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THE RIEMANNIAN CURVATURE TENSOR AND DIFFERENTIABLE SPACES

ADAM KOWALCZYK

Institute of Mathematics, Technical University of Warsaw, Warsaw, Poland

Introduction

In this paper we consistently use the notion of differential space introduced by R. Sikorski [13], [15] and we treat the subject from this point of view. As follows from papers of P. Walczak [16], [17], for the particular case of differentiable spaces which appear in this paper, our aproach is equivalent to that proposed by Aronszajn and Szeptycki [1] and developed in papers of C. Marshall and M. Breuer [2], [9], [10]. For all basic definitions we refer to Sikorski's papers [13], [15] or to a paper by Mostov [12], where a brief introduction to the theory of differentiable spaces is given.

If M is a differentiable space (for short a d-space), then $C^\infty(M)$ denotes the differential structure (for short the d-structure) of M. For a set $\{\beta_s\}$ of functions on M denote by $\mathrm{Gen}\{\beta_s\}$ the smallest d-structure on M containing $\{\beta_s\}$. The union TM of all tangent spaces M_p to the d-space M has the natural d-structure $C^\infty(TM)$ called the *structure of the tangent d-space to M* (see [6]). If $C^\infty(M) = \mathrm{Gen}\{\beta_s\}$, then $C^\infty(TM) = \mathrm{Gen}(\{\beta_s \circ \pi\} \cup \{d\beta_s\})$, where $\pi\colon TM \to M$ is the natural projection and

$$d\beta_s(v) := v(\beta_s) \quad \text{for any } v \in TM.$$

Denote by TTM the tangent d-space to TM and let $TM_v^\perp$, where $v \in TM$, denotes the set $\{V \in TM;\ \pi_* v = 0\}$, where $\pi_*\colon TTM \to TM$ is the tangent mapping to π.

Below, by a Euclidean space E^n we mean the set of all real n-sequences $p = (p^i) = (p^1, \ldots, p^n)$ with the usual scalar product. If it is convenient, the points of $E^n \times E^m$ (resp. E^{2n}, E^{3n}, etc.) are denoted by (p^i, q^j) (resp. (p^i, q^i), (p^i, q^i, w^i), etc.). For $0 < n \leqslant m$ the Euclidean space E^n is identified with the subspace of E^m spanned by the first n coordinate axes.

Let $f \in C^\infty(E^n)$, $h = (h^i) \in E^n$, $h' = (h'^i) \in E^n$ and $p \in E^n$. We denote

$$\partial_h f(p) := \sum_{i=1}^{n} f_{,i}(p) h^i,$$

$$\partial_{h'}\partial_h f(p) := \sum_{i,j=1}^{n} f_{,ij}(p) h^i h'^j,$$

$$\partial_h^2 f(p) := \partial_h \partial_h f(p),$$

where $f_{,i}(p)$, $f_{,ij}(p)$ denote the respective partial derivatives.

In this paper, by a d-space we mean a space which is, up to an imbedding, a differentiable subspace (for short a *d-subspace*) of a certain Euclidean space. In [2], [4], [11] it was proved that if a d-space M is Lindelöf, as a topological space, and each point $p \in M$ has a neighbourhood U_p, which may be imbedded in E^n, i.e., $\dim M \leqslant n$, then M may be imbedded in E^{2n+1}. For a point p of a d-space M we denote by $\mathcal{M}_p M$ the set of all imbeddings x of an open neighbourhood U_x of p into E^n, where $n = \dim M_p$ (we define E^0 as the d-space consisting of 1 point). It can be proved (see [4]) that $\mathcal{M}_p M \neq \emptyset$.

We define the mapping $\underset{\sim}{j}$: $TE^n \to E^n$ by the formula

$$\underset{\sim}{j}(v) := \big(v(\pi^i)\big) \quad \text{for any } v \in TE^n,$$

where $\pi^1, \ldots, \pi^n$ denote the canonical projections on coordinate axes. The mapping $TE^n \ni v \mapsto \big(\pi(v), \underset{\sim}{j}(v)\big) \in E^n \times E^n$ is a diffeomorphism (see [6]).

Let M be a d-subspace of E^n and let $\underset{\sim}{i}$: $M \to E^n$ denote the inclusion mapping. The set $C^\infty(M) = \operatorname{Gen}\{\pi^i|M\}$ consists of all functions $\alpha|M$ where α is a C^∞-function on an open neighbourhood U of M. We denote

$$\underline{v} := \underset{\sim}{j} \circ \underset{\sim}{i}_*(v) = \big(v(\pi^i|M)\big) \in E^n \quad \text{for any } v \in M_p,$$

$$\underline{M_p} := \underset{\sim}{j}(M_p) \subset E^n \quad \text{for any } p \in M,$$

$$\underline{TM} := \{\big(\pi(v), \underline{v}\big) \in E^n \times E^n;\ v \in TM\}.$$

It can be proved [4], [10] that $\underline{M_p} = \{h \in E^n;\ \partial_h \alpha(p) = 0$ for any $\alpha \in C^\infty(E^n)$, $\alpha|M = \text{const}\}$.

The mapping $TM \ni v \mapsto \big(\pi(v), \underset{\sim}{j} \circ \underset{\sim}{i}_*(v)\big) \in \underline{TM} \subset E^n \times E^n$ is the natural diffeomorphism of the d-space TM and the d-subspace $\underline{TM}$ of E^{2n}. Thus, in the above manner, the tangent differential space TTM to TM may be identified with the d-subspace $\underline{TTM}$ $(:= \underline{T(\underline{TM})})$ of E^{4n}, and TM_v, where $v \in TM$, may be identified with the corresponding linear subspace $\underline{TM_v}$ $(:= \underline{(\underline{TM})_v})$ of E^{2n}. These identifications are given by the formulas:

$$TM_v \ni V \mapsto \big(V(\pi^i \circ \pi), V\big(d(\pi^i|M)\big)\big) \in \underline{TM_v} \subset E^{2n},$$

$$TTM \ni (v, V) \mapsto \big(\pi(v), \underline{v}, V(\pi^i \circ \pi), V\big(d(\pi^i|M)\big)\big) \in \underline{TTM} \subset E^{4n}.$$

Let M be a d-space, $p \in M$, $n := \dim M_p$ and $x \in \mathcal{M}_p M$ (note that x: $U_x \to E^n$). For brevity we denote

$$\underline{x_*}\,\underline{M_p} := \underset{\sim}{j} \circ x_*(M_p) \quad \text{for any } p \in M,$$

$$\partial_h^i \alpha(p)|_x := \partial^i_{\underline{x_*h}} \alpha\big(x(p)\big) \quad \text{for any } p \in M, h \in M_p, \alpha \in C^\infty(E^n), \quad i = 1, 2.$$

We write $\partial_h \alpha(p)|_x$ instead of $\partial_h^1 \alpha(p)|_x$.

It is not difficult to verify that a sequence (v_n) of elements TM converges to a vector $v \in TM$ (in the topology of the tangent d-space, see [6]) if and only if for a map $x \in \mathcal{M}_{\pi(v)} M$ we have

$$x \circ \pi(v) = \lim_n x\pi(v_n),$$
$$\underline{x_* v} = \lim_n \underline{x_* v_n}.$$

§ 1. The second tangent space to a d-space

PROPOSITION 1.1. *Let M be a d-space, $p \in M$ and $v \in M_p$. The vector spaces M_p and $TM_v^\perp$ are isomorphic.*

In fact, the isomorphism $I\colon M_p \to TM_v^\perp$ may be defined as follows:

$$(Iw)(\alpha \circ \pi) := 0,$$
$$(Iw)(d\alpha) = \lim_{\varepsilon \to 0} \big(d\alpha(v+\varepsilon w) - d\alpha(v)\big)/\varepsilon,$$

for any $v \in M_p$ and any $\alpha \in C^\infty(M)$.

COROLLARY 1.1. *We have*

(a) $\dim TM_v^\perp = \dim M_p$ *for any* $v \in M_p$.

(b) $\dim TM_v = \dim M_p + \dim(\pi_* TM_v)$.

PROPOSITION 1.2. *For any d-subspace M of E^n, any point $p \in M$ and any $v \in M_p$ the following relations holds*:

$$\underline{\pi_* V} = \big(V(\pi^i \circ \pi)\big) \in E^n \quad \textit{for any } V \in TM_v, \tag{1}$$

$$\big(V(d(\pi^i|M))\big) \in \underline{M_p} \subset E^n \quad \textit{for any } V \in TM_v^\perp, \tag{2}$$

where $\pi^i\colon E^n \to E$ is the projection on the i-th coordinate axis.

Proof. The proof of (1) is trivial. For the proof of (2) take a vector $V \in TM_v$ and denote $\big(V(d\pi^i|M)\big) \in E^n$ by h. From the proof of Proposition 1.1 it follows that there exists a vector $w \in M_p$ such that

$$V(\beta) = (Iw)(\beta) = \lim_{\varepsilon \to 0} \big(d\beta(v+\varepsilon w) - d\beta(v)\big)/\varepsilon$$

for any $\beta \in TM$. For $\alpha \in C^\infty(E^n, M)$ we obtain

$$V(d\alpha) = \lim_{\varepsilon \to 0} \big((v+\varepsilon w)(\alpha|M) - v(\alpha|M)\big)/\varepsilon = w(\alpha|M) = 0.$$

Because $d(\alpha|M) = \sum_{i=1}^n \alpha_{,i} \circ \pi \, d(\pi^i|M)$, we have

$$0 = V\big(d(\alpha|M)\big) = \sum_{i=1}^n \alpha_{,i}(p)\, V\big(d(\pi^i|M)\big) = \partial_h \alpha(p).$$

Thus $h \in M_p \in E^n$ by an arbitrary choice of $\alpha \in C^\infty(E^n, M)$. ∎

The following example shows that, contrary to the case of a manifold, the equality

$$\dim TM_v = \dim TM_{v'} \quad \text{if} \quad v, v' \in M_p$$

does not always hold.

