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# CLOSED SETS OF UNIVERSAL HORN FORMULAS FOR MANY-SORTED (PARTIAL) ALGEBRAS

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In theoretical computer science universal Horn formulas play a growing role, as well in the algebraic specification of abstract data types as in programming languages like PROLOG. In this note we present and discuss an algebraic description of closed sets of universal Horn formulas – similar to the well-known description of closed sets of equations as fully invariant congruence relations. We include the treatment of negations of conjunctions of equations, since we think that mainly in connection with partial algebras they might be quite useful at some occasions – in particular negations of term-existence expressions. But we state and prove the main result only for many-sorted total algebras; the corresponding techniques of proof for partial algebras can be taken from [5], where we already indicated in Section 14.4 results of the kind stated here without formulating the corresponding theorem explicitly. A description of closed sets of positive Horn formulas has already been presented – for homogeneous partial algebras – in [1], Theorem 3.4.5, and in [5], Theorem 14.3.3, where even infinitary operations are allowed, while we want to restrict considerations in this note to finitary similarity types, although there are no great differences to a treatment of infinitary similarity types (see [5]). Our methods are based, say, on [3], [8] or [2], where an algebraic encoding of implications and of their satisfaction is presented. We include a list of derivation rules for universal Horn formulas and a description of the closure operator of consequences corresponding to some axiom system in the case of positive Horn formulas. This may help with respect to a more specific algorithmic treatment.

Let us recall from [1] or [5] that a class $K$ of (heterogeneous partial) algebras is definable by positive universal Horn formulas iff it is closed with respect to the formation of isomorphic copies, subalgebras and reduced products of $K$-algebras, and it is definable by universal Horn formulas –

including negative ones — iff it is closed with respect to the formation of isomorphic copies, subalgebras and reduced products with respect to proper filters and non-empty families of $K$-algebras. In each case such a class contains an initial object.

## 1. Some basic concepts

Although we hope that the reader is familiar with the basic concepts needed (see [1], [4], [9] or [5]), we briefly sketch those which we think most important to understand this note.

Let $\Omega$ be a set which will be called the *set of operation symbols*. A (*finitary*) *similarity type* $\tau$ is a sequence $(\tau =)\ (n_\varphi)_{\varphi\in\Omega}$ of natural numbers — including zero — such that each $\varphi\in\Omega$ designates an $n_\varphi$-ary operation symbol. Moreover, let $S$ be any set, chosen as *set of sorts*. A *heterogeneous signature* $\Sigma$ (with respect to the similarity type $\tau$ and the set $S$ of sorts) then combines with each operation symbol $\varphi$ and $n_\varphi$-ary sequence (INPUT$(\varphi)$: =) $(s_{1,\varphi}, s_{2,\varphi}, \ldots, s_{n_\varphi,\varphi})\in S^{n_\varphi}$ (of the "input sorts" for the arguments of $\varphi$) of elements of $S$ and an element (OUTPUT$(\varphi)$: =) $s_\varphi$ of $S$, the (*output*) *sort* of $\varphi$. A *heterogeneous $\Sigma$-algebra* $\boldsymbol{A} = (A, (\varphi^{\boldsymbol{A}})_{\varphi\in\Omega})$ consists of a family $A = (A_s)_{s\in S}$ of sets, and for each operation symbol $\varphi$ one has an $n_\varphi$-ary heterogeneous operation $\varphi^{\boldsymbol{A}}$: $A_{s_{1,\varphi}} \times \ldots \times A_{s_{n_\varphi,\varphi}} \to A_{s_\varphi}$. $A_s$ is called the *phylum* of $\boldsymbol{A}$ of sort $s$ and $A$ is called a *sorted set*; the elements of $A_s$ are called the *elements of $\boldsymbol{A}$ of sort $s$*, and sometimes we shall treat $A$ just as if it were the disjoint union of the family $(A_s)_{s\in S}$ (see also the remark below). Observe that we allow phyla to be empty as is done in [6], [9] or [5].

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two $\Sigma$-algebras; a ($\Sigma$-) *homomorphism* $f$: $\boldsymbol{A}\to\boldsymbol{B}$ is a family $(f_s\colon A_s \to B_s)_{s\in S}$ of mappings between the corresponding phyla such that for each operation symbol $\varphi$ and for each sequence $(a_1, \ldots, a_{n_\varphi})\in A_{s_{1,\varphi}} \times \times \ldots \times A_{s_{n_\varphi,\varphi}}$ in $A$ of the appropriate sorts one has

$$f_{s_\varphi}(\varphi^{\boldsymbol{A}}(a_1, \ldots, a_{n_\varphi})) = (\varphi^{\boldsymbol{B}}(f_{s_{1,\varphi}}(a_1), \ldots, f_{s_{n_\varphi,\varphi}}(a_{n_\varphi})).$$

A *congruence relation* $R$ of a $\Sigma$-algebra $\boldsymbol{A}$ is a family $(R_s)_{s\in S}$ of equivalence relations, $R_s$ being an equivalence relation on $A_s$, such that for each fundamental operation $\varphi^{\boldsymbol{A}}$ and any sequences $(a_1, \ldots, a_{n_\varphi})$ and $(b_1, \ldots, b_{n_\varphi})$ in $A$ of elements of the appropriate sorts one has:

$$a_i\, R_{s_{i,\varphi}}\, b_i \quad \text{for} \quad 1\leqslant i\leqslant n_\varphi \quad \text{imply} \quad \varphi^{\boldsymbol{A}}(a_1, \ldots, a_{n_\varphi})\, R_{s_\varphi}\, \varphi^{\boldsymbol{A}}(b_1, \ldots, b_{n_\varphi}).$$

The intersection of congruence relations means the family of intersections of the corresponding equivalence relations on the phyla.

