_q^{eK''}(\boldsymbol{R}^m)$, $K \subset \overline{m}$, $\hat{v} \in \mathrm{pu}_0^m$, then $\langle F, \mathrm{sm}_K^-\hat{v}\rangle \in l_q(r^m)$ and*

$$\|\langle F, \mathrm{sm}_K^-\hat{v} - \hat{v}\rangle\|_q \leqslant \mathrm{VL}\,\hat{v} \sum_{\emptyset \neq M \subset K} \mathrm{cv}_M \|D_M F\|_q,$$

where the number VL *is defined by* $\mathrm{VL}\hat{v} = (\mathrm{UP}\hat{v} - \mathrm{LW}\hat{v})^e$.

Proof. According to Lemma (27.5),

$$\|\langle F, \mathrm{sm}_K^-\hat{v} - \hat{v}\rangle\|_q \leqslant \sum_{\emptyset \neq M \subset K} \|\langle D_M^* F, \hat{y}_M\rangle\|_q. \tag{30.1}$$

From the estimations in (27.5) for $\hat{y}_M$ and Hölder's inequality we have in the case $1 \leqslant q < \infty$

$$\forall x \in r^m \quad |\langle D_M^* F, \hat{y}_M(x)\rangle|^q \leqslant \int_{\mathrm{NB}(x,\mathrm{ST}\hat{v})} |D_M^* F(X)|^q dX \Big(\int_{\mathrm{NB}(x,\mathrm{ST}\hat{v})} \mathrm{cv}_{\setminus M}^{-q/(q-1)} dX\Big)^{q-1}.$$

Because of (18.9) the volume of $\mathrm{NB}(x, \mathrm{ST}\hat{v})$ can be estimated by $\mathrm{mw}\,\mathrm{VL}\hat{v}$; hence we obtain the inequality

$$\begin{aligned}\|\langle D_M^* F, \hat{y}_M\rangle\|_q^q &\leqslant \mathrm{cv} \sum_{x \in r^m} \int_{\mathrm{NB}(x,\mathrm{ST}\hat{v})} |D_M^* F(X)|^q dX (\mathrm{mw}\,\mathrm{VL}\,\hat{v})^{q-1} \mathrm{cv}_{\setminus M}^{-q} \\ &\leqslant \mathrm{cv}\,\mathrm{VL}\,\hat{v}\, \|D_M F\|_q^q (\mathrm{VL}\,\hat{v})^{q-1} \mathrm{cv}^{q-1} \mathrm{cv}_{\setminus M}^{-q}.\end{aligned}$$

Thus, by substituting this inequality into (30.1) we obtain

$$\|\langle F, \mathrm{sm}_K^-\hat{v} - \hat{v}\rangle\|_q \leqslant \mathrm{VL}\,\hat{v} \sum_{\emptyset \neq M \subset K} \mathrm{cv}_M \|D_M F\|_q.$$

If $q = \infty$ then the required inequality directly follows from (30.1) and (27.5). If $q = (qx, qi, L)$ then the proof can be carried out similarly as in Lemma (20.3), starting from inequality (30.1) and using the estimations from (27.5). ∎

(31) LEMMA. *Assume that $p \in \mathrm{SL}^m$ and the vectors $k, l, s \in I^m$ are such that $\mathrm{UB}(p) \leqslant k$, $s \geqslant (l-e)^+$ and $\forall i \in p\ (l-i)^+ + i \in p$. Let $\hat{v} \in \mathrm{pu}_0^m$, $\hat{w} \in \mathrm{pu}_s^m$. If $F \in W_q^p(\boldsymbol{R}^m)$ then*

$$\langle F, \mathrm{sm}^{-l}\hat{w}\rangle \in w_q^p(r^m), \qquad [\mathrm{sm}^k\hat{v}, \langle F, \mathrm{sm}^{-l}\hat{w}\rangle] \in W_q^p(\boldsymbol{R}^m).$$

Moreover, there exist constants KA, KB, KC $\in \boldsymbol{R}$, *and* KD $\in \boldsymbol{R}^m$, *such that for every* $F \in W_q^p(\boldsymbol{R}^m)^{\cdot}$ *and* $i \in p$ *the inequalities hold:*

$$\|\partial^i \langle F, \mathrm{sm}^{-l}\hat{w}\rangle\|_q \leqslant \|D^i F\|_q + \sum_{n\in(p\ominus i)\setminus\{0\}} \mathrm{mw}^n \mathrm{KA}_{p\ominus i}(\hat{w}, s, l-i, n)\|D^{n+i}F\|_q + \\ + \sum_{n\in(p\ominus i)^+} \mathrm{mw}^n \sum_{0<r\leqslant s-(l-i-e)^+} \mathrm{KB}_{p\ominus i}(\hat{w}, s, l-i, n, r)\mathrm{MC}_q(D^{n+i}F; \mathrm{mw}, r);$$

$$\|D^i(F - [\mathrm{sm}^k\hat{v}, \langle F, \mathrm{sm}^{-l}\hat{w}\rangle])\|_q \\ \leqslant \mathrm{KC}(\hat{v}, \hat{w}, q, k-i, l-i, s)\mathrm{MC}_q\big(D^i F; \mathrm{KD}(\hat{v}, \hat{w}, k-i, l-i, s)\circ \mathrm{mw}\big) + \\ + \sum_{n\in(p\ominus i)\setminus\{0\}} \mathrm{mw}^n \mathrm{KA}_{p\ominus i}(\hat{w}, s, l-i, n)\|D^{n+i}F\|_q + \\ + \sum_{n\in(p\ominus i)^+} \mathrm{mw}^n \sum_{0<r\leqslant s-(l-i-e)^+} \mathrm{KB}_{p\ominus i}(\hat{w}, s, l-i, n, r)\mathrm{MC}_q(D^{n+i}F; \mathrm{mw}, r).$$

(KA, KB *can be derived from inequality* (31.6) *and fulfil estimation* (31.9), KC, KD *are defined by* (31.10)).