EXAMPLE 1.1. Consider a d-space $M := \{0\} \cup \{1/n;\ n = 1, 2, \dots\}$ of E. Then $\dim M_0 = 1$ and $\dim M_x = 0$ for other points x of M. Therefore

$$\underline{TM} = \{(1/n, 0) \in E^2;\ n = 1, 2, \dots\} \cup \{(0, y);\ y \in E\}$$

and

$$\underline{TTM} = \{(1/n, 0, 0, 0);\ n = 1, 2, \dots\} \cup \{(0, y, 0, \eta);\ 0 \neq y \in E, \eta \in E\} \cup \cup \{(0, 0, \xi, \eta);\ \xi, \eta \in E\}.$$

Thus $\dim TM_{(0,0)} = 2 \neq 1 = \dim TM_{(0,1)}$.

PROPOSITION 1.3. *Let M be a d-space, $p \in M$, $v \in TM$. Then*

$$\pi_*(TM_v) = \pi_*(TM_{\alpha v})$$

for any real $\alpha \neq 0$.

In fact, the mapping $TM \ni v \to \alpha v \in TM$ is a diffeomorphism for any $\alpha \neq 0$.

PROPOSITION 1.4. *Let X be a smooth vector field on a d-space M and $p \in M$. Then*

$$\pi_* TM_{X(p)} = M_p.$$

Proof. According to [6], the vector field X can be identified with a smooth mapping $X\colon M \to TM$ such that $\pi \circ X = \mathrm{id}_M$; so $\pi_* \circ X_*(M_p) = M_p$. ∎

COROLLARY 1.2. *Let 0 be the zero-vector of M_p. Then*

$$\pi_*(TM)_0 = M_p.$$

DEFINITION. A point p of a d-space M is called *regular* if there exists an open neighbourhood U of p such that

$$\dim M_q = \dim M_p$$

for any $q \in U$. The set of all regular points of M we denote by $M_=$.

It can be proved (see [6], [10]) that

1° $M_=$ is an open and dense subset of M,

2° $p \in M_=$ iff any vector $v \in M_p$ can be extended to a smooth vector field X on M.

COROLLARY 1.3. *We have*

(a) *$M_p = \pi_*(TM_v)$ for any $p \in M_=$ and any $v \in M_p$.*

(b) *The set $\{p \in M;\ M_p = \pi_*(TM_v)$ for any $v \in M_p\}$ contains an open and dense subset of M.*

PROPOSITION 1.5. *Let M be a d-space and $v \in TM$. The following two conditions are equivalent:*

(a) $\pi_*(TM_v) \neq \{0\}$,

(b) *There exists a sequence* (v_n) *of elements of TM such that* $\lim v_n = v$ *and* $\pi(v_n) \neq \pi(v)$.

Proof. It is sufficient to consider the particular case where M is a d-subspace of E^n and $n = \dim M_{\pi(v)}$. Suppose (b) and denote $p := \pi(v)$, $p_n := \pi(v_n)$ for $n = 1, 2, \ldots$ Passing if necessary, to a subsequence, we may assume that the sequence $((p-p_n)/|p-p_n|)$ converges (in E^n) to a limit h ($\neq 0$). Note that

$$\partial_{(h,0)}\alpha(p, v) = \lim_{n\to\infty} (\alpha(p, \underline{v}_n)-\alpha(p_n, \underline{v}_n))/|p-p_n|$$

for any $\alpha \in C^\infty(E^{2n})$, and $\partial_{(h,0)}\alpha(p, v) = 0$ if $\alpha|\underline{TM} = \text{const}$. Consequently $(h, 0) \in \underline{TM}_v$ and $\pi_* \underline{TM}_v \neq \{0\}$.

Suppose that (b) is not satisfied, i.e., there exists an open neighbourhood U of v in TM such that $TM \cap U \subset M_p$. Consequently, if $\alpha \in C^\infty(M)$, then $\alpha \circ \pi(w) = \text{const} = \alpha(v)$ *for any* $w \in U$. Hence, for any vector $V \in TM_v$ we get $\pi_* V(\alpha) = V(\alpha \circ \pi) = 0$; so (a) cannot be satisfied. ■

§ 2. The set $\mathscr{P}(TTM)$

We define $C_p^\infty := C^\infty(E^n)$, $C_{p,x} := \{\alpha \in C^\infty(E^n);\ \alpha|x(U_x) = 0\}$ and $\partial_h^2 \alpha(p)|_x := \partial^2_{x_*h}\alpha(x, p)$ for any $\alpha \in C_p^\infty$, $h \in M_p$.

NOTATION. Let M be a d-space, $p \in M$ and $x \in \mathscr{M}_p M$. We denote

$$\mathscr{P}_x M_p := \{h \in M_p;\ \partial_h^2 \alpha(p)|_x = 0 \text{ for any } \alpha \in C^\infty_{p,x}\},$$
$$\mathscr{P}M_p := \bigcap_{x \in \mathscr{M}_p M} \mathscr{P}_x M_p,$$
$$\mathscr{P}TM := \bigcup_{p \in M} \mathscr{P}M_p,$$
$$\mathscr{P}M := \{p \in M;\ M_p = \mathscr{P}M_p\}.$$

PROPOSITION 2.1. *For any d-space M and any vector $v \in TM$ the following inclusion holds:*

$$TM_v^\perp \subset \mathscr{P}(TM_v).$$

In fact, from the proof of Proposition 1.2, for any $V \in TM_v^\perp$, follows the existence of a smooth curve c: $(-\varepsilon, \varepsilon) \to TM$ such that $c(0) = v$ and $\dot{c}(0) = V$. By [8] V belongs to $\mathscr{P}(TM_v)$.

COROLLARY 2.1. *If V_1 and V_2 are vectors of $\mathscr{P}(TM_v)$ such that $\pi_* V_1 = \pi_* V_2$, then* $\operatorname{span}\{V_1, V_2\} \subset \mathscr{P}(TM_v)$.

Proof. The only interesting case is that of $\dim(\operatorname{span}\{V_1, V_2\}) = 2$ and $\pi_* V_i \neq 0$. As $V_1 - V_2 \in TM_v^\perp \subset \mathscr{P}TM_v$, the plane $\operatorname{span}\{V_1, V_2\}$ contains three .vectors such that any two of them are linearly independent. Thus $\operatorname{span}\{V_1, V_2\} \subset \mathscr{P}TM_v$ (see [8]). ■

PROPOSITION 2.2. *Let p be a point of a d-subspace M of E^n, let $(p_n, \underline{v}_n)$ be a sequence of elements of TM tending to $(p, \underline{v}) \in \underline{TM}$, $p \neq p_n$ for any n and let*

$$h = \lim((p_n-p)/|p_n-p|) \in E^n.$$

Then $(p, h) \in \underline{\pi}_*(\underline{\mathscr{P}TM}_v)$.

Proof. From [8] it follows that there exists a subsequence $(p_{n_i}, \underline{v}_{n_i})$ such that the vector

$$\underline{V} = \lim\left((p_{n_i}-p, \underline{v}_{n_i}-\underline{v})/\sqrt{|p_{n_i}-p|^2+|\underline{v}_{n_i}-\underline{v}|^2}\right)$$

belongs to $\underline{\mathscr{P}TM}_v$. Now notice that there exists a number c such that $(p, ch) = \underline{\pi}_* V$.

PROPOSITION 2.3. *Let p be a regular point of a d-space M. Then $\pi_*(\mathscr{P}TM_v) = M_p$ for any $v \in M_p$.*

In fact, if $\dim N_p = n$ for any $p \in N$, then $\dim TN_v = 2n$ for any $v \in TN$; hence, $T(TN) = \mathscr{P}TN$ (see [8]), which implies the above statement.

COROLLARY 2.2. *The set $\{p;\ \pi_*\mathscr{P}(TM_v) = M_p \text{ for any } v \in M_p\}$ contains a dense and open subset of M.*

In fact, the subset $M_{\underline{\ }}$ of M is open and dense in M.

§ 3. First and second derivatives of the coefficients of a Riemannian metric

Let U an open subset in E^n, $W \subset U$. We denote by $Q(U, W)$ the set of all smooth functions $\omega(q, v)$ on $TU = U \times E^n$ such that

$$(*) \qquad \omega(q, \alpha v) = \alpha^2 \omega(x, v)$$

for any $(q, v) \in U \times E^n$ and any real $\alpha \geqslant 0$,

$$(**) \qquad \omega(q, v) = 0 \quad \text{for any } (q, v) \in \underline{TW} \subset U \times E^n.$$

Denote by $Q(U)$ the set of all functions satisfying only $(*)$.

If M is a d-space, $p \in M$, $n = \dim M_p$ and $x \in \mathscr{M}_p M$, then we denote by $Q_{p,x}$, Q_p the sets $Q(E^n, x(U_x))$ and $Q(E^n)$, respectively. It can be proved [8] that each $\omega \in Q(E^n)$ is a smooth quadratic form on E^n, i.e., there exist $\omega_{ij} \in C^\infty(E^n)$ such that $\omega(x, y) = \sum \omega_{ij}(x) y^i y^j$.