If $R$ is a sorted binary relation on $A$, i.e., a family $(R_s)_{s\in S}$ such that each $R_s$ is a binary relation on $A_s$, then we denote by $\mathrm{Cong}_{\mathbf{A}}(R)$ (or simply by $\mathrm{Cong}(R)$ if the reference to the algebra under consideration is clear) the smallest congruence relation of $\mathbf{A}$ which contains $R$, i.e., the intersection of all congruence relations on $\mathbf{A}$ which contain $R$. For a $\Sigma$-algebra $\mathbf{A}$ and a congruence relation $R$ on $\mathbf{A}$ the *factor algebra* of $\mathbf{A}$ with respect to $R$ is denoted by $\mathbf{A}/R$.

If $f$: $\mathbf{A}\to\mathbf{B}$ is a homomorphism, then $\ker f$ designates the congruence relation on $A$ induced by $f$, i.e., $(\ker f)_s := \{(a, a')|\ a, a'\in A_s, f_s(a) = f_s(a')\}$ for each $s\in S$.

*Remark.* For those who are familiar with the concepts concerning partial algebras we want to hint upon the observations in [4] and [5] that a heterogeneous signature can be considered as a partial algebraic structure $(\varphi^S)_{\varphi\in\Omega}$ on the set $S$ of sorts such that each partial operation $\varphi^S$ has as domain exactly one sequence, namely $(s_{1,\varphi}, s_{2,\varphi}, \ldots, s_{n_\varphi,\varphi})$, and the corresponding value is $s_\varphi$, i.e., $\varphi^S(s_{1,\varphi}, s_{2,\varphi}, \ldots, s_{n_\varphi,\varphi}) = s_\varphi$.

Let us consider for a $\Sigma$-algebra $\mathbf{A}$ the carrier $A$ as the disjoint union of the phyla; then we get a mapping $v_A$: $A\to S$, the *sort mapping*, assigning to each element of $A_s$ its sort $s$, and $\varphi^A$ can be considered as a partial operation on $A$ in such a way that $v_A$ becomes a closed homomorphism from $\mathbf{A}$ into $\mathbf{S}$ (i.e., $\varphi^A$ is defined on a sequence in $A$ iff $\varphi^S$ is defined on its image sequence in $S$ with respect to $v_A$). A homomorphism $f$: $\mathbf{A}\to\mathbf{B}$ between $\Sigma$-algebras then becomes a homomorphism between partial algebras *satisfying* $v_B\cdot f = v_A$, *i.e. being compatible with the sort mappings*; conversely, each closed homomorphism between the partial algebras $\mathbf{A}$ and $\mathbf{B}$ which is compatible with their sort mappings is a homomorphism between the corresponding $\Sigma$-algebras. Thus a $\Sigma$-algebra can be represented by a partial algebra together with its sort mapping – and conversely.

A ($\Sigma$-) *term* of sort $s$ on a sorted set $X = (X_s)_{s\in S}$ of variables is defined as follows:

(1) Each $x\in X_s$ is a term of sort $s$.

(2) If $\varphi$ is an operation symbol of sort $s_\varphi = s$ with input sequence $(s_{1,\varphi}, s_{2,\varphi}, \ldots, s_{n_\varphi,\varphi})$, and if – for $1\leqslant i\leqslant n_\varphi$ – $t_i$ is a term of sort $s_{i,\varphi}$, then $\varphi t_1\ldots t_{n_\varphi}$ – the concatenation of the symbol $\varphi$ with the terms under consideration – is again a term of sort $s$.

$T_X$ designates the $S$-family of the smallest sets of terms on $X$ (of sort $s$ for each $s$ in $S$) satisfying (1) and (2). The corresponding $\Sigma$-algebra of terms on $X$ is denoted by $\mathbf{T}_X$ $(\varphi^{\mathbf{T}_X}$: $(t_1, \ldots, t_{n_\varphi})\mapsto\varphi t_1\ldots t_{n_\varphi})$.

It is well known that each sort respecting mapping $f$: $X\to A$ allows a homomorphic extension $\bar{f}$: $\mathbf{T}_X\to\mathbf{A}$.

Let $X$ be a sorted set of variables, and let $t$ and $t'$ be any two terms on $X$; an *equation* $(X\colon t \overset{e}{=} t')$ can be formed by them iff $t$ and $t'$ are terms of the same sort — we use the symbol "$\overset{e}{=}$" as equality symbol of the object language, since "$^e$" shall remind the reader of the fact that in the case of partial algebras the corresponding semantics also makes a statement about the fact that for each term involved the interpretation has to exist (see [1], [4], or [5]). An equation $(X\colon t \overset{e}{=} t')$ is valid in some $\Sigma$-algebra $A$ iff each homomorphism $f\colon T_X \to A$ identifies $t$ and $t'$.

Universal first order formulas on $X$ are built in the usual way with the symbols "$\neg$" for negation, "$\wedge$" for conjunction and "$\Rightarrow$" for implication, i.e., if $(M\colon \Phi)$ and $(M\colon \Psi)$ are any first order formulas on $M$, then so are $(M\colon \neg\Phi)$, $(M\colon \Phi \wedge \Psi)$ and $(M\colon \Phi \Rightarrow \Psi)$ — as usual we omit brackets whenever possible, and $\neg$ is binding stronger than $\wedge$, which is binding stronger than $\Rightarrow$.