Proof. Let us denote $(l-e)^+$ by j, $l^+ - (l-e)^+$ by e'_K. Following formula (28.2),

$$\langle F, \mathrm{sm}^{-l}\hat{w}\rangle = \sum_{0\leqslant n\leqslant j} b_{nl}\mathrm{mw}^n \langle D^{*n}F, \mathrm{sm}_K^-\hat{w}^{nl}\rangle. \tag{31.1}$$

Let us thus investigate the function $\hat{w}^{nl}$. First, if $i \in \overline{m}$, $\hat{z} \in \mathrm{pu}_i^m$, $G \in L_q(\boldsymbol{R}^m)^{\mathrm{loc}}$, then (18.5) and (9.6) yield

$$\langle \Delta_i^* \hat{z}, G\rangle = \langle \mathrm{ts}_i^- \hat{z} - \hat{z}, G\rangle = \langle \mathrm{TS}^{\mathrm{mw}\circ e'_i}\hat{z} - \hat{z}, G\rangle = \langle \hat{z}, \mathrm{DF}_{-\mathrm{mw}}^{e'_i} G\rangle,$$

hence we obtain by induction

$$\langle \Delta^{*k}\hat{z}, G\rangle = \langle \hat{z}, \mathrm{DF}_{-\mathrm{mw}}^k G\rangle \quad \text{for every } k \in I^m,\ k \geqslant 0. \tag{31.2}$$

As can easily be checked, (31.2) remains true if we replace $\hat{z}$ by $\hat{y}_M$ occurring in Lemma (27.5). Hence,

$$\langle F, \mathrm{sm}^- \hat{w}^{0l}\rangle = \langle F, \mathrm{sm}_K^- \mathrm{sm}^{s-j+(-l)^+}\hat{w}^0\rangle + \\ + \sum_{0<i\leqslant s-j} t_i \langle \mathrm{DF}_{-\mathrm{mw}}^i F, \mathrm{sm}_K^- \mathrm{sm}^{s-j+(-l)^+-i}\hat{w}^i\rangle, \tag{31.3}$$

and if $0 < n \leqslant j$ then

$$\langle D^{*n}F, \mathrm{sm}_K^-\hat{w}^{nl}\rangle = \langle \mathrm{DF}_{-\mathrm{mw}}^{s-j} D^{*n}F, \mathrm{sm}_K^- \mathrm{sm}^{(-l)^+}\hat{w}^{s-j+n}\rangle. \tag{31.4}$$

For every $G \in W_q^{e'_K}(\boldsymbol{R}^m)^{\mathrm{loc}}$ and $\hat{z} \in \mathrm{pu}_0^m$ by applying Lemma (30) and (20.5) we obtain the following inequality

$$\|\langle G, \mathrm{sm}_K^-\hat{z}\rangle\|_q \leqslant \|\langle G, \hat{z}\rangle\|_q + \|\langle G, \mathrm{sm}_K^-\hat{z} - \hat{z}\rangle\|_q \\ \leqslant \|G\|_q + \mathrm{VL}\,\hat{z} \sum_{\emptyset\neq M\subset K} \mathrm{cv}_M \|D_M G\|_q. \tag{31.5}$$

If we estimate the right-hand sides of (31.3,4) according to (31.5), substitute the obtained results into (31.1), and introduce the notation

$$\mathrm{CS}(\hat{w}, s, l) = \max\big(\{\mathrm{VL}(\mathrm{sm}^{s-j+(-l)^+-i}\hat{w}^i):\ 0 < i \leqslant s-j\}\cup \\ \cup\ \{\mathrm{VL}(\mathrm{sm}^{(-l)^+}\hat{v}^{s-j+i}):\ 0 < i \leqslant j\}\big),$$

then we obtain

$$(31.6)\quad \|\langle F, \mathrm{sm}^{-l}\hat{w}\rangle\|_q \leqslant \|F\|_q + \mathrm{CS}(\hat{w}, s, l) \sum_{\emptyset \neq M \subset K} \mathrm{cv}_M \|D_M F\|_q + \\ + \sum_{0<i\leqslant s-j} |t_i| \Big\{\|\mathrm{DF}^i_{-\mathrm{mw}} F\|_q + \mathrm{CS}(\hat{w}, s, l) \sum_{\emptyset \neq M \subset K} \mathrm{cv}_M \|\mathrm{DF}^i_{-\mathrm{mw}} D_M F\|_q\Big\} + \\ + \sum_{0<n\leqslant j} |b_{nl}|\,\mathrm{mw}^n \Big\{\|\mathrm{DF}^{s-j}_{-\mathrm{mw}} D^n F\|_q + \mathrm{CS}(\hat{w}, s, l) \sum_{\emptyset \neq M \subset K} \mathrm{cv}_M \|\mathrm{DF}^{s-j}_{-\mathrm{mw}} D_M D^n F\|_q\Big\}.$$

The applying of Lemma (10.6) yields

$$(31.7)\qquad \|\mathrm{DF}^i_{-\mathrm{mw}} D^n F\|_q \leqslant \mathrm{mw}^i \|D^{n+i} F\|_q \quad \text{if} \quad n+i \in p,$$

$$\|\mathrm{DF}^i_{-\mathrm{mw}} D^n F\|_q \leqslant \mathrm{mw}^g \|\mathrm{DF}^r_{-\mathrm{mw}} D^{n+g} F\|_q \quad \text{if} \quad n+g \in p^+,\ r \geqslant 0,\ r+g = i.$$

By substituting (31.7) into (31.6) and changing the order of summation we obtain

$$(31.8)\quad \|\langle F, \mathrm{sm}^{-l}\hat{w}\rangle\|_q \leqslant \|F\|_q + \sum_{n\in p\setminus\{0\}} \mathrm{mw}^n \mathrm{KA}_p(\hat{w}, s, l, n)\|D^n F\|_q + \\ + \sum_{n\in p^+} \mathrm{mw}^n \sum_{0<r\leqslant s-j} \mathrm{KB}_p(\hat{w}, s, l, n, r)\mathrm{MC}_q(D^n F; \mathrm{mw}, r)$$

(KB is not uniquely determined), where

$$(31.9)\quad \sum_{n\in p\setminus\{0\}} \mathrm{KA}_p(\hat{w}, s, l, n) + \sum_{n\in p^+} \sum_{0<r\leqslant s-j} \mathrm{KB}_p(\hat{w}, s, l, n, r) \\ \leqslant \mathrm{CS}(\hat{w}, s, l) 2^{\mathrm{ne} K} \sum_{0\leqslant i\leqslant s} |t_i|.$$

To obtain an estimation of $\|\partial^i \langle F, \mathrm{sm}^{-l}\hat{w}\rangle\|_q$ let us remark that according to formula (28.5) for every $i \in p$

$$\partial^{*i}\langle F, \mathrm{sm}^{-l}\hat{w}\rangle = \langle D^{*i}F, \mathrm{sm}^{i-l}\hat{w}\rangle,$$

and then apply inequality (31.8) with F, l, p replaced by $D^iF, l-i, p\ominus i$, respectively. Now, by applying formulas (31.1,3) we obtain

$$\|F - [\mathrm{sm}^k\hat{v}, \langle F, \mathrm{sm}^{-l}\hat{w}\rangle]\|_q \leqslant \|F - [\mathrm{sm}^k\hat{v}, \langle F, \mathrm{sm}^{s-j+(-l)^+}\hat{w}^0\rangle]\|_q + \\ + \|[\mathrm{sm}^k\hat{v}, \langle F, \mathrm{sm}^{s-j+(-l)^+}\hat{w}^0 - \mathrm{sm}_K^- \mathrm{sm}^{s-j+(-l)^+}\hat{w}^0\rangle]\|_q + \\ + \sum_{0<i\leqslant s-j} |t_i|\, \|[\mathrm{sm}^k\hat{v}, \langle \mathrm{DF}^i_{-\mathrm{mw}} F, \mathrm{sm}_K^- \mathrm{sm}^{s-j+(-l)^+-i}\hat{w}^i\rangle]\|_q + \\ + \sum_{0<n\leqslant j} |b_{nl}|\,\mathrm{mw}^n \|[\mathrm{sm}^k\hat{v}, \langle D^{*n}F, \mathrm{sm}_K^- \hat{w}^{nl}\rangle]\|_q.$$