DEFINITION 3.1. Let p be a point of a d-space M. For a map $x \in \mathscr{M}_p M$ we denote by $\mathscr{F}_x M_p$ (resp. $\mathscr{S}_x M_p$) the set of all pairs $(h, v) \in M_p \times M_p$ such that

$$\partial_h \omega(p, v)|_x := \partial_{\underline{x_* h}} \omega(x(p), \underline{x_* v}) = 0$$

(resp.

$$\partial_h^2 \omega(p, v)|_x := \partial^2_{\underline{x_* h}} \omega(x(p), \underline{x_* v}) = 0)$$

for any $\omega \in Q_{p,x}$. We set

$$\mathscr{F}M_p := \bigcap_{x \in \mathscr{M}_p M} \mathscr{F}_x M_p,\ \mathscr{S}M_p := \bigcap_{x \in \mathscr{M}_p M} \mathscr{S}_x M_p,\ \mathscr{F}M := \{p \in M; M_p \times M_p = \mathscr{F}M_p\}$$

and

$$\mathscr{S}M := \{p \in M;\ M_p \times M_p = \mathscr{S}M_p\}.$$

PROPOSITION 3.1. *Let M and N be d-spaces, let $f\colon M \to N$ be a diffeomorphism and $p \in M$. Then $(h, v) \in \mathscr{F}M_p$ $(\in \mathscr{S}M_p)$ if and only if $(f_* h, f_* v) \in \mathscr{F}N_{f(p)}(\in \mathscr{S}N_{f(p)})$. Consequently, the equalities $f(\mathscr{F}M) = \mathscr{F}N$ and $f(\mathscr{S}M) = \mathscr{S}N$ hold.*

PROPOSITION 3.2. *Let M be a d-space, $p \in M$ and $x \in \mathscr{M}_p M$. Then:*

(a) $\mathscr{F}M_p = \mathscr{F}_x M_p$,

(b) *If $p \in \mathscr{F}M$, then $\mathscr{S}_x M_p = \mathscr{S}M_p$.*

The proof is obtained by straightforward calculations.

COROLLARY 3.1. *Let N be a d-subspace of M, $p \in N$ and $x \in \mathscr{M}_p N$. Then $\mathscr{F}_x N_p \subset \mathscr{F}M_p$ and $\mathscr{F}N_p \subset \mathscr{F}M_p$. If $p \in \mathscr{F}M$, then $\mathscr{S}_x N_p \subset \mathscr{S}M_p$.*

PROPOSITION 3.3. *Let M be a d-space, $p \in M_p$ and $x \in \mathscr{M}_p M$. Then $\mathscr{S}_x M_p \subset \mathscr{F}_x M_p$. Consequently $\mathscr{S}M_p \subset \mathscr{F}M_p$.*

Proof. Let $n = \dim M_p > 0$, $(h, v) \in \mathscr{S}_x M_p$ and $\omega \in Q_{p,x}$. We show that $\partial_h \omega(p, v)|_x = 0$. Denote by $(|)$ the scalar product in E^n. It is evident that the function $\omega'\colon E^n \times E^n \to E$ defined by the formula

$$\omega'(\xi, u) := \omega(\xi, u)(\xi|\underline{x_* h}) \quad \text{for any } (\xi, u) \in E^n \times E^n$$

belongs to $Q_{p,x}$. As $\partial_h^2 \omega(p, v)|_x = 0$ for any $v \in M_p$, we obtain

$$0 = \partial_h^2 \omega'(p, v)|_x = 2(h|h)\, \partial_h \omega(p, v)|_x$$

for any $v \in M_p$; hence $\partial_h \omega(p, v)|_x = 0$. ∎

COROLLARY 3.2. *$\mathscr{S}M \subset \mathscr{F}M$ for any d-space M.*

COROLLARY 3.3. *If there exists a map $x \in \mathscr{M}_p M$ such that $\mathscr{S}_x M_p = M_p \times M_p$, then $p \in \mathscr{S}M$.*

PROPOSITION 3.4. *Let M be a d-space, $p \in M$, $(U, x) \in \mathscr{M}_p M$,* $\dim M_p = n$.

(a) *The following conditions are equivalent:*

$1°$ *$p \in \mathscr{F}M$.*

$2°$ *If g and g' are two Riemannian metrics on E^n such that*

$$x^* g = x^* g', \tag{1}$$

then

$$g_{,i}\big(x(p)\big)(v, w) = 0$$

for any $(v, w) \in E^n$ and $i = 1, 2, \ldots, n$.

(b) *The following two conditions are equivalent:*

$3°$ *$p \in \mathscr{S}M$.*

4° *If g and g' are two Riemannian metrics on E^n satisfying* (1), *then*

$$g_{,ij}(x(p))(v, w) = 0$$

for any (v, w) E^n and any i, j, $1 \leqslant i, j \leqslant n$.

Proof. We omit the details of the proof, which is straighforward. In the following we need only the implications 1° ⇒ 2° and 3° ⇒ 4°. To show them, it is sufficient to notice that for two Riemannian metrics g, g' on E^n satisfying (1) the map

$$\omega\big(E^n \times E^n \ni (q, v) \mapsto (g-g')(q)(v, v)\big)$$

belongs to $Q(E^n, x(U))$. Hence, if $p \in \mathscr{F}M$, then

$$\begin{aligned}(g_{,i}-g'_{,i})\,(x(p))\,(v, w) & \\ &= \tfrac{1}{2}\big(\omega_{,i}(x(p))\,(v+w)-\omega_{,i}(x(p))\,(v)-\omega_{,i}(x(p))\,(w)\big) = 0,\\ (g_{,ij}-g'_{,ij})\,(x(p))\,(v, w) & \\ &= \tfrac{1}{2}\big(\omega_{,ij}(x(p))\,(v+w)-\omega_{,ij}(x(p))\,(v)-\omega_{,ij}(x(p))\,(w)\big) = 0. \ \blacksquare\end{aligned}$$

LEMMA 3.1. *Let M be a d-subspace of E^n, $p \in M$, $n = \dim M_p$, and let N and N' be two submanifolds of E^n of dimensions less than n and such that $p \in N \cap N'$.*

(a) *If $p \in \mathscr{F}M$, then for every $\omega \in Q(E^n, M-N)$*

$$\omega(p, u) = 0 \quad \textit{for any } u \in E^n. \tag{2}$$

(b) *If $p \in \mathscr{S}M$, then for every $\omega \in Q(E^n, M-N\cap N')$*

$$\omega(p, v) = 0 \quad \textit{for any } v \in E^n. \tag{3}$$

Proof. Let $f, f' \in C^\infty(E^n)$ be such that $N \subset f^{-1}(0)$, $N' \subset f'^{-1}(0)$ and there exist vectors $h, h' \in M_p$ such that $h(f|M) = \partial_{\underline{h}} f(p) \neq 0$, $h'(f'|M) = \partial_{\underline{h}'} f(p) \neq 0$.

(a): Take $\omega \in Q(E^n, M-N)$ and put $\tilde{\omega}(q, u) := f(q)\,\omega(q, u)$ for any $(q, u) \in E^n \times E^n$. If $p \in \mathscr{F}M$, then $\{h\} \times M_p \subset \mathscr{F}M_p$, so $\partial_{\underline{h}}\tilde{\omega}(p, \underline{v}) = \partial_{\underline{h}} f(p)\,\omega(p, \underline{v}) + f(p)\,\partial_{\underline{h}}\omega(p, \underline{v}) = 0$ for any $v \in M_p = E^n$. As $f(p) = 0$, we obtain (2).

(b): Take $\omega \in Q(E^n, M-(N\cup N'))$ and put

$$\tilde{\omega}(q, \underline{v}) := f(q)\,f'(q)\,\omega(q, \underline{v}) \quad \text{for any } (q, v) \in E^n \times E^n. \tag{4}$$

If $p \in \mathscr{S}M$, then

$$\partial^2_{\underline{h}}\tilde{\omega}(p, \underline{v}) = 0 \quad \text{for any } \underline{v} \in \underline{M}_p = E^n.$$

From (4) and the above equality we immediately obtain (3). ∎

As a consequence of the above lemma we obtain

PROPOSITION 3.5. *Let M be a d-subspace of E^n, $n = \dim M_p$, and let g, g' be two Riemannian metrics on E^n such that*

$$\tilde{j}^* g = \tilde{j}^* g',$$

where $\tilde{j}\colon M-\{p\} \to E^n$ denotes the inclusion mapping. If $p \in \mathscr{F}M$, then

$$g(p) = g'(p).$$

PROPOSITION 3.6. *Let M be a d-subspace of E^n, $p \in M$, $m = \dim M_p$, and let N_1, N_2, N_3, N_4 be m-dimensional submanifolds in E^n such that $\{p\} = N_1 \cap N_2 \cap N_3 \cap N_4$ and no two of them have any other common points than p.*

(a) *If $p \in \mathscr{F}M$, then $M \not\subset N_1 \cup N_2 \cup N_3$.*

(b) *If $p \in \mathscr{S}M$, then $M \not\subset N_1 \cup N_2 \cup N_3 \cup N_4$.*

Proof. Without loss of generality we may assume that there exist functions $f_1, f_2, f_3, f_4 \in C^\infty(E^n)$ satisfying

$$(4') \qquad N_i \subset f_i^{-1}(0) \quad \text{for} \quad i = 1, 2, 3, 4,$$
$$\operatorname{grad} f_i(q) := (f_{i,1}(q), \ldots, f_{i,n}(q)) \neq 0$$

for $i = 1, 2, 3, 4$ and every $q \in E^n$. Put

$$\omega(q, u) := (\operatorname{grad} f_1(q)|u)(\operatorname{grad} f_2(q)|u) f_3(q),$$
$$\omega(q, u) := (\operatorname{grad} f_1(q)|u)(\operatorname{grad} f_2(q)|u) f_3(q) f_4(q),$$

for any $(q, u) \in E^n \times E^n$, where $(|)$ denotes the scalar product in E^n. Now pui $h_i := \operatorname{grad} f_i(p)$ for $i = 1, 2, 3, 4$. Note that it is always possible to choose f_1 and f_3 in a such way that

$$(5) \qquad (h_1|h_2) \geqslant 0,$$

$$(6) \qquad (h_3|h_4) \geqslant 0.$$

For $v := h_1 + h_2$ and $h := h_3 + h_4$, after easy calculations and considering (5), (6), we obtain

$$\partial_{h_3}\omega(p, v) = (h_1|h_1+h_2)(h_2|h_1+h_2)(h_3|h_3) \neq 0,$$
$$\partial_h^2 \tilde{\omega}(p, v) = (h_1|h_1+h_2)(h_2|h_1+h_2)(h_3|h_3+h_4)(h_4|h_3+h_4) \neq 0$$