Observe that each formula has to be connected with a finite set of variables containing all its free variables, since the semantics of a formula depends on the set of variables it is connected with because of the fact that we allow empty phyla: If $X$ does not allow a mapping into a $\Sigma$-algebra $A$ — since in $A$ some phylum is empty which is non-empty in $X$ —, then each universal first order formula on $X$ is trivially valid in $A$. In a universal formula $(X\colon \Phi)$ $X$ is called the *reference set of variables*, while $\mathrm{fvar}(\Phi)$ designates the set of all (free) variables really occurring in $\Phi$.

Semantics are extended from equations to other universal formulas in the usual way.

## 2. Universal Horn formulas

A *universal Horn formula* is now — as usual — either an elementary implication of the form

(pHf)

$(M\colon t_1 \overset{e}{=} t_1' \wedge \ldots \wedge t_n \overset{e}{=} t_n' \Rightarrow t \overset{e}{=} t')$ (positive universal Horn formula)

or a negation of a conjunction of equations:

(nHf)

$(M\colon \neg(t_1 \overset{e}{=} t_1' \wedge \ldots \wedge t_n \overset{e}{=} t_n'))$ (negative universal Horn formula).

Since we want to present an algebraic and set-theoretical description of closed sets of universal Horn formulas, we have first to give a set-theoretical description of universal Horn formulas:

The positive universal Horn formula (pHf) is represented by an ordered pair consisting of the set of the pairs of the terms occurring in the equations of the premise together with all the pairs representing the variables and a pair of terms representing the equation of the conclusion

$$\text{(pHf)}' \qquad (\{(x, x), (t_i, t_i')|\ x\in X,\ 1 \leqslant i \leqslant n\}, (t, t')),$$

and the negative universal Horn formula (nHf) is represented by the set of all pairs representing the equations in (nHf) and of pairs representing the variables in $X$:

$$\text{(nHf)}' \qquad \{(x, x), (t_i, t_i')|\ x\in X,\ 1 \leqslant i \leqslant n\}.$$

With respect to these representations we can say that the positive universal Horn formula (pHf) is valid in some $\Sigma$-algebra $A$ iff for every homomorphism $f\colon T_X \to A$ one has:

If $f$ identifies all pairs of terms representing the premise, then it also identifies the terms representing the conclusion, i.e.,

$$\{(t_i, t_i')|\ 1 \leqslant i \leqslant n\} \subseteq \ker f \quad \text{implies} \quad (t, t')\in \ker f.$$

Similarly the negative universal Horn formula (nHf) is valid in $A$ iff no homomorphism from $T_X$ into $A$ satisfies all the equations involved, i.e., iff one has for every homomorphism $f\colon T_X \to A$:

$$\{(t_i, t_i')|\ 1 \leqslant i \leqslant n\} \nsubseteq \ker f.$$

We shall call a pair $(t, t')$ of terms on some sorted set of variables *equational*, if $t$ and $t'$ are terms of the same sort.

Then let us finally observe that for any finite set $F = \{(t_i, t_i')|\ 1 \leqslant i \leqslant n\}$ of equational pairs of terms and for an equational pair $(t, t')$ on some sorted set $X$ of variables $(F, (t, t'))$ represents the positive universal Horn formula

$$(\mathrm{fvar}(\{t, t', t_i, t_i'|\ 1 \leqslant i \leqslant n\})\colon t_1 \stackrel{e}{=} t_1' \wedge \ldots \wedge t_n \stackrel{e}{=} t_n' \Rightarrow t \stackrel{e}{=} t'),$$

and $F$ represents the negative universal Horn formula

$$(\mathrm{fvar}(\{t_i, t_i'|\ 1 \leqslant i \leqslant n\})\colon \neg(t_1 \stackrel{e}{=} t_1' \wedge \ldots \wedge t_n \stackrel{e}{=} t_n')).$$

Moreover, if we say that $F$ is a set of equational pairs on a sorted set $M$, then this will usually mean that $M = \mathrm{fvar}(\{t, t'|\ (t, t')\in F\})$, i.e. that $M$ is exactly the set of all variables really occurring in some of the terms contained in an equational pair of $F$.

## 3. The main result

In what follows we shall always assume that the representations (pHf)′ respectively (nHf)′ really *are* the corresponding universal Horn formulas, i.e., when we speak of a universal Horn formula, then it is given either as (pHf)′ – in the positive case – or as (nHf)′ – in the negative case ((pHf) respectively (nHf) are just other codes for it). Moreover, we assume $X$ to be a sorted set of variables with a countably infinite phylum of each sort, and each sorted set of variables considered below is assumed to be a subset of $X$. Let $Q^*$ be any set of universal Horn formulas; then $Q$ will designate the set of all positive universal Horn formulas in $Q^*$, and $\mathrm{Neg}(Q^*)$ the set of all negative universal Horn formulas contained in $Q^*$, while for each finite set $F := \{(t_i, t_i') \mid 1 \leqslant i \leqslant n\}$ of equational pairs of terms we denote by $Q(F)$ the set of all equational pairs $(t, t')$ such that $(F, (t, t'))$ belongs to $Q^*$, i.e.,

$$Q(F) := \{(t, t') \mid (F, (t, t')) \in Q^*\}.$$

Moreover, for any class $K$ of $\Sigma$-algebras let $\mathbf{Horn}(K)$ designate the class of all universal Horn formulas which are valid in each $K$-algebra, while for some set $H$ of universal Horn formulas $\mathbf{Mod}(H)$ will designate the class of all $\Sigma$-algebras in which every Horn formula from $H$ is valid.