The first term on the right-hand side can be estimated by using Theorem (23), the second one by applying Lemmas (20.3) and (30), the last two — by taking into account Lemma (20.3) and further as in the first part of the present proof. Finally, we obtain that $\|F-[\mathrm{sm}^k\hat{v}, \langle F, \mathrm{sm}^{-l}\hat{w}\rangle]\|_q$ can be estimated by the sum on the right-hand side of inequality (31.6) with $\|F\|_q$ replaced by the number

(31.10) $\mathrm{CT}(q, \mathrm{sm}^k\hat{v}, \mathrm{sm}^{s-j+(-l)^+}\hat{w}^0)\mathrm{MC}_q\big(F; \mathrm{MX}(\mathrm{sm}^k\hat{v}, \mathrm{sm}^{s-j+(-l)^+}\hat{w}^0)\circ \mathrm{mw}\big)$,

which will be denoted by

$$\mathrm{KC}(\hat{v}, \hat{w}, q, k, l, s)\mathrm{MC}_q\big(F; \mathrm{KD}(\hat{v}, \hat{w}, k, l, s)\circ \mathrm{mw}\big).$$

Moreover, by applying formulas (28.5,6) we obtain for every $i \in p$

$$D^i(F-[\mathrm{sm}^k\hat{v}, \langle F, \mathrm{sm}^{-l}\hat{w}\rangle]) = D^iF-[\mathrm{sm}^{k-i}\hat{v}, \langle D^iF, \mathrm{ts}^i\mathrm{sm}^{i-l}\hat{v}\rangle].$$

Therefore our lemma is true. ■

Similarly as in the previous section, let us formulate the following corollary.

(32.1) Corollary. *Assume that $p, k, l, s, \hat{v}, \hat{w}$ fulfil the assumptions of Lemma* (31). *Let*

$$B \subset \boldsymbol{R}^m, \quad b = \mathrm{sb}\big(B, \bigcup_{0\leqslant n\leqslant s} \mathrm{ST}(\hat{w}^n)-(s+e-l)''\big),$$

$$B^-[\mathrm{mw}] = \mathrm{SB}(b, \mathrm{ST}\hat{v}-k'').$$

If $F \in W_q^p(B)$ then

$$\langle F, \mathrm{rd}_b\mathrm{sm}^{-l}\hat{w}\rangle \in w_q^p(b) \quad \textit{and} \quad [\mathrm{sm}^k\hat{v}, \langle F, \mathrm{rd}_b\mathrm{sm}^{-l}\hat{w}\rangle] \in W_q^p(\boldsymbol{R}^m).$$

Moreover, the numbers

$$\|\partial^i\langle F, \mathrm{rd}_b\mathrm{sm}^{-l}\hat{w}\rangle\|_q, \quad \|D^i(F-\mathrm{RD}_B[\mathrm{sm}^k\hat{v}, \langle F, \mathrm{rd}_b\mathrm{sm}^{-l}\hat{w}\rangle])\|_q^{(B^-[\mathrm{mw}])}$$

can be estimated by the terms on the right-hand sides of inequalities from Lemma (31).

Proof. By representing $\mathrm{sm}^{-l}\hat{w}$ in the form (28.2) and applying Lemmas (27.2) and (27.5), and formula (13.9) we come to the conclusion that if $x \in b$ then $\mathrm{SUPP}\,\mathrm{sm}^{-l}\hat{w}(x) \subset B$, and if $X \in B^-[\mathrm{mw}]$ then $\mathrm{supp}\big(\mathrm{sm}^k\hat{v}(\cdot, X)\big) \subset b$. Therefore the corollary can be proved similarly as Lemma (31) using Definition (28.4). ■

(32.2) *Remark.* The condition occurring in the assumptions of Lemma (31), $\forall i \in p\ (l-i)^+ + i \in p$, is equivalent to the condition $l \leqslant \mathrm{LB}(p^+)$. Hence if $p = \{k \in I^m\colon |k| \leqslant n\}$, $m \geqslant 2$, then the only vector satisfying this condition is $l = 0$.

Proof. Let us take an arbitrary $i \in p^+$. Then $(l-i)^+ + i \in p$ if and only if $(l-i)^+ = 0$, that is, $l \leqslant i$, what was to be shown. ∎

(33) LEMMA. *Assume that* $F \in W_q^{-p}(\boldsymbol{R}^m)$, $F = \mathrm{EC}_{q}^{p}(\vec{F}_p)$, *and* $k \geqslant \mathrm{UB}(p)$. *Let* $\hat{w} \in \mathrm{pu}_0^m$. *Then* $\langle \mathrm{sm}^k \hat{w}, F\rangle \in w_q^{-p}(r^m)$, $\langle \mathrm{sm}^k\hat{w}, F\rangle = \mathrm{ec}_q^p(\vec{z}_p)$, *where* $z_l = \langle \mathrm{sm}^{k-l}\hat{w}, F_l\rangle$, *and for every* $f \in w_{q'}^p(r^m)$, $[f, \langle \mathrm{sm}^k\hat{w}, F\rangle] = \langle [\mathrm{sm}^k\hat{w}, f], F\rangle$.

Proof. Let $f \in w_{q'}^p(r^m)$. By applying Lemma (29), $[\mathrm{sm}^k\hat{w}, f] \in W_{q'}^p(\boldsymbol{R}^m)$. Thus, since $F = \mathrm{EC}_q^p(\vec{F}_p)$, we have

$$\langle [\mathrm{sm}^k\hat{w}, f],\ F\rangle = \sum_{l\in p} \langle D^l[\mathrm{sm}^k\hat{w}, f],\ F_l\rangle = \sum_{l\in p}\langle [D^l\mathrm{sm}^k\hat{w}, f],\ F_l\rangle,$$

and by using Lemma (20.6) combined with Lemma (27.2) we obtain

$$\sum_{l\in p}\langle [D^l\mathrm{sm}^k\hat{w}, f], F_l\rangle = \Big[\sum_{l\in p}\langle D^l\mathrm{sm}^k\hat{w}, F_l\rangle,\ f\Big] = [\langle \mathrm{sm}^k\hat{w}, F\rangle,\ f].$$

Hence

$$[\langle \mathrm{sm}^k\hat{w}, F\rangle, f] = \langle [\mathrm{sm}^k\hat{w}, f],\ F\rangle.$$