(note that $f_3(p) = f_4(p) = 0$). From (4') it follows that $\omega \in Q(E^n, N_1 \cup N_2 \cup N_3)$ and $\tilde{\omega} \in Q(E^n, N_1 \cup N_2 \cup N_3 \cup N_4)$; hence, if $p \in \mathscr{F}M$, then $M \not\subset N_1 \cup N_2 \cup N_3$ and, if $p \in \mathscr{S}M$, then $M \not\subset N_1 \cup N_2 \cup N_3 \cup N_4$. ∎

EXAMPLE 3.1. This example shows that the assumption that $p \in M$ in Proposition 3.2 (b) is essential. Let $M = \{(\zeta, \eta);\ \zeta\eta = 0\}$ be a d-subspace of E^2, and $p = (0, 0)$. Let i: $M \to E^2$ denote the inclusion mapping and let x: $M \to E^2$ be defined as $(\zeta, \eta) \mapsto (\zeta, \eta + \zeta^2)$. Consider $v \in M_p$ such that $\underline{v} = (1, 0)$. It is obvious that $(v, v) \in \mathscr{S}_{\tilde{i}} M_p$, but $(v, v) \notin \mathscr{S}_x M_p$ because $E^2 \times E^2 \ni ((\zeta, \eta)(\zeta', \eta')) \mapsto (\eta - -\zeta^2)(\zeta')^2$ belongs to $Q_{p,x}$ and $\partial_v^2 \omega(v)|_x = -2$. In this case

$$\underline{\mathscr{F}M_p} = \{((0, a), (0, 0));\ a \in E\} \cup \{(a, 0), (a, 0);\ a \in E\}.$$

§ 4. Algebraic criteria

Let V be a real vector space. For any non-empty subset W of V we denote by span W the smallest vector subspace of V containing W. By $v \otimes w$ we shall denote the tensor product and by $v \odot w$ $(= v \otimes w + w \otimes v)$ the symmetric product of vectors $v, w \in V$. Consequently, $V \otimes V$ $(V \odot V)$ denotes the tensor product (the symmetric product) of V.

PROPOSITION 4.1. *Let M be a d-space, $p \in M$, $x \in \mathcal{M}_p M$ and let*

$$\varphi\colon M_p \times M_p \to M_p \otimes (M_p \odot M_p), \qquad \psi\colon M_p \times M_p \to (M_p \odot M_p) \otimes (M_p \odot M_p)$$

be mappings defined as follows:

$$\varphi(h, v) := h \otimes (v \odot v) \quad \text{and} \quad \psi(h, v) := (h \odot h) \otimes (v \odot v)$$

for any $(h, v) \in M_p \times M_p$. Then:

$$\varphi^{-1}\big(\operatorname{span}\varphi(\mathcal{F}_x M_p)\big) = \mathcal{F}_x M_p, \tag{1}$$

$$\psi^{-1}\big(\operatorname{span}\psi(\mathcal{S}_x M_p)\big) = \mathcal{S}_x M_p. \tag{2}$$

Proof. We prove equality (1). The proof of (2) is similar. We should prove the inclusion

$$\varphi^{-1}\big(\operatorname{span}\varphi(\mathcal{F}_x M_p)\big) \subset \mathcal{F}_x M_p. \tag{3}$$

Let us take $\omega \in Q_{p,x}$ and notice that the mapping $\Omega\colon M_p \times M_p \times M_p \to E$ defined by the formula

$$\Omega(h, v, w) := \tfrac{1}{2}\big(\partial_h \omega(p, v+w)|_x - \partial_h \omega(p, v)|_x - \partial_h \omega(p, w)|_x\big)$$

for any $(h, v, w) \in M_p \times M_p \times M_p$ is 3-linear on M_p and symmetric with respect to v, w. Consequently, there exists linear mapping $\tilde{\Omega}\colon M_p \otimes (M_p \odot M_p) \to E$ satisfying the equality

$$\Omega(h, v, w) = \tilde{\Omega}(h \otimes v \odot w) \quad \text{for any } h, v, w \in M_p.$$

Now one should notice that $\partial_h \omega(p, v)|_x = \Omega(h, v, v)$ for any $h, v \in M_p$; hence

$$\partial_h \omega(p, v)|_x = \Omega \circ \varphi(h, v) \quad \text{for any } h, v \in M_p. \tag{4}$$

If $(h, v) \in \varphi^{-1}(\operatorname{span}\varphi(\mathcal{F}_x M_p))$, then there exist a collection $\alpha^1, \ldots, \alpha^s$ of real numbers and a collection $(h_1, v_1), \ldots, (h_s, v_s)$ of elements of $\mathcal{F}_x M_p$ such that

$$\varphi(h, v) = \sum_{l=1}^{s} \alpha^l \varphi(h_l, v_l).$$

Therefore, from (4) it follows that

$$\partial_h \omega(p, v)|_x = \sum_{l=1}^{s} \alpha^l \tilde{\Omega} \circ \varphi(h_l, v_l) = \sum_{l=1}^{s} \alpha^l \partial_{h_l} \omega(p, v_l)|_x = 0.$$

As $\omega \in Q_{p,x}$ and $(h, v) \in \varphi^{-1}(\operatorname{span}\varphi(\mathcal{F}_x M_p))$ were chosen arbitrarily, the above equality implies (3). ∎

In the following corollaries we assume that p is a point of a d-space M such that $n := \dim M_p > 0$.

COROLLARY 4.1. *We have $p \in \mathcal{F}M$ (resp. $p \in \mathcal{S}M$) if and only if there exists a map $x \in \mathcal{M}_p M$ and a collection (h_l, v_l) of $n^2(n+1)/2$ (resp. $n^2(n+1)^2/4$) of elements of $\mathcal{F}_x M_p$ (resp. $\mathcal{S}_x M_p$) such that the set $\{h_l \otimes v_l \odot v_l\}$ (resp. $\{h_l \odot h_l \otimes v_l \odot v_l\}$) forms a basis of $M_p \otimes M_p \odot M_p$ (resp. of $M_p \odot M_p \otimes M_p \odot M_p$).*

COROLLARY 4.2. *If* $(h_i, v) \in \mathscr{F}M_p$ *for* $i = 1, \ldots, k$, *then* $(\operatorname{span}\{h_i;\ i = 1, \ldots \ldots, k\}) \times \{v\} \subset \mathscr{F}M_p$.

COROLLARY 4.3. *If* $(h_i, v) \in \mathscr{S}M_p$, $i = 1, 2, \ldots, k$ *and* $\operatorname{span}(h_i \odot h_i) = \operatorname{span}\{h_i\} \odot \operatorname{span}\{h_i\}$, *then* $(\operatorname{span}\{h_i\}) \times \{v\} \subset \mathscr{S}M_p$.

COROLLARY 4.4. *Let* π *be a* 2*-dimensional linear subspace of* M_p. *If*

(i) $(h, v_i) \in \mathscr{F}M_p$ *(resp.* $\in \mathscr{S}M_p$*) for* $i = 1, 2, 3$,

(ii) $\operatorname{span}\{v_1\} \cap \operatorname{span}\{v_2\} \cap \operatorname{span}\{v_3\} = \{0\}$,

(iii) $0 \neq v_i \in \pi$ *for* $i = 1, 2, 3$,

then

$$\{h\} \times \pi \subset \mathscr{F}M_p \quad (\subset \mathscr{S}M_p).$$

Proof. If the assumptions are satisfied, then $\operatorname{span}\{v_i \odot v_i\} = \pi \odot \pi$. Hence if $\{(h, v_i)\} \subset \mathscr{F}M_p$, we obtain

$$\begin{aligned}\varphi(\{h\} \times \pi) &= \{h\} \otimes \pi \odot \pi = \{h\} \otimes \operatorname{span}\{v_i \odot v_i\}\\ &= \operatorname{span}\{h \otimes v_i \odot v_i\} \subset \operatorname{span} \varphi(\mathscr{F}M_p) \subset \varphi(\mathscr{F}M_p).\end{aligned}$$

Similarly we obtain the inclusion

$$\psi(\{h\} \times \pi) \subset \psi(\mathscr{S}M_p).$$

By (1) and (2) this completes the proof. ∎

§ 5. Relations among $\pi_* TM_v$, $\mathscr{P}M_p$, $\mathscr{S}M_p$ and $\mathscr{F}M_p$

PROPOSITION 5.1. *For any point* p *of a d-space* M *the following conditions hold*:

(a) *If* $(h, v) \in \mathscr{S}M_p$, $v \neq 0$, *then* $h \in \mathscr{P}M_p$.

(b) $\mathscr{S}M \subset \mathscr{P}M$.