In this terminology closed sets of universal Horn formulas can be described as follows:

THEOREM. *Let $Q^* = Q \cup \mathrm{Neg}(Q^*)$ be a set of universal Horn formulas. Then the following statements are equivalent*:

(i) $Q^* = \mathbf{Horn\,Mod}(Q^*)$.

(ii) *For any finite sets $F$, $F'$ of equational pairs on sorted sets $M$ and $M'$ of variables, respectively, one has*:

(I1) *$Q(F)$ is a congruence relation on $T_M$.*

(I2) $F \subseteq Q(F)$.

(I3) *For every homomorphism $f\colon T_M \to T_{M'}$ which satisfies $(f \times f)(F) \subseteq Q(F')$, i.e., which maps $F$ into the $Q^*$-consequences of $F'$, one also has $(f \times f)(Q(F)) \subseteq Q(F')$, i.e., $f$ then also maps all the $Q^*$-consequences of $F$ into those of $F'$.*

(N1) *$F \in \mathrm{Neg}(Q^*)$ implies $Q(F) = T_M \times T_M$.*

(N2) *If $F \in \mathrm{Neg}(Q^*)$, and if there is a homomorphism $f\colon T_M \to T_{M'}$ which maps $F$ into the $Q^*$-consequences of $F'$, then one also has $F' \in \mathrm{Neg}(Q^*)$.*

Notice that in the case when $F = F' = \emptyset$ in the above theorem in (ii), and if we are in the homogeneous case and forget about the possible empty algebra, then we can choose $M = M' = X$, and (I1) and (I3) just describe the set of all equations valid in $\mathbf{Mod}(Q^*)$, in particular (I3) just describes full invariance of the congruence relation $Q(\emptyset)$. Thus this theorem is indeed a direct generalization of the well-known Birkhoff Theorem for equational theories of homogeneous algebras. In the heterogeneous equational case (I1) and (I3) are just the algebraic counterpart of the description of closed sets of equations for heterogeneous algebras as given by Goguen and Meseguer in [6]. If "**Horn**" only stands for positive Horn formulas, i.e., for elementary implications with one conclusion (i.e., for quasi-equations), then (i) is obviously equivalent to properties (I1), (I2) and (I3) of (ii).

*Proof.* Let us first show that (i) implies (ii). Thus assume that $Q^* = \mathbf{Horn\,Mod}(Q^*)$; then we have to show that (I1), (I2), (I3), (N1) and (N2) are true:

Let $F$ be a finite set of, say $n$, equational pairs on the finite sorted set $M$ of variables, and let $F^* := (M\colon p_1 \stackrel{e}{=} q_1 \wedge \ldots \wedge p_n \stackrel{e}{=} q_n)$ be the corresponding conjunction of equations contained in $F$. Moreover, let $F$ belong to $\mathrm{Neg}(Q^*)$. Then $F^*$ cannot be satisfied in any model of $Q^*$, and therefore $F^*$ implies any equation between terms with variables in $M$; thus (N1) is proved. Moreover, if $F'$ is another finite set of equational pairs on some finite sorted set $M'$ of variables, and if $f\colon T_M \to T_{M'}$ is any homomorphism mapping $F$ into the $Q^*$-consequences of $F'$, then the conjunction $F'^*$ of all equations contained in $F'$ cannot be satisfied for any valuation into any model of $Q^*$, since otherwise the restriction of $f$ to $M$ followed by the homomorphic extension to $T_{M'}$ of this valuation would yield a valuation of $M$ which satisfies $F^*$ in contradiction to the assumption $F \in \mathrm{Neg}(Q^*)$. Thus also $F'$ has to belong to $\mathrm{Neg}(Q^*)$, and (N2) has been proved.

(I1) and (I2) easily follow from the fact that for each set $F$ of equations on some sorted set $M$ of variables the set of all consequences of $F$ with respect to $\mathbf{Mod}(Q^*)$ is the intersection of all congruence relations induced by homomorphisms $g$ from $T_M$ into models of $Q^*$ such that $g$ identifies all pairs in $F$: this intersection is itself a congruence relation – containing $F$ – on $T_M$.

Finally, let $F$ and $F'$ be finite sets of equations on finite sorted sets $M$ and $M'$ respectively, and let $f\colon T_M \to T_{M'}$ be a homomorphism mapping $F$ into the consequences of $F'$ with respect to $\mathbf{Mod}(Q^*)$. Let $v$ be a valuation of $M'$ into some model of $Q^*$, and let $v'$ be its homomorphic extension to $T_{M'}$. Assume that $v'$ identifies all pairs in $F'$; then it also identifies all pairs in $Q(F')$, and therefore $v' \cdot f$ identifies all pairs in $F$ and therefore also all pairs in $Q(F)$. This shows that $f$ maps $Q(F)$ into the kernel of $v'$. Since this is true for each valuation of $M'$ identifying $F'$ in some model of $Q^*$, $f$ has to map $Q(F)$ into the intersection of all the corresponding kernels, i.e., into $Q(F')$. If there is no such valuation starting from $M'$, then $F'$ belongs to $\mathrm{Neg}(Q^*)$, $Q(F')$ equals $T_{M'} \times T_{M'}$ and the statement of (I3) is trivially true. Thus also (I3) and therefore all properties stated in (ii) have been proved for $Q^*$.

Conversely, let us assume that all the properties of (ii) are true for the set $Q^*$ encoding universal Horn formulas. Define

$$K := \{T_M/Q(F) \mid F \notin \mathrm{Neg}(Q^*) \text{ is a finite set of equations with variables in a finite sorted subset } M \text{ of the sorted set } X \text{ of variables}\}.$$

If we can show that $Q^* = \mathbf{Horn}(K)$, then the general properties of a Galois correspondence imply that (i) is true:

Let $(F, (t, t'))$ be any element of $Q^*$ with a finite sorted set $M$ of variables, i.e., $F$ is a finite set of equational pairs of terms from $T_M$.