Moreover, formula (28.6) and Lemma (20.6) yield

$$\sum_{l\in p}\langle [D^l\mathrm{sm}^k\hat{w}, f],\ F_l\rangle = \sum_{l\in p}\langle [\mathrm{sm}^{k-l}\hat{w},\ \partial^l f],\ F_l\rangle = \sum_{l\in p}[\langle \mathrm{sm}^{k-l}\hat{w}, F_l\rangle,\ \partial^l f].$$

Thus, the lemma is proved. ∎

(34) LEMMA. *Assume that* $f \in w_q^{-p}(r^m)$, $\hat{v} \in \mathrm{pu}_s^m$, *and* $l \in I^m$ *is such that* $l \leqslant s+e$, $l \leqslant \mathrm{LB}(p^+)$. *Then* $[f, \mathrm{sm}^{-l}\hat{v}] \in W_q^{-p}(\boldsymbol{R}^m)$ *and for every* $F \in W_{q'}^p(\boldsymbol{R}^m)$

$$\langle F,\ [f, \mathrm{sm}^{-l}\hat{v}]\rangle = [f, \langle F, \mathrm{sm}^{-l}\hat{v}\rangle].$$

Proof. If $f = \mathrm{ec}_q^p(\vec{f}_p)$ then following Definition (27.8)

$$[\mathrm{sm}^{-l}\hat{v}, f] = \sum_{k\in p}[\partial^k\mathrm{sm}^{-l}\hat{v}, f_k] = \sum_{k\in p} D^{*k}[\mathrm{ts}^k\mathrm{sm}^{k-l}\hat{v}, f_k].$$

If for every $k \in p$ we represent $\mathrm{sm}^{k-l}\hat{v}$ in the form (28.2) and denote $(l-k)^+ - (l-k-e)^+$ by $e'_{K(k)}$, we obtain

$$[\mathrm{ts}^k\mathrm{sm}^{k-l}\hat{v}, f_k] = \sum_{0\leqslant n\leqslant (l-k-e)^+} b_{n,l-k}\mathrm{mw}^n[D^n\mathrm{sm}^-_{K(k)}\hat{v}^{n,l-k}, \mathrm{ts}^{-k}f_k],$$

and hence by applying Lemma (27.5),

$$[\mathrm{sm}^{-l}\hat{v}, f]$$

$$= \sum_{k\in p}\ \sum_{0\leqslant n\leqslant (l-k-e)^+}\ \sum_{M\subset K(k)} (-1)^{|n+e'_M|} b_{n,l-k}\mathrm{mw}^n D^{*k+n+e'_M}[\hat{y}_M^{n,l-k}, \mathrm{ts}^{-k}f_k].$$

Therefore we have the formula

(34.1) $\langle F, [\mathrm{sm}^{-l}\hat{v}, f]\rangle$

$$= \sum_{k\in p} \sum_{0\leqslant n\leqslant (l-k-e)^+} \sum_{M\subset K(k)} (-1)^{|n+e'_M|} b_{n,l-k}\, \mathrm{mw}^n \langle D^{k+n+e'_M} F, [\hat{y}_M^{n,l-k}, \mathrm{ts}^{-k} f_k]\rangle,$$

from which the representation of $[\mathrm{sm}^{-l}\hat{v}, f]$ can be derived, since for every (k, n, M) from the domain of summation, $k+n+e'_M \in p$. Moreover, because of the estimation of $\hat{y}_M$ given in (27.5) we can prove similarly as Lemma (20.6) that for every k, n, M

$$\langle D^{k+n+e'_M} F, [\hat{y}_M^{n,l-k}, \mathrm{ts}^{-k} f_k]\rangle = [\langle D^{k+n+e'_M} F, \hat{y}_M^{n,l-k}\rangle, \ \mathrm{ts}^{-k} f_k].$$

Hence, applying (27.5) and (28.2) once more we obtain

$$\sum_{0\leqslant n\leqslant (l-k-e)^+} \sum_{M\subset K(k)} (-1)^{|n+e'_M|} b_{n,l-k}\, \mathrm{mw}^n \langle D^{k+n+e'_M} F, \hat{y}_M^{n,l-k}\rangle = \langle D^k F, \mathrm{sm}^{k-l}\hat{v}\rangle.$$

Thus, combining the above formula with (34.1) and (28.5) we get

(34.2)
$$\langle F, [\mathrm{sm}^{-l}\hat{v}, f]\rangle = \sum_{k\in p} [\langle D^k F, \mathrm{sm}^{k-l}\hat{v}\rangle, \ \mathrm{ts}^{-k} f_k] = \sum_{k\in p} [\partial^k \langle F, \mathrm{sm}^{-l}\hat{v}\rangle, f_k] = [\langle F, \mathrm{sm}^{-l}\hat{v}\rangle, \ f],$$

what was to be shown. ∎

(35) LEMMA. *Assume that* $F \in W_q^{-p}(\boldsymbol{R}^m)$ *and the vectors* k, l, s *satisfy the inequalities* $k \geqslant \mathrm{UB}(p)$, $l \leqslant s+e$, $l \leqslant \mathrm{LB}(p^+)$. *Let* $\hat{v} \in \mathrm{pu}_0^m$, $\hat{w} \in \mathrm{pu}_s^m$. *Then* $[\mathrm{sm}^{-l}\hat{w}, \ \langle \mathrm{sm}^k \hat{v}, F\rangle] \in W_q^{-p}(\boldsymbol{R}^m)$ *and there exist constants* $\mathrm{KE}(\hat{w}, \hat{v}, k, l, s) \in \boldsymbol{R}$, $\mathrm{KF}(\hat{w}, \hat{v}, k, l, s) \in \boldsymbol{R}^m$ *and* $\mathrm{KG}(\hat{w}, l, s) \in \boldsymbol{R}$ *such that the following inequality holds*

$$\|F - [\mathrm{sm}^{-l}\hat{w}, \ \langle \mathrm{sm}^k \hat{v}, F\rangle]\|_{-p,q} \leqslant \mathrm{KE}\cdot \mathrm{MC}_{-p,q}(F; \ \mathrm{KF}\circ \mathrm{mw}) + \mathrm{KG}\,|\mathrm{mw}|\cdot \|F\|_{-p,q}.$$

Proof. The first part of the lemma is a simple consequence of Lemmas (33), (34). To obtain the estimation we have to investigate the term $\langle G, F - [\mathrm{sm}^{-l}\hat{w}, \ \langle \mathrm{sm}^k \hat{v}, F\rangle]\rangle$ for every $G \in W_{q'}^p(\boldsymbol{R}^m)$ according to the definition of the norm. Let us assume that $F = \mathrm{EC}_q^p(\vec{F}_p)$, i.e. $\langle G, F\rangle = \sum_{i\in p} \langle D^i G, F_i\rangle$. Following formula (34.1) and Lemma (33) we have