Proof. Let $(h, v) \in \mathscr{S}M_p$, $x \in \mathscr{M}_p M$, $n = \dim M_p$ and $(|)$ denote the scalar product in E^n. For any $\varphi \in C^{\infty}_{p,x}$ the mapping $\omega\colon E^n \times E^n \to E$ defined by

$$\omega(q, w) := \varphi(q)(w|w) \quad \text{for any } (q, w) \in E^n \times E^n,$$

is an element of $Q_{p,x}$. Hence

$$0 = \partial_h^2 \omega(p, v)|_x = \partial_h^2 \varphi(p)|_x \, (\underline{x_* v} | \underline{x_* v}),$$

which proves (a). Condition (b) follows immediately from (a). ∎

LEMMA 5.1. *Let* M *be a d-space,* $p \in M$ *and* $(h, v) \in M_p \times M_p$.

(a) *If* $h \in \pi_* TM_v$, *then* $(h, v) \in \mathscr{F}M_p$.

(b) *If*

(1) $$p \in \mathscr{F}M,$$

(2) $$h \in \pi_* \mathscr{P}TM_v,$$

then

(3) $$(h, v) \in \mathscr{S}M_p.$$

Proof. It will be convenient and involve no loss of generality to consider the case where M is a d-subspace of E^n, $n = \dim M_p$. Take $\omega \in Q(E^n, M)$. Then

$$\omega(p, u) = 0 \quad \text{for any } u \in E^n. \tag{4}$$

Now notice that ω may be interpreted as a function on $E^n \times E^n$ and then it satisfies the condition

$$\omega | \underline{TM} = 0. \tag{5}$$

(a): Assume that $h \in \pi_* TM_v$, i.e., that there exists a vector $H \in TM_v$ such that

$$\underline{H} = (\underline{h}, \underline{V}) \in \underline{TM_v} \subset E^n \times E^n.$$

From (5) we get

$$0 = \partial_{\underline{H}} \omega(p, \underline{v}) = \omega(p, \underline{v} + \underline{V}) - \omega(p, \underline{v}) - \omega(p, \underline{V}) + \partial_{\underline{h}} \omega(p, \underline{v}).$$

By virtue of (4), it follows that

$$0 = \partial_{\underline{h}} \omega(p, \underline{v}) = \partial_h \omega(p, v)|_{\underset{\sim}{i}},$$

where $\underset{\sim}{i}: M \to E^n$ stands for the inclusion mapping. As $\omega \in Q(E^n, M)$ was chosen arbitrarily, we obtain $(h, v) \in \mathscr{F}_{\underset{\sim}{i}} M_p$ (for the last equality see Proposition 3.2).

(b): Let (1) and (2) be satisfied, i.e., there exists a vector $H \in \mathscr{P}TM_v$ such that

$$\underline{H} = (\underline{h}, \underline{V}) \in \underline{\mathscr{P}TM_v} \tag{6}$$

and

$$\partial_{\underline{h}} \omega(p, \underline{u}) = \partial_{\underline{u}} \omega(p, \underline{v}) = 0 \quad \text{for any } \underline{u} \in E^n. \tag{7}$$

By virtue of (7) and (5), there exists a C^∞-curve $(-\varepsilon, \varepsilon) \ni t \mapsto (p(t), v(t)) \in E^n \times E^n$ such that $(c(0), u(0)) = (p, \underline{v})$, $(\dot{c}(0), \dot{u}(0)) = (\underline{h}, \underline{V})$, where dots denote derivatives with respect to the parameter t, and

$$d^2\omega \circ (c, u)/dt^2(0) = 0.$$

According to the last equality we obtain

$$\partial^2_{\underline{h}} \omega(p, \underline{v}) + 2\partial_{\underline{h}} \omega(p, \underline{v} + \underline{V}) - 2\partial_{\underline{h}} \omega(p, \underline{v}) - 2\partial_{\underline{h}} \omega(p, \underline{V}) +$$
$$+ \partial_{\ddot{c}(0)} \omega(p, \underline{V}) + 2\omega(p, \underline{V}) + \omega(p, \underline{v} + \ddot{u}(0)) - \omega(p, \underline{v}) - \omega(p, \ddot{u}(0)) = 0.$$

Now considering (4), (7) we get

$$\partial^2_{\underline{h}} \omega(p, \underline{v}) = 0.$$

By virtue of Proposition 3.2, the above equality implies (3). ■

PROPOSITION 5.2. *Let p be a point of a d-space M. Then*:

(a) $\{(h, v);\ h \in \pi_* TM_v,\ v \in M_p\} \subset \mathscr{F} M_p$.

(b) *If $p \in \mathscr{F} M$, then* $\{(h, v);\ h \in \pi_* \mathscr{P}TM_v,\ v \in M_p\} \subset \mathscr{S} M_p$.

COROLLARY 5.1. *Let M be a d-space. Then*:

(a) $M_{=} \subset \mathscr{S}M \cap \mathscr{P}M$.

(b) *$\mathscr{S}M \cap \mathscr{P}M$ contains an open and dense subset of M.*

§ 6. Limit criteria

Let $|\ |$ denote the Euclid n norm in E^n. For a d-space M and $p \in M$, denote by $\mathscr{L}M_p$ the set of all pairs $(h, v) \in M_p \times M_p$ such that there exists a sequence (v_n) of elements of TM and a map $x \in \mathscr{M}_p M$ satisfying

$$\begin{aligned} &p \neq \pi v_n \quad \text{for any } n, \\ &v = \lim v_n, \\ &\underline{x_* h} = \lim_n \big(x(\pi v_n) - x(p)\big)/|x(\pi v_n) - x(p)|, \end{aligned}$$

Proposition 6.1. *Let M be a d-space, $p \in M$, $x \in \mathscr{M}_p M$. Then*:

(a) $\mathscr{L}M_p \subset \mathscr{F}M_p \cap \mathscr{S}_x M_p$.

(b) *If $p \in \mathscr{F}M$, then $\mathscr{L}M_p \subset \mathscr{S}M_p$.*

This statement easily results from Propositions 2.2 and 5.2.

Corollary 6.1. *If a point p of a d-space M belongs to the closure of the set $\{q \in M;\ \dim M_q \geqslant 1\}$, then*:

(a) *There exists $(h, v) \in \mathscr{F}M_p$ such that $v \neq 0$, $h \neq 0$.*

(b) *If $p \in \mathscr{F}M$, then there exists $(h, v) \in \mathscr{S}M_p$ such that $h \neq 0$ and $v \neq 0$.*

It suffices to notice that if the assumptions are satisfied, $\mathscr{L}M_p$ contains (h, v) as above.

Proposition 6.2. *Let M be a d-space, $p \in M$, $h, v \in M_p$ and let (v_n), (h_n) be two sequences of elements of TM such that*

$$p_n := \pi v_n = \pi h_n \quad \text{for} \quad n = 1, 2, \ldots, \tag{1}$$

$$(p, h, v) = (\lim_n p_n, \lim_n h_n, \lim_n v_n), \tag{2}$$

$$\dim M_{p_n} = \dim M_p \quad \text{for} \quad n = 1, 2, \ldots \tag{3}$$

Then the following conditions hold:

(a) *If $(h_n, v_n) \in \mathscr{F}M_{p_n}$ for any n, then $(h, v) \in \mathscr{F}M_p$.*

(b) *If $(h_n, v_n) \in \mathscr{S}M_{p_n}$ for any n, then $(h, v) \in \mathscr{S}M_p$.*

Proof. Suppose that $x \in \mathscr{M}_p M$ is a map such that $p_n \in U_x$ for any n. For any $\omega \in Q_{p,x}$ we have

$$\partial_h \omega(p, v)|_x = \lim \partial_{h_n} \omega(p_n, v_n)|_x, \tag{4}$$

$$\partial_h^2 \omega(p, v)|_x = \lim \partial_{h_n}^2 \omega(p_n, v_n)|_x. \tag{5}$$

By (3), if $(h_n, v_n) \in \mathscr{F}M_{p_n}$ (resp. $\in \mathscr{S}M_{p_n}$), then the right-hand side of (4) (resp. of (5)) is zero, which completes the proof. ∎

Example 6.1. Let us consider the cone $M := \{(x, y, z);\ z^2 = x^2 + z^2 = x^2 + y^2\}$. Put $p = (0, 0, 0)$, $p_n = (0, 1/n, 1/n)$, $\underline{h} = \underline{v} = \underline{v}_n = (1, 0, 0)$ for any $n = 1, 2, \ldots$ Then $(M - \{p\}) \subset \mathscr{S}M \subset \mathscr{F}M$, $\dim M_p = 3$ and $\dim M_q = 2$ for any

$q \in M-\{p\}$, $(\underline{h}, \underline{v}_n) \in \mathscr{S}\underline{M}_{p_n}$ for any n. But $(\underline{h}, \underline{v}) \notin \mathscr{S}\underline{M}_p$. In fact, the function $\omega\colon E^n \times E^n \to E$ defined as follows:

$$\omega(p, u) := acz-aby-a^2x \quad \text{for any } p = (x, y, z), \quad u = (a, b, c) \in E^3,$$

belongs to $Q(E^3, C)$ but $\partial_{\underline{h}}^2 \omega(p, \underline{v}) = -1$. We conclude that assumption (3) in Proposition 6.2 is essential.