– Let us first assume that $F$ belongs to $\mathrm{Neg}(Q^*)$, and as above let $F^*$

designate the conjunction of the equations contained in $F$. Let us assume that there exists some finite set $F'$ of equations on some finite heterogeneous set $M'$ of variables such that $F'$ does not belong to $\mathrm{Neg}(Q^*)$ and that $F^*$ can be satisfied in $T_{M'}/Q(F')$. Then this means that there exists a homomorphism $g\colon T_M/\mathrm{Cong}(F) \to T_{M'}/Q(F')$ and therefore there exists a homomorphism $f\colon T_M \to T_{M'}$ mapping $F$ into $Q(F')$. Thus (N2) implies $F' \in \mathrm{Neg}(Q^*)$ contrary to our assumption on $F'$. Hence the negation of $F^*$ as well as the implication $(F, (t, t'))$ are valid in $\boldsymbol{K}$.

– Next, assume that $(F, (t, t')) \in Q$, and that $F$ does not belong to $\mathrm{Neg}(Q^*)$. Let, for some finite subset $F'$ of $T_{M'} \times T_{M'}$ not belonging to $\mathrm{Neg}(Q^*)$ ($M'$ a finite sorted subset of our sorted set of variables), $f\colon T_M \to T_{M'}$ be any homomorphism mapping $F$ into $Q(F')$, i.e., $f$ induces – respectively is induced by – a homomorphism $g\colon T_M/\mathrm{Cong}(F) \to T_{M'}/Q(F')$. Then (I3) tells us that $f$ also maps $(t, t')$ into $Q(F')$, i.e., the implication corresponding to $(F, (t, t'))$ is satisfied in $T_{M'}/Q(F')$ with respect to the chosen valuation of $M$ into $T_{M'}/Q(F')$. Since this is true for any such valuation, the implication $(F, (t, t'))$ is valid in $\boldsymbol{K}$.

These arguments show that $Q^*$ is a subset of **Horn Mod**$(\boldsymbol{K})$.

Finally, let $(F, (t, t'))$ be an elementary implication which is valid in $\boldsymbol{K}$ – with finite sorted set $M$ of variables:

– If $F^*$ – the conjunction of equations contained in $F$ – cannot be satisfied in $\boldsymbol{K}$, then $F$ belongs to $\mathrm{Neg}(Q^*)$, since (I2) implies that each finite set $F'$ – not belonging to $\mathrm{Neg}(Q^*)$ – of equational pairs is at least satisfied in $T_{\mathrm{var}(F')}/Q(F')$. (N1) then tells us, too, that $(F, (t, t'))$ belongs to $Q^*$.

– If $F^*$ above can be satisfied in $\boldsymbol{K}$, then $F$ cannot belong to $\mathrm{Neg}(Q^*)$ – for else we would get a contradiction with (N2). Therefore let us consider the identity homomorphism $f\colon T_M \to T_M$. $f$ maps $F$ into $Q(F)$, by (I2), i.e., the equations in $F$ are satisfied in the object $T_M/Q(F)$ of $\boldsymbol{K}$. The valuation corresponding to $f$ followed by the quotient mapping with respect to $Q(F)$ only satisfies equations represented in $Q(F)$, i.e., $(t, t')$ has to belong to $Q(F)$, and therefore $(F, (t, t'))$ belongs to $Q^*$.

Thus equality between $Q^*$ and **Horn**$(\boldsymbol{K})$ has been established; and the theorem has altogether been proved.

*Remark.* Observe that (N1) does not say that "$Q(F) = T_M \times T_M$" implies that $F$ belongs to $\mathrm{Neg}(Q^*)$, even when $\mathrm{Neg}(Q^*)$ is non-empty.

However, let us define for a closed set $Q^*$ of universal Horn formulas the set

$\mathrm{Possible\,Neg}(Q^*) := \{F \mid F$ is a finite set of equational pairs on some sorted subset $M := \mathrm{fvar}(F)$ of $X$ of variables such that $Q(F) = T_M \times T_M\}$ of all "candidates" for negative Horn formulas with respect to the positive

Horn formulas in $Q^*$. Let, for any sorted set $B$, Sorts$(B)$ be the set of all sorts $s$ of $S$ for which $B$ has a non-empty phylum.

Moreover, define

$$\text{Sorts Possible Neg}(Q^*) := \{S' \mid S' = \text{Sorts}(T_{\text{fvar}(F)}),\ F \in \text{Possible Neg}(Q^*)\}.$$

Then (N1) and (N2) together imply that the set $V$ of all sets Sorts$(T_{\text{fvar}(F)})$ for some $F$ in Neg$(Q^*)$ forms an upper end of Sorts Possible Neg$(Q^*)$ with respect to set-theoretic inclusion, and we have

$$F \in \text{Neg}(Q^*) \quad \text{iff} \quad F \in \text{Possible Neg}(Q^*) \quad \text{and} \quad \text{Sorts}(T_{\text{fvar}(F)}) \quad \text{belongs to } V.$$

Conversely, it is easy to realize (see also [5], Proposition 14.4.4) that for any closed set $Q$ of positive universal Horn formulas and any upper end $V$ of Sorts Possible Neg$(Q)$ one has:

The union $Q^*$ of $Q$ with the set of all sets $F$ in Possible Neg$(Q)$ for which Sorts$(T_{\text{fvar}(F)})$ belongs to $V$ is a closed set of universal Horn formulas.