(35.1)
$$\langle G, [\mathrm{sm}^{-l}\hat{w}, \ \langle \mathrm{sm}^k \hat{v}, F\rangle]\rangle = \sum_{i\in p} \sum_{0\leqslant n\leqslant (l-i-e)^+} \sum_{M\subset K(i)} (-1)^{|n+e'_M|} \times b_{n,l-i}\, \mathrm{mw}^n \langle D^{i+n+e'_M} G, \ [\hat{y}_M^{n,l-i}, \mathrm{ts}^{-i}\langle \mathrm{sm}^{k-i}\hat{v}, F_i\rangle]\rangle.$$

Let us first consider the term

$$s_0 = \sum_{i\in p} \{\langle D^i G, F_i\rangle - b_{0,l-i}\langle D^i G, [\hat{y}_{\varnothing}^{0,l-i}, \mathrm{ts}^{-i}\langle \mathrm{sm}^{k-i}\hat{v}, F_i\rangle]\rangle\}.$$

Following Lemma (27.5), $\hat{y}_{\varnothing}^{0,l-i} = \hat{w}^{0,l-i} \in \mathrm{pc}_0^m$, and according to formula (28.1), $b_{0,l-i} = 1$, therefore

$$(35.2)\qquad |s_0| \leqslant \sum_{i\in p} \|D^i G\|_{q'} \|F_i - [\hat{w}^{0,l-i}, \ \mathrm{ts}^{-i}\langle \mathrm{sm}^{k-i}\hat{v}, F_i\rangle]\|_q$$

$$\leqslant \sum_{i\in p} \|D^i G\|_{q'} \mathrm{md}_0(\hat{w}^{0,l-i}) \mathrm{CT}(q, \hat{w}^{0,l-i}, \ \mathrm{sm}^{k-i}\hat{v}) \times$$

$$\times \mathrm{MC}_q\big(F_i; \ \mathrm{MX}(\hat{w}^{0,l-i}, \mathrm{sm}^{k-i}\hat{v}) \circ \mathrm{mw}\big),$$

what follows from Theorem (23). Following Lemma (27.5), for every i, n, M

$$\|[\hat{y}_M^{n,l-i}, \mathrm{ts}^{-i}\langle \mathrm{sm}^{k-i}\hat{v}, F_i\rangle]\|_q \leqslant \mathrm{VL}(\hat{w}^{0,l-i}) \mathrm{cv}_M \|\langle \mathrm{sm}^{k-i}\hat{v}, F_i\rangle\|_q,$$

hence

$$(35.3)\qquad |b_{n,l-i}\mathrm{mw}^n \langle D^{i+n+e'_M} G, \ [\hat{y}^{n,l-i}, \ \mathrm{ts}^{-i}\langle \mathrm{sm}^{k-i}\hat{v}, F_i\rangle]\rangle|$$

$$\leqslant \mathrm{mw}^{n+e'_M} |b_{n,l-i}| \mathrm{VL}(\hat{w}^{0,l-i}) \|D^{i+n+e'_M} G\|_{q'} \|F_i\|_q.$$

By substituting estimations (35.2) and (35.3) with $n + e'_M \neq 0$ into (35.1) we obtain the required result. ∎

With the aid of the lemmas just proven the following theorems can be stated.

(36) THEOREM. *Assume that* $B \subset \boldsymbol{R}^m$, $p \in \mathrm{SL}^m$ *and* q *is finite. Suppose further that* $\hat{v} \in \mathrm{pc}_0^m$, $\hat{w} \in \mathrm{pc}_s^m$ *and the vectors* k, l *from* I^m *fulfil the inequalities* $k \geqslant \mathrm{UB}(p)$, $l \leqslant \mathrm{LB}(p^+)$, $l \leqslant s+e$. *Let there exist a constant* MD *independent of the mesh width such that* $\mathrm{md}_0(\hat{v}) \leqslant \mathrm{MD}$, $\mathrm{md}_s(\hat{w}) \leqslant \mathrm{MD}$. *Let* $b = \mathrm{sb}\big(B, \bigcup_{0\leqslant n\leqslant s} \mathrm{ST}(\hat{w}^n) - (s+e-l)''\big)$, $B^-[\mathrm{mw}] = \mathrm{SB}(b, \mathrm{ST}\hat{v} - k'')$. *If we define the operators* $\mathrm{rs}_i^{-l}\colon W_q^p(B) \to w_q^p(b)$, $\mathrm{PR}_i^k\colon w_q^p(b) \to W_q^p(B)$, $\mathrm{rs}_e^{-l}\colon \mathring{W}_q^p(B) \to w_q^p(r^m)$, $\mathrm{PR}_e^k\colon w_q^p(r^m) \to W_q^p(B)$, *by the formulas*

$$\mathrm{rs}_i^{-l} F = \langle \mathrm{rd}_b \mathrm{sm}^{-l}\hat{w}, F\rangle, \qquad \mathrm{PR}_i^k f = \mathrm{RD}_B[\mathrm{sm}^k\hat{v}, f],$$

$$\mathrm{rs}_e^{-l} F = \langle \mathrm{sm}^{-l}\hat{w}, F\rangle, \qquad \mathrm{PR}_e^k f = \mathrm{RD}_B[\mathrm{sm}^k\hat{v}, f],$$

then

$$\mathrm{Ap}_i(p, q, B, \hat{v}, k, \hat{w}, -l) = \big\{\big(w_q^p(b)_h, \ \mathrm{rs}_{ih}^{-l}, \ \mathrm{PR}_{ih}^k\big)\big\}_{h\in H}$$

is an internal approximation of $W^p_q(B)$, *and*

$$\mathrm{Ap}_e(p, q, B, \hat v, k, \hat w, -l) = \{(w^p_q(r^m)_h, \mathrm{rs}^{-l}_{eh}, \mathrm{PR}^k_{eh})\}_{e\in H}$$

is an external approximation of $\mathring{W}^p_q(B)$. *Moreover, there exist constants* $\mathrm{KH}(\hat w, l, s)$, $\mathrm{KI}(\hat v, \hat w, k, l, s) \in \boldsymbol{R}$ *and* $\mathrm{KJ}(\hat v, \hat w, k, l, s) \in \boldsymbol{R}^m$, *such that*

$$\|\mathrm{rs}_j^{-l}F\|_{p,q} \leqslant \mathrm{MD}(1+\mathrm{KH}\,|\mathrm{mw}|)\,\|F\|_{p,q},$$

$$\|\mathrm{PR}^k_j f\|_{p,q} \leqslant \mathrm{MD}\,\|f\|_{p,q}, \quad \text{for} \quad j = i, e;$$

$$\|F - \mathrm{PR}^k_i \mathrm{rs}_i^{-l}F\|_{p,q}^{(B^-[\mathrm{mw}])} \leqslant \mathrm{MD}^2\,\mathrm{KI}\cdot \mathrm{MC}_{p,q}(F;\ \mathrm{KJ}\circ \mathrm{mw}),$$

$$\|F - \mathrm{PR}^k_e \mathrm{rs}_e^{-l}F\|_{p,q} \leqslant \mathrm{MD}^2\,\mathrm{KI}\cdot \mathrm{MC}_{p,q}(\mathrm{EX}_m F;\ \mathrm{KJ}\circ \mathrm{mw}).$$

The first inequality can be replaced by an inequality following from Corollary (22.4), *similarly as in Theorem* (23).