§ 7. The products of d-spaces

For every point (p, q) of the product $M \times N$ of d-spaces M and N, the tangent space $(M \times N)_{(p,q)}$ is identified with $M_p \times N_p$ (see [14], [15]). Let the mapping

$$\underset{\sim}{j}\colon (M \times N)_{(p,q)} \times (M \times N)_{(p,q)} = M_p \times N_q \times M_p \times N_q \to M_p \times M_p \times N_q \times N_q$$

be defined as follows:

$$\underset{\sim}{j}(v, w, v', w') = (v, v', w, w') \quad \text{for any } v, v' \in M_p \text{ and } w, w' \in N_q.$$

Proposition 7.1. *For any point (p, q) of the product $M \times N$ of d-spaces M and N the following equality holds*:

$$\underline{j}(\mathscr{F}(M \times N)_{(q,p)}) = \mathscr{F}M_p \times \mathscr{F}N_q. \tag{1}$$

Moreover, if $p \in \mathscr{F}M$ and $q \in \mathscr{F}N$, then

$$\underline{j}(\mathscr{S}(M \times N)_{(p,q)}) = \mathscr{S}M_p \times \mathscr{S}N_q. \tag{2}$$

Proof. As the proofs of the two equalities are similar, we shall prove only equality (2). To simplify the notation below we identify the elements v of M_p (or $w \in N_q$) with $\underline{v} \in \underline{M}_p$ (or $\underline{w} \in \underline{N}_q$, respectively) and denote all these objects without underlying, e.g., $(M, N)_{(p,q)} = E^n \times E^n$. Let $M \subset E^m$, $N \subset E^n$ and $m = \dim M_p$, $n = \dim N_q$. First we shall prove the inclusion

$$\underline{j}(\mathscr{S}(M \times N)_{(p,q)}) \subset \mathscr{S}M_p \times \mathscr{S}N_q. \tag{1'}$$

Let

$$(h, g, v, w) \in \mathscr{S}(M \times N)_{(p,q)} \subset E^m \times E^n \times E^m \times E^n \tag{3}$$

and $\omega \in Q(E^n, N)$. We let

$$\Omega(p', q', v', w') := \omega(p', v')$$

for any $(p', q', v', w') \in E^n \times E^m \times E^n \times E^m$. As Ω belongs to $Q(E^m \times E^n, M \times N)$, it follows from (3) that $\partial_h^2 \omega(p, v) = \partial_{(h,g)}^2 \Omega(p, q; v, w) = 0$; hence $(h, v) \in \mathscr{S}M_p$ (see Proposition 3.2). An analogous reasoning shows that $(g, w) \in \mathscr{S}N_q$.

Now we proceed to the proof of the opposite inclusion, i.e.,

$$\underset{\sim}{j}(\mathscr{S}(M \times N)_{(p,q)}) \supset \mathscr{S}M_p \times \mathscr{S}N_q. \tag{4}$$

Assume that

$$(p, q) \in \mathscr{F}M \times \mathscr{F}N, \tag{5}$$

$$(h, v) \in \mathscr{S}M_p \subset E^m \times E^m, \tag{6}$$

$$(g, w) \in \mathcal{S}N_q \subset E^n \times E^n, \tag{7}$$

$$\alpha \in Q(E^m \times E^n, M \times N). \tag{8}$$

Denote by β a smooth bilinear symmetric form on $E^m \times E^n$ such that

$$\alpha(p', q'; v', w') = \beta(p', q'; v', w'; v', w')$$

for any pairs (p', q'), $(v', w') \in E^m \times E^n$. For the proof of (4) it is sufficient to show the following equality:

$$\partial^2_{(h,g)} \alpha(p, q; v, w) = 0. \tag{9}$$

By simple computations we obtain the equality

$$\begin{aligned}\partial^2_{(h,g)} \alpha(p, q; v, w) = {} & (2\partial_{(h,0)} \partial_{(0,g)} \alpha + \partial^2_{(h,0)} \alpha + \partial^2_{(0,g)} \alpha)(p, q; v, 0) + \\ & + (4\partial_{(h,0)} \partial_{(0,g)} \beta + 2\partial^2_{(h,0)} \beta + 2\partial^2_{(0,g)} \beta)(p, q; v, 0; 0, w) + \\ & + (\partial_{(h,0)} \partial_{(0,g)} \alpha + \partial^2_{(h,0)} \alpha + \partial^2_{(0,g)} \alpha)(p, q; 0, w).\end{aligned}$$

We shall prove that each of the above nine terms is equal to zero. It is easy to notice that, by symmetry, it is sufficient to show the following five equalities:

$$\partial_{(h,0)} \partial_{(0,g)} \alpha(p, q; v, 0) = 0, \tag{10}$$

$$\partial^2_{(h,0)} \alpha(p, q; v, 0) = 0, \tag{11}$$

$$\partial^2_{(0,g)} \alpha(p, q; v, 0) = 0, \tag{12}$$

$$\partial_{(h,0)} \partial_{(0,g)} \beta(p, q; v, 0; 0, w) = 0, \tag{13}$$

$$\partial^2_{(h,0)} \beta(p, q; v, 0; 0, w) = 0. \tag{14}$$

For every point $q' \in N$ the function

$$\alpha^{q'}(E^m \times E^m \ni (p', v') \mapsto \alpha(p', q'; v', 0))$$

belongs, by (7), to $Q(E^m, M)$. According to (5) and Proposition 3.4 we obtain

$$\partial_h \alpha^{q'}(p, v) = 0$$

for any $q' \in N$. The function $\alpha'(E^n \ni q' \mapsto \partial_h \alpha^{q'}(p, v))$ belongs to $C^\infty(E^n, N)$. Therefore

$$\partial_{(0,g)} \partial_{(h,0)} \alpha(p, q; v, 0) = \partial_g \alpha'(q) = 0.$$

This completes the proof of equality (10).

For the proof of (11), it is sufficient to notice that $\alpha^q \in Q(E^m, M)$, $\partial^2_{(h,0)} \alpha(p, q; v, 0) = \partial^2_h \alpha^q(p, v)$ and make use of (5).

Now observe that

$$\alpha^{p,v}(q') := \alpha(p, q'; v, 0)$$

for any $q' \in E^n$ belongs to $C^\infty(E^n, N)$. From (6) and Proposition 5.1 we obtain

$$\partial^2_{(0,g)} \alpha(p, q; v, 0) = \partial^2_g \alpha^{p,v}(q) = 0,$$

and so we get equality (12). Now we need

LEMMA 7.1. *Let X be a d-subspace of E^n, $p \in X$, $\dim X_p = n$. Denote by* Lin(E^n, X) *the set of all linear forms $\omega(q, w)$ on E^n vanishing for $(q, w) \in TX$.*

(a) *If* $(h, v) \in \mathscr{F}X_p$, *then* $\partial_h \omega(p, v) = 0$ *for any* $\omega \in \mathrm{Lin}(E^n, X)$.

(b) *If* $(h, v) \in \mathscr{S}M_p$, *then* $\partial_h^2 \omega(p, v) = 0$.

Proof of Lemma 7.1. If $\omega \in \mathrm{Lin}(E^n, X)$, then $\Omega(q, w) := \omega(q, w)(h|w)$ belongs to $Q(E^n, X)$. Consequently $0 = \partial_h^i \Omega(p, v) = \partial_h^i \omega(p, v)(v|w)$, $i = 1, 2$. This completes the proof of the lemma.

We proceed to the proof of the proposition. For every $(q', w') \in E^n \times E^n$ the formula

$$\beta^{q', w'}(p', v') := \beta(p', q'; v', 0; 0, w'),$$

for any $(p', v') \in E^m \times E^m$, defines an element $\beta^{q', w'}$ of $\mathrm{Lin}(E^m, M)$. Hence $\partial_{(h, 0)} \beta^{q', w'}(p, v) = 0$ for any $(q', w') \in TN$. Finally, applying the lemma to a function

$$\tilde{\beta}\big(E^n \times E^n \ni (q', w') \mapsto \partial_{(h, 0)} \beta^{q', w'}(p, v)\big)$$

belonging to $\mathrm{Lin}(E^n, N)$, we obtain the equality

$$\partial_{(0, g)} \partial_{(h, 0)} \beta(p, q; v, 0; 0, w) = \partial_g \tilde{\beta}(q, w) = 0,$$

which completes the proof of (13).

Finally, in order to prove (14) note that $\beta^{q, v} \in \mathrm{Lin}(E^n, M)$. Making use of (5) and the lemma we get the equality

$$\partial_{(h, 0)}^2 \beta(p, q; v, 0; 0, w) = \partial_h^2 \beta^{q, w}(p, v) = 0.$$

This completes the proof. ■

COROLLARY 7.1. *For d-spaces M and N the following equalities hold*:

$$\mathscr{F}(M \times N) = \mathscr{F}M \times \mathscr{F}N,$$
$$\mathscr{S}(M \times N) = \mathscr{S}M \times \mathscr{S}N.$$

§ 8. The Riemannian curvature tensor of a Riemannian d-space

DEFINITION. Let M be a d-space. A map g $(M \ni p \mapsto g(p))$, where $g(p)$ is an inner product in the vector space M_p, is called a *Riemannian metric on a d-space* M if a mapping $TM \ni v \mapsto g(p)(v, v)$ is smooth. The pair (M, g) is called a *Riemannian d-space*. If (M, g) and (N, g') are two Riemannian d-spaces, then a smooth map $f\colon M \to N$ is called an *isometry* if $g = f^* g'$, where $f^* g'(v, w)$ is defined as $g'(f(p))\,(f_* v, f_* w)$ for any $p \in M$ and $v, w \in M_p$.