## 4. Corresponding rules of derivation

From the above theorem it is quite easy to deduce the well-known rules of derivation for elementary implications with respect to a given implicational axiom system.

The following statements are equivalent for some set $Q^*$ of universal Horn formulas:

(i) $Q^*$ is closed, i.e., $Q^* = \mathbf{Horn\,Mod}(Q)$.

(ii) $Q$ is closed with respect to the following special derivation rules (general rules are the transitivity of derivation, and finiteness of derivation), where the premises $F$ and $F'$ are in each case finite sets of equational pairs on sorted sets $M, M' \subseteq X$ of variables, respectively, and in particular $F^* = t_1 \stackrel{e}{=} t_1' \wedge \ldots \wedge t_n \stackrel{e}{=} t_n'$.

(IR1) $\vdash (M\colon F^* \Rightarrow t \stackrel{e}{=} t)$ for all $t \in T_M$ ((I1), reflexivity).

(IR2) $(M\colon F^* \Rightarrow t \stackrel{e}{=} t') \vdash (M\colon F^* \Rightarrow t' \stackrel{e}{=} t)$ ((I1), symmetry).

(IR3) $(M\colon F^* \Rightarrow t \stackrel{e}{=} t'),\ (M\colon F^* \Rightarrow t' \stackrel{e}{=} t'') \vdash (M\colon F^* \Rightarrow t \stackrel{e}{=} t'')$ ((I1), transitivity).

(IR4) $\{(M\colon F^* \Rightarrow t_k \stackrel{e}{=} t_k') \mid 1 \leqslant k \leqslant n_\varphi\} \vdash (M\colon F^* \Rightarrow \varphi t_1 \ldots t_{n_\varphi} \stackrel{e}{=} \varphi t_1' \ldots t_{n_\varphi}')$, $\varphi \in \Omega$ ((I1), compatibility).

(IR5) $\vdash (M\colon F^* \Rightarrow t_i \stackrel{e}{=} t_i')$, $1 \leqslant i \leqslant n$ ((I2)).

(IR6) $(M\colon F^* \Rightarrow t \stackrel{e}{=} t')$,
$\{(M'\colon F'^* \Rightarrow t_i(q_m \mid m \in M) \stackrel{e}{=} t_i'(q_m \mid m \in M)) \mid 1 \leqslant i \leqslant n\}$
$\vdash (M'\colon F'^* \Rightarrow t(q_m \mid m \in M) \stackrel{e}{=} t'(q_m \mid m \in M))$,
$t, t', t_i, t_i', q_m$ arbitrary terms ($1 \leqslant i \leqslant n$, $m \in M$) on the

corresponding sets of variables, each $q_m$ being a term on $M'$ of the same sort as $m$, and $t(q_m|\ m \in M)$ — and each of the corresponding other terms — is obtained from $t$ by replacing each occurrence of $m$ in $t$ by the term $q_m$ ((I3)).

(NR1) $(M\colon \neg F^*) \vdash (M\colon F^* \Rightarrow t \stackrel{e}{=} t')$, $(t, t')$ any equational pair of $T_M$ ((N1)).

(NR2) $(M\colon \neg F^*)$, $\{(M'\colon F'^* \Rightarrow t_i(q_m|\ m \in M) \stackrel{e}{=} t_i'(q_m|\ m \in M))|\ 1 \leqslant i \leqslant n\}$ (for some family $(q_m|\ m \in M)$ of terms in $T_{M'}$) $\vdash (M'\colon \neg F'^*)$ ((N2)).

It is easily derivable from the above theorem that this system of rules is complete and sound. And it is also obvious that in the case of positive universal Horn formulas we need only rules (IR1) through (IR6).

## 5. The closure system of consequences for a given axiom system

For the case of total $\Sigma$-algebras we add another description of the classes $Q(F)$ for a closed set $Q$ of positive universal Horn formulas, when $Q$ is generated by an axiom system, say $(I_1 := (F^1, (t^1, t'^1)), \ldots, I_n := (F^n, (t^n, t'^n)))$, of positive universal Horn formulas. Naturally, the remarks of this section will also work for an infinite axiom system, but they will not work so easily in the partial case, where not only the congruence relations are to be generated, but also their domains. We shall show the

Proposition. *For each finite sorted set $M$ of variables the sets $Q(F)$, $F$ a set of equational pairs with set $M$ of variables, form an algebraic closure system of congruence relations on $T_M$.*

In the proof a system of finitary partial operations on $T_M \times T_M$ is given such that the system of $Q(F)$ becomes the system of subalgebras of the corresponding partial algebra.

For more information of this kind see [5], Section 14, or [7]; here we only want to list the partial operations which prove the above statements:

For each term $t$ in $T_M$ we introduce a fundamental constant $c_t^M := (t, t)$ (the $c_m^M$ $(m \in M)$ would suffice).

Moreover, let $\sigma$ be a binary operation defined for any two equational pairs of terms of the same sort by

$$\sigma^M((a, b), (c, d)) := \begin{cases} (a, d) & \text{if } b = c, \\ (b, a) & \text{else.} \end{cases}$$

Thus $\sigma$ takes care as well of transitivity (first part of the definition) as of symmetry (second part of the definition). The operations defined so far will take care of the fact that we deal with equivalence relations.

For each fundamental operation symbol $\varphi$ we define $\varphi^M$ to be the induced product operation of $T_M \times T_M$. This takes care of the fact that the closed sets will be congruence relations.