Proof. The part of the theorem concerning Ap_i directly follows from Definition (28.4), Lemma (29) and Corollary (32.1), the second part can be proved by applying Definition (28.3), Lemmas (29) and (31). ∎

(37) THEOREM. *Suppose that the assumptions of Theorem* (36) *are satisfied. Let*

$$b = \mathrm{sb}(B, \mathrm{ST}\hat v - k''), \qquad B^-[\mathrm{mw}] = \mathrm{SB}\Big(b, \bigcup_{0\leqslant n\leqslant s} \mathrm{ST}(\hat w^n) - (s+e-l)''\Big).$$

If we define the operators rs^k_i: $W^{-p}_q(B)\to w^{-p}_q(b)$, PR^{-l}_i: $w^{-p}_q(b) \to W^{-p}_q(B)$, rs^k_e: $W^{-p}_q(B)\to w^{-p}_q(r^m)$, PR^{-l}_e: $w^{-p}_q(r^m)\to W^{-p}_q(B)$, *by the formulas*

$$\mathrm{rs}^k_i F = \mathrm{rd}_b\langle \mathrm{sm}^k\hat v, F\rangle, \quad \mathrm{PR}^{-l}_i f = \mathrm{RD}_B[\mathrm{sm}^{-l}\hat w, f],$$

$$\mathrm{rs}^k_e F = \langle \mathrm{sm}^k\hat v, F\rangle, \qquad \mathrm{PR}^{-l}_e f = \mathrm{RD}_B[\mathrm{sm}^{-l}\hat w, f],$$

then

$$\mathrm{Ap}_i(-p, q, B, \hat w, -l, \hat v, k) = \{(w^{-p}_q(b)_h, \mathrm{rs}^k_{ih}, \mathrm{PR}^{-l}_{ih})\}_{h\in H}$$

and

$$\mathrm{Ap}_e(-p, q, B, \hat w, -l, \hat v, k) = \{(w^{-p}_q(r^m)_h, \mathrm{rs}^k_{eh}, \mathrm{PR}^{-l}_{eh})\}_{h\in H}$$

are internal approximations of $W^{-p}_q(B)$. *Moreover, the following estimations are true*:

$$\|\mathrm{rs}^k_j F\|_{-p,q} \leqslant \mathrm{MD}\,\|F\|_{-p,q}, \qquad \|\mathrm{PR}^{-l}_j f\|_{-p,q} \leqslant \mathrm{MD}(1+\mathrm{KH}\,|\mathrm{mw}|)\|f\|_{-p,q}, \quad j = i, e;$$

$$\|F - \mathrm{PR}^{-l}_i \mathrm{rs}^k_i F\|^{(B^-[\mathrm{mw}])}_{-p,q} \leqslant \mathrm{KE}\cdot \mathrm{MC}_{-p,q}(F;\ \mathrm{KF}\circ \mathrm{mw}) + \mathrm{KG}\,|\mathrm{mw}|\cdot\|F\|_{-p,q},$$

$$\|F - \mathrm{PR}^{-l}_e \mathrm{rs}^k_e F\|_{-p,q} \leqslant \mathrm{KE}\cdot \mathrm{MC}_{-p,q}(\mathrm{EX}_m F;\ \mathrm{KF}\circ \mathrm{mw}) + \mathrm{KG}\,|\mathrm{mw}|\cdot\|F\|_{-p,q},$$

where the constants KE, KF, KG, KH *are same as in Lemma* (35) *and Theorem* (36).

Proof. First, let us observe that according to (9.3),

$$\langle G, F\rangle = \langle G, \mathrm{EX}_m F\rangle \quad \text{for every } G \in \mathring{W}^p_{q'}(B),\ F \in W^{-p}_q(B),$$
$$[g, f] = [g, \mathrm{ex}_m f] \quad \text{for every } g \in \mathring{w}^p_{q'}(b),\ f \in w^{-p}_q(b).$$

Thus, following Lemmas (33) and (34) we have

$$\langle G, \mathrm{PR}_j^{-l} f\rangle = [f, \mathrm{rs}_j^{-l} G] \quad \text{and} \quad [g, \mathrm{rs}_j^k F] = \langle F, \mathrm{PR}_j^k g\rangle,$$

where PR_j^k and rs_j^{-l} $(j = i, e)$ are the same as in Theorem (36). Hence the first two estimates can be obtained as in Theorem (36), the last two follow from Lemma (35). ∎

Let us now consider an approximation of the spaces $C^r(B)$. In the end of the previous section it was said that to build an internal approximation of $C^0(B)$ we need a function $\hat{v} \in \mathrm{pc}_l^m \cap (r^m \to C^0(\boldsymbol{R}^m))$. As can easily be seen, such a condition is fulfilled if we take $\hat{v} = \mathrm{sm}^e \hat{z}$, $\hat{z} \in \mathrm{pc}_l^m$. But at the same time we can take a larger class of the functions $\hat{w}$ used for building the restriction operator for an internal as well as external approximation. Indeed, if $\hat{z} \in \mathrm{pc}_0^m$ then for each $x \in r^m$ the distribution $\mathrm{sm}^{-e}\hat{z}(x)$, which is a linear continuous functional on the space $W_1^{e''}(\boldsymbol{R}^m)$, can be prolongated in natural way onto the space $C^0(\boldsymbol{R}^m)$. An example of such a distribution is $sm^{-e}\hat{c}$ (where $\hat{c}$ is the characteristic function defined by (18.7)), which is the Dirac distribution

$$\langle \mathrm{sm}^{-e}\hat{c}(x), F\rangle = F(x + \theta \circ \mathrm{mw}).$$

Therefore the following corollaries are true.

(38.1) COROLLARY. *Let $B \subset \boldsymbol{R}^m$. Assume that $\hat{v} \in \mathrm{pc}_0^m$, $\hat{w} \in \mathrm{pc}_s^m$, and the vectors k, l from I^m fulfil the inequalities*

if $q = \infty$, $p \in \mathrm{SL}^m$, then $k \geqslant \mathrm{UB}(p) + e$, $l \leqslant \mathrm{LB}(p^+) + e$, $l \leqslant s + e$;
if $q = (\infty, qi, K)$, $p \in \mathrm{SL}_K^m$, then $k \geqslant \mathrm{UB}(p) + e'_{\setminus K}$, $l \leqslant \mathrm{LB}(p^+) + e'_{\setminus K}$, $l \leqslant s + e$;
if $q = (qx, \infty, K)$, $p \in \mathrm{SL}_K^m$, then $k \geqslant \mathrm{UB}(p) + e'_K$, $l \leqslant \mathrm{LB}(p^+) + e'_K$, $l \leqslant s + e$.