It is easily proved (see [6]) that if M is a manifold, then our definition of a Riemannian metric is equivalent to the classical one. Moreover, if (M, g) is a paracompact Riemannian d-space, N is a manifold and $f\colon M \to N$ is an imbedding, then there exists an open subset U in N and a Riemannian metric g' on U such that $g = f^* g'$ (see [6]). In particular, we have:

LEMMA 8.1. *Let* (M, g) *be a Riemannian d-space*, $p \in M$, $n = \dim M_p$, $x \in \mathscr{M}_p M$. *Then there exists an open set* U *in* E^n *and a Riemannian metric* g' *on* U *such that*

(1) $$x(U_x) \subset U,$$

(2) $$x^* g' = g \quad on \quad U_x.$$

DEFINITION. Let (M, g) be a Riemannian d-space, $p \in M$, $n = \dim M_p > 0$. A real 4-linear mapping R_p on M_p is called a *Riemannian curvature tensor of* (M, g) *at the point* p if there exists a map $x \in \mathcal{M}_p M$ such that

$$R_p = x_p^* R^{g'} \tag{3}$$

for any Riemannian metric g' on an open subset U of E^n satisfying (1) and (2), where

$$x_p^* R^{g'}(v_1, \ldots, v_4) := R^{g'}(x_* v_1, \ldots, x_* v_4)$$

for any $v_1, \ldots, v_4 \in M_p$ and $R^{g'}$ is a Riemannian curvature tensor of (U, g'). Similarly we define the *sectional curvature* k on the set of all 2-dimensional linear subspaces of M_p, the *scalar curvature*, etc. The set of all points $p \in M$ for whose the Riemannian curvature of (M, g) may be defined in the above manner we denote by $\mathrm{Riem}(M, g)$.

Remark. As the sectional curvature determines the Riemannian curvature tensor (see [3]), $\mathrm{Riem}(M, g)$ is exactly the set of all points $p \in M$ for which the sectional curvature may be defined in the above manner.

Immediately from the above definition we obtain

LEMMA 8.2. *If R_p is a Riemannian curvature tensor on a Riemannian d-space (M, g) at a point $p \in M$, then* (3) *is satisfied for any map $x \in \mathcal{M}_p M$ and any Riemannian metric g' on any open submanifold U in E^n, provided that* (1), (2) *are satisfied.*

LEMMA 8.3. *If g, g_1 are two Riemannian metrics on a d-space M, then*

$$\mathrm{Riem}(M, g) = \mathrm{Riem}(M, g_1).$$

Proof. Take

$$p \notin \mathrm{Riem}(M, g) \tag{4}$$

and fix a map $x \in \mathcal{M}_p M$. We shall show that $p \in \mathrm{Riem}(M, g_1)$. Let g', g'', g_1' be Riemannian metrics on U', U'', U_1, respectively, such that $x(U_x) \subset U' \cap U'' \cap U_1$, $x_q^* g' = x_q^* g'' = g(q)$, $x_q^* g_1' = g_1(q)$ for any $q \in U_x$ and $x_p^* R^{g'} \neq x_p^* R^{g''}$. Now notice that $(g_1' + g' - g'')$ is a Riemannian metric on an open neighbourhood V of $x(U_x)$ in E^n and such that $x_q^*(g_1' + g' - g'') = x_q^* g_1' = g_1(q)$ for any $q \in U_x$ and $x_p^* R^{g_1'} + x_p^* R^{g'} - x_p^* R^{g''} \neq x_p^* R^{g_1'}$; so $p \notin \mathrm{Riem}(M, g_1)$. ∎

By virtue of Lemma 8.3, for any d-space M, the set $\mathrm{Riem}(M, g) \subset M$ is independent of the choice of the Riemannian metric g; so we denote this set by $\mathrm{Riem}\, M$.

Now note that in local coordinates on a manifold the coefficients of a Riemannian curvature tensor may be expressed by means of first and second partial derivatives of the coefficients of the Riemannian metric. Hence, from Proposition 3.4 we immediately obtain:

THEOREM 8.1. *$\mathcal{S} M \subset \mathrm{Riem}\, M$ for any d-space M.*

Now we present an example of a d-space M such that $\mathrm{Riem}\, M \neq \mathcal{S} M$.

EXAMPLE 8.1. Let $M = M' \cup M''$, where $M' = \{(x^1, x^2);\ x^1 x^2 = 0\}$ and $M'' = \{(x^1, x^2);\ |x^1| < (x^2)^2,\ |x^2| < (x^1)^2,\ (x^1)^2 + (x^2)^2 = 1/n,\ n = 1, 2, \ldots\}$ (see

Fig. 8.1). We shall show that $p = (0,0) \in \operatorname{Riem} M - \mathscr{S}M$. It is easily seen that $\{((a,b), (-b,a));\ (a,b) \in E^2\} \cup \{((1,0), (1,0)), ((0,1), (0,1))\} \subset \mathscr{L}M_p$. We recall that according to Proposition 6.1

$$(5) \qquad \mathscr{L}M_p \subset \mathscr{F}M_p \subset \mathscr{S}_{\underset{\sim}{i}}M_p,$$

where $\underset{\sim}{i}: M \to E^n$ denotes the inclusion mapping. It is not difficult to prove that substituting in $h \otimes v \odot v$ the pair (h, v) successively by the following six elements of $\mathscr{L}M_p$: $((1,0), (1,0))$, $((0,1), (0,1))$, $((1,1), (-1,1))$, $((1,2), (-2,1))$, $((-1,1), (1,1))$ and $((2,1), (-1,2))$, we obtain a basis of $E^2 \otimes E^2 \odot E^2$; hence, by virtue of Corollary 4.1, we obtain $p \in \mathscr{F}M$.

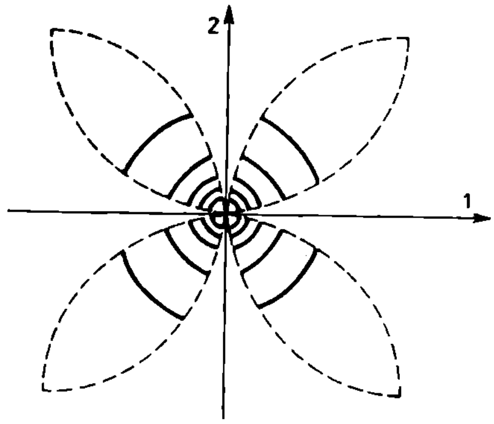

Fig. 8.1

Let two Riemannian metrics g and g' on E^2 satisfy the equality $i^*g = i^*g'$. We set

$$(6) \qquad \omega(q, v) = \sum_{i,j=1}^{2} \omega_{ij}(q) v^i v^j := g(q)(v, v) - g'(q)(v, v)$$

for any $(q, v) \in TE^2 = E^2 \times E^2$. It is obvious that $\omega \in Q(E^2, M)$. According to (5) we get

$$(7) \qquad \partial_h^2 \omega(p, v) = \sum \omega_{ij,kl}(p) h^k h^l v^i v^j = 0$$

for any $(h, v) \in \mathscr{L}M_p$. For (h, v) equal to $((1,0), (1,0))$ and $((0,1), (0,1))$ we obtain

$$(8) \qquad \omega_{11,11}(p) = 0,$$

$$(9) \qquad \omega_{22,22}(p) = 0.$$

Similarly, taking (h, v) equal to $((1, a), (-a, 1)) \in \mathscr{S}_{\underset{\sim}{i}}M_p$, we obtain the equation

$$0 = \partial_h^2 \omega(p, v) = \omega_{22,11}(p) + 2(\omega_{22,12} + \omega_{12,11})(p)a + (\omega_{11,11} - 4\omega_{12,12} + \omega_{22,22})(p)a^2 + 2(\omega_{11,12} - \omega_{12,22})(p)a^3 + \omega_{11,22}(p)a^4.$$

As there is no restriction on a, all the coefficients of the above polynomial with respect to a have to vanish. In particular, we get

$$\omega_{22,11}(p) = \omega_{11,22}(p) = 0, \tag{10}$$
$$\omega_{11,11}(p) - 4\omega_{12,12}(p) + \omega_{22,22}(p) = 0.$$

By virtue of (8), (9) the last equality is equivalent to

$$\omega_{12,12}(p) = 0. \tag{11}$$

Now observe that the Riemannian curvature tensor R_p of (E^2, g) at the point p expressed in the natural coordinates in E^2 has up to the sign only one non-trivial component, namely

$$R_{1212}(p) = \Big(g_{12,12} - g_{11,22}/2 - g_{22,11}/2 - \\ - \sum_{r,s} (\Gamma^r_{11}\Gamma^s_{22} - \Gamma^r_{12}\Gamma^s_{12}) g_{rs}\Big) (g_{11}g_{22} - g_{12}^2)^{-1}(p),$$

where $\Gamma^k_{ij} = g^{kl}(g_{ik,j} + g_{jl,i} - g_{ij,l})$. As $p \in \mathscr{F}M$, it follows from Proposition 3.4 that $\Gamma^k_{ij}(p) = \Gamma'^k_{ij}(p)$ for any i, j, k, where Γ'^k_{ij} denote the Christoffel symbols of the Riemannian connection on (E^2, g'). Therefore, according to (6), the following equality holds:

$$R^g_{1212}(p) = R^{g'}_{1212}(p) + (\omega_{12,12} - 0.5\omega_{11,22} - 0.5\omega_{22,11})(g_{11}g_{22} - g_{12}^2)^{-1}(p).$$

Thus from (7), (8) and (10) we obtain the equality

$$R_{1212}(p) = R'_{1212}(p).$$

Consequently, $p \in \operatorname{Riem} M$ and $M = \operatorname{Riem} M$.

To see that $p \notin \mathscr{S}M$ let us notice that the formula

$$\Omega(x, v) := x^1x^2(v^1)^2 + (x^2)^2v^1v^1,$$

for any $(x, v) \in E^2 \times E^2$, defines an element of $Q(E^2, M)$ such that $\partial^2_{(0,1)}\omega(p, (1, 1)) = 2$.