Now consider the implications

$$I_k = (M_k\colon t_1^k \stackrel{e}{=} t_1'^k \wedge \ldots \wedge t_{l_k}^k \stackrel{e}{=} t_{l_k}'^k \Rightarrow t^k \stackrel{e}{=} t'^k), \qquad 1 \leqslant k \leqslant n.$$

We define for it an $n_k$-ary partial operation $\iota_k$ whose graph is the following relation:

$$\operatorname{graph}(\iota_k) := \{((f(t_1^k), f(t_1'^k)), \ldots, (f(t_{l_k}^k), f(t_{l_k}'^k))), (f(t^k), f(t'^k)) \mid f\colon T_{M_k} \to T_M \text{ is any homomorphism}\},$$

i.e., $\iota_k$ maps any sequence in $T_M \times T_M$ obtained from the premise of $I_k$ by substituting each of its variables $m$ by $f(m)$ onto $(f(t^k), f(t'^k))$. This takes care of the substitution rule (I3).

The closed sets of $T_M \times T_M$ with respect to this algebraic structure are now easily seen to be congruence relations $R$ on $T_M$ such that $T_M/R$ is a model of the given axioms; and it is also obvious that each congruence relation with this last property has to be a closed set with respect to the operations defined above.

## 6. Partial case

Although we have proved the main result only for total $\Sigma$-algebras, we think that the treatment of negative Horn formulas becomes especially important in the case of partial $\Sigma$-algebras, where one forbids e.g. the existence of special terms in some objects (e.g. in rings with partial inverse zero must not have a partial inverse, or when considering projective planes as partial two-sorted algebras the connecting line of two identical points and the intersection point of two identical lines must not exist). Here mainly the case of universal Horn formulas is of importance, which we have called in [5] "existentially conditioned universal Horn formulas" (ECH-formulas), where the premise of a positive ECH-formula only contains existence equations where both terms are equal, and the same is true for negative ECH-formulas. Since we do not want in this note do develop the techniques of dealing with partial algebras, we only present to the interested reader who is familiar with these techniques the differences of the results for total and for partial $\Sigma$-algebras, and only in the case of universal Horn formulas (the case of ECH-formulas reduces for total $\Sigma$-algebras to the equational case).

THEOREM (partial case). *Let $Q^* = Q \cup \operatorname{Neg}(Q^*)$ be a set of universal Horn formulas. Then the following statements are equivalent*:

(i) $Q^* = \mathbf{Horn\,Mod}(Q^*)$.

(ii) *For any finite sets $F$, $F'$ of equational pairs on sorted sets $M$ and $M'$ of variables, respectively, one has*:

(I1p) *$Q(F)$ is a closed congruence relation on the relative subalgebra $\downarrow Q(F)$ of $T_M$ which is formed by all subterms of terms occurring in $Q(F)$; $\downarrow Q(F)$ is generated by $M$.*

(I2p) $F \subseteq Q(F)$.

(I3p) *For every homomorphism $f\colon \downarrow F \to \downarrow Q(F)$ (where $\downarrow F$ is the relative subalgebra of $T_M$ of all subterms of terms in $F$ joined with $M$) which satisfies $(f \times f)(F) \subseteq Q(F')$, i.e., which maps $F$ into the $Q^*$-consequences of $F'$, one also has $(f \times f)(Q(F)) \subseteq Q(F')$, i.e., $f$ then also maps all the $Q^*$-consequences of $F$ into those of $F'$.*

(N1p) *$F \in \mathrm{Neg}(Q^*)$ implies $Q(F) = T_M \times T_M$.*

(N2p) *If $F \in \mathrm{Neg}(Q^*)$, and if there is a homomorphism $f\colon \downarrow F \to \downarrow Q(F)$ which maps $F$ into the $Q^*$-consequences of $F'$, then one also has $F' \in \mathrm{Neg}(Q^*)$.*

Similarly the rules (IR1) through (IR5) become only a little bit more complicated and can be formulated as follows, where

$$F^* = t_1 \stackrel{e}{=} t'_1 \wedge \ldots \wedge t_n \stackrel{e}{=} t'_n,$$

and where $F'^*$ is also a finite conjunction of equations:

(IR1p) $\vdash (M\colon F^* \Rightarrow x \stackrel{e}{=} x)$ for all $x \in M$ ((I1p), reflexivity).

(IR2p) $(M\colon F^* \Rightarrow t \stackrel{e}{=} t') \vdash (M\colon F^* \Rightarrow t \stackrel{e}{=} t)$ ((I1p), reflexivity).

(IR3p) $(M\colon F^* \Rightarrow t \stackrel{e}{=} t') \vdash (M\colon F^* \Rightarrow t' \stackrel{e}{=} t)$ ((I1p), symmetry).

(IR4p) $(M\colon F^* \Rightarrow t \stackrel{e}{=} t')$, $(M\colon F^* \Rightarrow t' \stackrel{e}{=} t'') \vdash (M\colon F^* \Rightarrow t \stackrel{e}{=} t'')$ ((I1p), transitivity).

(IR5p) $\{(M\colon F^* \Rightarrow t_k \stackrel{e}{=} t'_k) \mid 1 \leqslant k \leqslant n_\varphi\}$,
$(M\colon F^* \Rightarrow \varphi t_1 \ldots t_{n_\varphi} \stackrel{e}{=} \varphi t_1 \ldots t_{n_\varphi}) \vdash (M\colon F^* \Rightarrow \varphi t_1 \ldots t_{n_\varphi} \stackrel{e}{=} \varphi t'_1 \ldots t'_{n_\varphi})$,
$\varphi \in \Omega$ ((I1p), closed congruence relation).