Let the symbol qc be obtained from q by setting $*$ instead of ∞. If we define the sets b, $B^-[\mathrm{mw}]$, and the operators rs_i^{-l}: $W^p_{qc}(B) \to w^p_q(b)$, PR_i^k: $w^p_q(b) \to W^p_{qc}(B)$, rs_e^- : $\mathring{W}^p_{qc}(B) \to w^p_q(r^m)$, PR_e^k: $w^p_q(r^m) \to W^p_{qc}(B)$, by the same formulas as in Theorem (36), then $\mathrm{Ap}_i(p, q, B, \hat{v}, k, \hat{w}, -l)$ is an internal approximation of $W^p_{qc}(B)$, and $\mathrm{Ap}_e(p, q, B, \hat{v}, k, \hat{w}, -l)$ is an external approximation of $\mathring{W}^p_{qc}(B)$. The estimations given in (36) are satisfied. Moreover, if the vector k satisfies only the inequality $k \geqslant \mathrm{UB}(p)$, then PR_i^k: $w^p_q(b) \to W^p_q(B)$, PR_e^k: $w^p_q(r^m) \to W^p_q(B)$, and the operators quoted above form only an external approximation of $W^p_{qc}(B)$.

(38.2) COROLLARY. *Let $B \subset \boldsymbol{R}^m$. Assume that $\hat{v} \in \mathrm{pc}_0^m$, $\hat{w} \in \mathrm{pc}_s^m$, and the vectors k, l from I^m fulfil the inequalities*

if $q = \infty$, $p \in \mathrm{SL}^m$, then $k \geqslant \mathrm{UB}(p) - e$, $l \leqslant \mathrm{LB}(p^+) - e$, $l \leqslant s+e$;

if $q = (\infty, qi, K)$, $p \in \mathrm{SL}^m_K$, then $k \geqslant \mathrm{UB}(p) - e'_{\setminus K}$, $l \leqslant \mathrm{LB}(p^+) - e'_{\setminus K}$, $l \leqslant s+e$;

if $q = (qx, \infty, K)$, $p \in \mathrm{SL}^m_K$, then $k \geqslant \mathrm{UB}(p) - e'_K$, $l \leqslant \mathrm{LB}(p^+) - e'_K$, $l \leqslant s+e$.

Let the symbol qc be obtained from q by setting $*$ instead of ∞. If we define the sets b, B^- [mw], and the operators $\mathrm{rs}_i^k\colon W_{qc}^{-p}(B) \to w_q^{-p}(b)$, $\mathrm{PR}_i^{-l}\colon w_q^{-p}(b) \to W_{qc}^{-p}(B)$, $\mathrm{rs}_e^k\colon W_{qc}^{-p}(B) \to w_q^{-p}(r^m)$, $\mathrm{PR}_e^{-l}\colon w_q^{-p}(r^m) \to W_{qc}^{-p}(B)$, by the same formulas as in Theorem (37), *then $\mathrm{Ap}_i(-p, q, B, \hat{w}, -l, \hat{v}, k)$ and $\mathrm{Ap}_e(-p, q, B, \hat{w}, -l, \hat{v}, k)$ are internal approximations of $W_{qc}^{-p}(B)$. The estimations given in* (37) *are true. Moreover, if the vector l satisfies only the inequalities $l \leqslant \mathrm{LB}(p^+)$, $l \leqslant s+e$, then $\mathrm{PR}_i^{-l}\colon w_q^{-p}(b) \to W_q^{-p}(B)$, $\mathrm{PR}_e^{-l}\colon w_q^{-p}(r^m) \to W_q^{-p}(B)$, and the operators quoted above form only an external approximation of $W_{qc}^{-p}(B)$.*

Similarly as Theorem (24), the following result concerning the approximation of local spaces can be proved.

(38.3) COROLLARY. *Theorems* (36), (37) *and Corollaries* (38.1), (38.2) *remain true if we substitute all the spaces occurring there by the local spaces with the same indices. The estimations of the seminorms are analogous to those obtained in Theorem* (24).

The last question discussed in the present section is the existence of a reflexive approximation of $W_q^p(\boldsymbol{R}^m)$, i.e. an approximation satisfying (2.4). The problem is solved, however, only in the case when $p = n''$, $k = l = n$.

Let us thus consider $\mathrm{Ap}_i(n'', q, \boldsymbol{R}^m, \hat{v}, n, \hat{w}, -n)$. According to the definition of Ap_i, condition (2.4) is equivalent to

$$\forall f \in w_q^{n''}(r^m) \qquad \langle \mathrm{sm}^{-n}\hat{w}, [\mathrm{sm}^n\hat{v}, f]\rangle = f,$$

and hence to

$$\forall x, y \in r^m \quad \langle \mathrm{sm}^{-n}\hat{w}(x), \mathrm{sm}^n\hat{v}(y)\rangle = \mathrm{cv}^{-1}\delta(x=y), \tag{39.1}$$

what can be proved by using Lemma (33). Let us take

$$\hat{v} = \hat{c}, \quad \hat{w} = \sum_{0 \leqslant i \leqslant (n-e)^+} t_i \Delta^{*i} \mathrm{ts}^{(n-e)^+ - n} \mathrm{sm}^{n-i} \hat{c} \quad \text{with} \quad t_0 = 1. \tag{39.2}$$

Following formulas (28.1,2),

$$\mathrm{sm}^{-n}\hat{w} = \sum_{0 \leqslant i \leqslant (n-e)^+} t_i \mathrm{mw}^i D^i \mathrm{ts}_K^- \mathrm{sm}_K^- \hat{c}, \quad \text{where } K = \{j \in \overline{m}\colon n_j \geqslant 1\}.$$

Let us compute the left-hand side of the equality in (39.1). The application of formula (9.2), Lemma (27.2) and Definition (13.8) yields

$$\langle \mathrm{sm}^n \hat{c}(x),\ D^i \mathrm{sm}_{\bar{K}} \hat{c}\rangle = \langle D^{*i} \mathrm{sm}^n \hat{c}(x),\ \mathrm{sm}_{\bar{K}} \hat{c}\rangle = (-1)^{|i|} \langle \partial^{*i} \mathrm{sm}^{n-i} \hat{c}(x),\ \mathrm{sm}_{\bar{K}} \hat{c}\rangle$$
$$= \langle \mathrm{ts}^{-i} \partial^i \mathrm{sm}^{n-i} \hat{c}(x),\ \mathrm{sm}_{\bar{K}} \hat{c}\rangle.$$