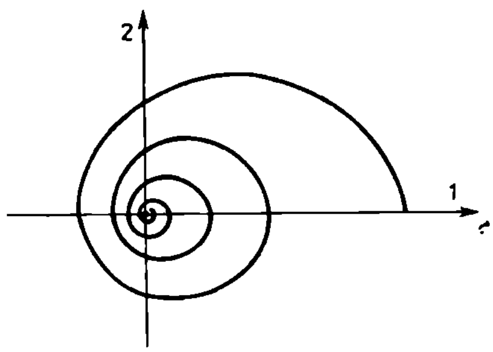

Fig. 8.2

EXAMPLE 8.2. Let $M := \{(0.5 + e^{it})/t \in E^2;\ t > 2\} \cup \{(0, 0)\}$ (see Fig. 8.2). Let $p = (0, 0)$ and let $\underset{\sim}{i}\colon M \to E^2$ the inclusion map. We show that $p \in \mathscr{S}M$, and so $M = \mathscr{S}M$. In fact, for any α we have

$$(h, v) := (0.5e^{i\alpha}, e^{i(\alpha+\pi/2)}) \in \underline{\mathscr{L}M}_p \subset E^2 \times E^2.$$

Putting $\alpha = 0, \pi/2, -\pi/2, \pi/3, 2\pi/3, 4\pi/3, -\pi/3, \pi/4, -\pi/4$ we obtain nine elements (h_i, v_i), $i = 1, \dots, 9$, of $\underline{\mathscr{L}M_p} \subset \underline{\mathscr{S}_{\tilde{l}}M_p}$ such that $\{h_i \odot h_i \otimes v_i \odot v_i;\ i = 1, \dots, 9\}$ is a basis of $E^2 \odot E^2 \otimes E^2 \odot E^2$. Hence, by Corollary 4.1 and Proposition 3.3, we obtain $p \in \mathscr{S}M$.

EXAMPLE 8.3. Consider a cone $C_\alpha = f_\alpha^{-1}(0)$ in E^3, where $f_\alpha(x, y, z) := \alpha^2 x^2 - -y^2 - z^2$, $\alpha \neq 0$. The point $p = (0, 0, 0) \in C_\alpha$ does not belong to $\mathscr{P}C_\alpha$ (e.g., $\partial^2 f/(\partial x)^2(p) = 2 \neq 0$), and so, by Proposition 5.1, $p \notin \mathscr{S}C_\alpha$. By means of quadratic forms $E^3 \times E^3 \ni \big((x, y, z), (X, Y, Z)\big) \mapsto (x^2 - y^2 - z^2)(X^2 + Y^2 + Z^2)$ and $E^3 \times E^3 \ni \big((x, y, z), (X, Y, Z)\big) \mapsto (\alpha^2 xX - yY - zZ)^2$ it can be proved that $\mathscr{S}(C_\alpha)_p = \mathscr{L}(C_\alpha)_p = \big\{\big((1, \alpha\sin\beta, \alpha\cos\beta), (X, Y, Z)\big);\ \alpha^2 X - \alpha\sin\beta Y - \alpha\cos\beta Z = 0,\ \beta \in E\big\}$, and that $\dim\big(\operatorname{span}\{h \odot h \otimes v \odot v;\ (h, v) \in \mathscr{S}(C_\alpha)_p\}\big) = 28$. As $\dim(M_p \odot M_p \otimes M_p \odot M_p) = \dim(E^3 \odot E^3 \otimes E^3 \odot E^3) = 36$, the following conjecture arises: If π is a plane "in the general position" containing p, then $p \in \mathscr{S}(C_\alpha \cup \pi)$ (note that $\dim E^2 \odot E^2 \otimes \otimes E^2 \odot E^2 = 9 > 36 - 28$). But it can be proved that for the (y, z)-plane this is not true.

EXAMPLE 8.4. Let $C = C_\alpha \cup C_\beta \subset E^3$, where $\alpha \neq \beta$. It can be proved that $p = (0, 0, 0) \in \mathscr{S}C$, and so $C = \mathscr{S}C$.

§ 9. Isometric immersions of a Riemannian d-space with a non-positive sectional curvature

For a d-space M denote by $\mathscr{E}^2 M$ the set of all points $p \in M$ such that for any map $x \in \mathscr{M}_p M$ and any $\varphi \in C^\infty(E^n)$, where $n = \dim M_p$, the following statement holds:

If the function $\varphi | x(U_x)$ has a local maximum at $x(p)$, then

$$d\varphi|_p = 0, \tag{1}$$

$$X^2\varphi(p) \leqslant 0 \quad \text{for any smooth vector field } X \text{ on } E^n. \tag{2}$$

It has been proved in [8] that $\mathscr{E}^2 M \subset \mathscr{P}M$ and $\mathscr{E}^2 M \times \mathscr{E}^2 N = \mathscr{E}^2(M \times N)$ for any d-spaces M and N.

Below, by an immersion we mean a smooth mapping with the 1-1 tangent map at each point. It can be proved that each immersion is a local diffeomorphism onto its image (see [10]). Moreover, it has been proved that every Lindelöf Riemannian d-space (M, g), satisfying, for certain natural n, the condition

$$\dim M_p \leqslant n \quad \text{for every } p \in M,$$

can be isometrically immersed into a Euclidean space E^m of sufficiently large dimension m [5]. The following theorem gives a lower estimation for m.

THEOREM 9.1. *Let (M, g) be a compact Riemannian d-space and let n be a natural number. If*

(a) $M = \mathscr{E}^2 M = \operatorname{Riem} M$,

(b) *the sectional curvature of (M, g) is non-positive,*

(c) $\dim M_p \geqslant n$ *for any $p \in M$,*

then M cannot be isometrically immersed into E^{2n-1}.

Proof. The case $n = 0$ is trivial; hence assume $n > 0$. Let $f\colon M \to E^m$ be an isometric immersion. We let

$$\varphi(y) = (y|y) \quad \text{for any } y \in E^m,$$

where $(\ |\ )$ stands for the inner product in E^m. Let $q = f(p) \in f(M)$ be a point at which φ has a maximum on $f(M)$ and $n' := \dim M_p$. Let N be an n'-dimensional submanifold of E^m such that $N \cap f(M)$ is an open neighbourhood of q in $f(M)$. By (a), $d(\varphi|N)|_p = 0$, $X^2(\varphi|N)(p) \leqslant 0$ for any smooth vector field X on N. Hence, from Weingarten's formula we get (see [3], vol. II, p. 28)

$$X^2\varphi|M = \big(\alpha^N(X, X)|f\big) + (X|X) \quad \text{at} \quad p$$

for any smooth vector field X on N, where α^N stands for the second fundamental form of the submanifold N of E^m. Therefore

$$\big(\alpha_q^N(v, v)|q\big) \leqslant (v|v) \quad \text{for any } v \in N_q.$$

Consequently

$$\alpha_q^N(v, v) \neq 0 \quad \text{for any } q \in N. \tag{3}$$

According to (a), (b) the sectional curvature of N at q is non-positive. In [3] (vol. II, p. 28) it has been proved that, in this case, (3) holds only if

$$m \geqslant 2n' \geqslant 2n$$

which completes the proof. ■

EXAMPLE 9.1. Let W be a d-subspace of E^2 and let p, q, q' be three different points of W such that $\tilde{W} = W - \{q, q'\}$ is a 1-dimensional submanifold of E^2 and $W - \{p\}$ contains two componets which are diffeomorphic to the d-space constructed in Example 8.2 (see Fig. 9.1).

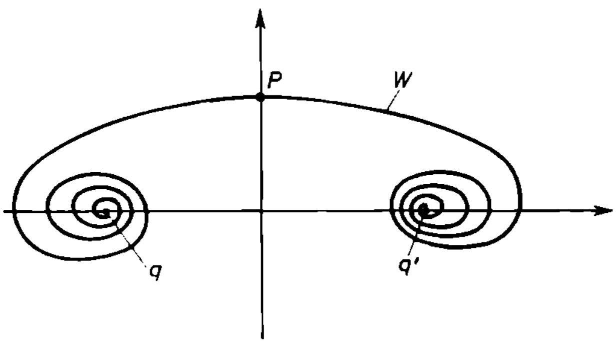

Fig. 9.1

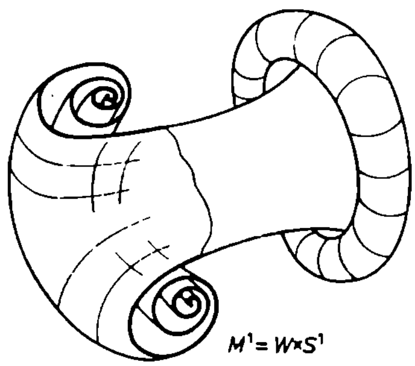

Fig. 9.2

The d-space W is compact and, according to Example 8.2, $W = \mathscr{S}W = \operatorname{Riem} W$. It can be proved that $W = \mathscr{E}^2 W$. Denote by S^1 the unit circle in E^2. For a natural k let us set

$$M^k := W \times \underbrace{S^1 \times \ldots \times S^1}_{k} \subset E^{2(k+1)}$$

and denote by g^k the Riemannian metric on M^k induced from $E^{2(k+1)}$. According to Proposition 7.1 and [8] the Riemannian d-space (M^k, g^k) satisfies the assumptions of Theorem 9.1 with $n = k+1$; thus there is no isometric immersion f of (M^k, g^k) into E^{2k+1} (notice that it is essential that f is an isometric immersion, because M^k can be imbedded into E^{k+3}). For $k = 1$ it is easily seen that M^1 (see Fig. 9.2) may be imbedded in E^3. According to the above theorem, (M^1, g^1) cannot be isometrically immersed into E^3. It can be proved that the last sentence is equivalent to the following statement:

There exists no smooth map $f\colon E^4 \to E^3$ such that

$$\lim_n |f(p_n)-f(p)|/|p_n-p| = 1$$

for any sequence p_n of points of M^1 converging to a point $p \in M^1$.

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