Further, by applying Definitions (25.1) and (27.1), condition (18.5), formula (28.5) and Lemma (9.10), we obtain the sequence of equalities (for $j \geqslant e$)

$$\langle \mathrm{sm}^j \hat{c}(x),\ \mathrm{ts}_{\bar{K}} \mathrm{sm}_{\bar{K}} \hat{c}\rangle = \langle \partial^*_K \mathrm{IN}_K \mathrm{sm}^{j-e'_K} \hat{c}(x),\ \mathrm{ts}_{\bar{K}} \mathrm{sm}_{\bar{K}} \hat{c}\rangle$$
$$= \langle \mathrm{IN}_K \mathrm{sm}^{j-e'_K} \hat{c}(x),\ \partial_K \mathrm{ts}_{\bar{K}} \mathrm{sm}_{\bar{K}} \hat{c}\rangle$$
$$= \langle D_K \mathrm{IN}_K \mathrm{sm}^{j-e'_K} \hat{c}(x),\ \hat{c}\rangle = \langle \mathrm{sm}^{j-e'_K} \hat{c}(x),\ c\rangle$$

(the third term should be understood in the sense of Definition (28.4)). Hence

$$\langle \mathrm{sm}^n \hat{c}(x),\ D^i \mathrm{sm}_{\bar{K}} \hat{c}(y)\rangle = \langle \mathrm{ts}^{-i} \partial^i \mathrm{sm}^{n-i-e'_K} \hat{c}(x),\ \hat{c}(y)\rangle$$

and thus condition (39.1) is equivalent to

$$\forall x,\ y \in r^m \qquad \sum_{0 \leqslant i \leqslant (n-e)^+} t_i \langle \mathrm{ts}^{-i} \Delta^i \mathrm{sm}^{n-i-e'_K} \hat{c}(x),\ \hat{c}(y)\rangle = \mathrm{mw}^{-e}\, \delta(x = y).$$

Following (18.5), it is sufficient to consider an arbitrary but fixed point x, e.g. $x = 0$, and following Lemma (27.2) and property (13.9), it suffices to take y from

$$\bigcup_{0 \leqslant i \leqslant (n-e)^+} \mathrm{ts}^{-i} \mathrm{nb}\big(0,\ i'' - (n-i-e'_K)''\big) = \{y \in r^m:\ -(n-e)^+ \circ \mathrm{mw} \leqslant y \leqslant 0\}.$$

Hence we have obtained the system of $\prod_{j \in K} n_j$ linear equations with the same number of unknowns t_i (including t_0). It is sufficient, however, to consider $\hat{w}$ as the product of one-dimensional functions,

$$\hat{w}(x,\ X) = \prod_{j \in \bar{m}} \hat{w}_j(x_j,\ X_j), \qquad \text{where} \qquad \hat{w}_j = \sum_{0 \leqslant k \leqslant (n_j-1)^+} s^j_k \mathrm{ts}^{(n_j-1)^+ - n_j} \mathrm{sm}^{n_j-k} \hat{c}_j.$$

We have thus for each $j \in K$ the system of linear equations

$$\text{(39.3)} \qquad \sum_{0 \leqslant k \leqslant n_j - 1} s^j_k \mathrm{mw}_j \langle \mathrm{ts}^{-k} \Delta^k \mathrm{sm}^{n_j-k-1} \hat{c}_j(0),\ \hat{c}_j(-l \cdot \mathrm{mw}_j)\rangle = \delta_{0l},$$
$$l = 0, 1, \ldots, n_j - 1,$$

and for $j \in \setminus K$ — the single equation

$$s^j_0 \mathrm{mw}_j \langle \hat{c}_j(0),\ \hat{c}_j(0)\rangle = 1,$$

which is satisfied with $s_0^j = 1$. As can easily be checked, the transposition of the matrix of system (39.3) satisfies the assumptions of Lemma (40), formulated at the end of the present section, with $n = n_j$ and $b_k = 1$ for each k. By applying this lemma we conclude that system (39.3) is uniquely solvable and that $s_j^0 = 1$, hence $t_0 = 1$ as has been required.

Similar results can be obtained for $\mathrm{Ap}_i(-n'', q, \boldsymbol{R}^m, \hat{w}, -n, \hat{v}, n)$, therefore we have shown the following corollary.

(39.4) COROLLARY. *The approximations* $\mathrm{Ap}_i(-n'', q, \boldsymbol{R}^m, \hat{w}, -n, \hat{v}, n)$ *and* $\mathrm{Ap}_i(n'', q, \boldsymbol{R}^m, \hat{v}, n, \hat{w}, -n)$ *are reflexive, if* $\hat{v}, \hat{w}$ *are defined as in* (39.2) *with the coefficients* $t_i = \prod_{j\in\overline{m}} s_{i_j}^j$ *and* s_k^j *determined by linear system* (39.3).

At last, let us prove the algebraic lemma which has been used in the proof of corollary (39.4).

(40) ALGEBRAIC LEMMA. *Let us construct the sequence of matrices as follows.*

$$A_n = [a_{ij}^{(n)}]_{i,j=0,\ldots,n-1},$$

where

$$\sum_{k=0}^{n-1} a_{0k}^{(n)} = b_n, \quad a_{ij}^{(n+1)} = a_{i-1,j}^{(n)} - a_{i-1,j-1}^{(n)} \quad \textit{for} \quad n \geqslant 1,$$
$$1 \leqslant i \leqslant n, \ 0 \leqslant j \leqslant n;$$

$$a_{ij}^{(n)} = 0 \quad \textit{for} \quad 0 \leqslant i \leqslant n-1, \ j = -1 \ \textit{or} \ j = n.$$

Then $\det A_n = \prod_{i=1}^{n} \big((-1)^{i-1} b_i\big)$.

Proof. The proof is carried out by induction with respect to n. If $n = 1$ then the equality holds. Let us assume that it is true for $n = k$. Then

$$\det A_{k+1} = \sum_{j=0}^{k} (-1)^j a_{0j}^{(k+1)} \det A_{k+1,j},$$

where A_{nj} is obtained from A_n by striking out the row number 0 and jth column. Let us denote the jth column of A_n by $c_j^{(n)}$. Then following the definition of A_n,

$$A_{k+1,j} = [c_0^{(k)}, c_1^{(k)} - c_0^{(k)}, \ldots, c_{j-1}^{(k)} - c_{j-2}^{(k)}, c_{j+1}^{(k)} - c_j^{(k)}, \ldots, c_{k-1}^{(k)} - c_{k-2}^{(k)}, -c_{k-1}^{(k)}].$$

By the elementary transformations of the determinant we obtain that

$$\det A_{k+1,j} = (-1)^{k-j} \det A_k,$$

therefore

$$\det A_{k+1} = \sum_{j=0}^{k} a_{0j}^{(k+1)} (-1)^k \det A_k = (-1)^k b_{k+1} \det A_k,$$

and our lemma can be proved by applying the inductive assumption. ∎

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Presented to the Semester
Computational Mathematics
February 20 – May 30, 1980