(IR6p) $(M\colon F^* \Rightarrow \varphi t_1 \ldots t_{n_\varphi} \stackrel{e}{=} \varphi t_1 \ldots t_{n_\varphi}) \vdash (M\colon F^* \Rightarrow t_k \stackrel{e}{=} t_k)$ $1 \leqslant k \leqslant n_\varphi$, $\varphi \in \Omega$ ((I1p), $M$-generated)

(IR7p) $\vdash (M\colon F^* \Rightarrow t_i \stackrel{e}{=} t'_i)$, $1 \leqslant i \leqslant n$ ((I2p)).

(IR8p) $(M\colon F^* \Rightarrow t \stackrel{e}{=} t')$, $\{(M'\colon F'^* \Rightarrow q_m \stackrel{e}{=} q_m) \mid m \in M\}$,
$\{(M'\colon F'^* \Rightarrow t_i(q_m \mid m \in M) \stackrel{e}{=} t'_i(q_m \mid m \in M)) \mid 1 \leqslant i \leqslant n\}$
$\vdash \big(M'\colon F'^* \Rightarrow t(q_m \mid m \in M) \stackrel{e}{=} t'(q_m \mid m \in M)\big)$ ($t$, $t'$, $t_i$, $t'_i$, $q_m$ arbitrary terms ($1 \leqslant i \leqslant n$, $m \in M$) on the corresponding sets of variables, each $q_m$ being a term on $M'$ of the same sort as $m$, and $t(q_m \mid m \in M)$ – and each of the corresponding other terms – is obtained from $t$ by replacing each occurrence of $m$ in $t$ by the term $q_m$) ((I3p)).

(NR1p) $(M\colon \neg F^*) \vdash (M\colon F^* \Rightarrow t \stackrel{e}{=} t')$, $(t, t')$ any equational pair of $T_M$ ((N1p)).

(NR2p) $(M\colon \neg F^*)$, $\{(M'\colon F'^* \Rightarrow q_m \stackrel{e}{=} q_m) \mid m \in M\}$,
$\{(M'\colon F'^* \Rightarrow t_i(q_m \mid m \in M) \stackrel{e}{=} t_i'(q_m \mid m \in M)) \mid 1 \leqslant i \leqslant n\}$
for some family $(q_m \mid m \in M)$ of terms in $T_{M'} \vdash (M'\colon \neg F'^*)$ ($m$ and $q_m$ have to have the same sort) ((N2p)).

## 7

Let us add some examples for the concepts and results presented in this note:

EXAMPLE 1. Let us first briefly illustrate for the cases of equations and implications the dependence of the semantics of a formula in the heterogeneous case on its reference set of variables:

Let $S := \{s, s', s'', s'''\}$, $\Omega := (\varphi, \varphi', \varphi'', \varphi''')$, $\tau := (0, 1, 2, 2)$,

$\Sigma\colon \varphi \mapsto (( \,) \mapsto s)$ (nullary constant of sort $s$),

$\varphi' \mapsto ((s''') \mapsto s)$ (unary operation of sort $s$ with input sort $s'''$),

$\varphi'' \mapsto ((s, s') \mapsto s'')$,

$\varphi''' \mapsto ((s, s') \mapsto s'')$ (binary operations of sort $s''$ with input sequence $(s, s')$).

Without any axioms around this specification means for a $\Sigma$-algebra $A$:
$A_s$ is always non-empty, since it contains the nullary constant $\varphi^A$ (in the partial case this is achieved by the axiom $\varphi \stackrel{e}{=} \varphi$).

If $A_{s'}$ is also non-empty, then – because of $A_s \neq \emptyset$ and of the specification of $\varphi''$ – $A_{s''}$ is non-empty, too.

Now, let $x$, $x'$, $x''$, $x'''$ be variables of sort $s$, $s'$, $s''$ and $s'''$, respectively. Consider the equations

$$E := (\{x, x'\}\colon \varphi''(x, x') \stackrel{e}{=} \varphi'''(x, x')),$$
$$E' := (\{x, x', x'''\}\colon \varphi''(x, x') \stackrel{e}{=} \varphi'''(x, x')).$$

Then $E'$ is trivially valid in the term algebra $T_{\{x,x'\}}$, while $E$ is not valid in $T_{\{x,x'\}}$.

Notice that with respect to its interpretations $E'$ is equivalent to the implication (= positive universal Horn formula – here we do not need any additional reference to the set of variables)

$$x \stackrel{e}{=} x \wedge x' \stackrel{e}{=} x' \wedge x''' \stackrel{e}{=} x''' \Rightarrow \varphi''(x, x') \stackrel{e}{=} \varphi'''(x, x').$$

EXAMPLE 2. As an example of a total $\Sigma$-algebra where positive Horn formulas are needed, let us consider a signature $\Sigma$, where it shall be specified that a phylum of sort $s$ is (behaves like) a subset of the phylum of sort $s'$, and that all operations $\varphi$ applying to elements of sort $s$ at some place have a corresponding operation $\varphi'$ for which in the input sequence of $\varphi$ each occurrence of $s$ is replaced by $s'$.

For the representation of the inclusion we need a unary total operation $\varphi_0$ mapping elements of sort $s$ onto elements of sort $s'$ and being injective, i.e., satisfying the implication

$$(\{x, y\}_s\colon \varphi_0(x) \stackrel{e}{=} \varphi_0(y) \Rightarrow x \stackrel{e}{=} y).$$

Moreover, if $\varphi$ has arguments $x_1, \ldots, x_k$ of sort $s$ and arguments $x_{k+1}, \ldots, x_{k+l}$ of other sorts, then $\varphi'$ has arguments $y_1, \ldots, y_k$ of sort $s'$ and arguments $y_{k+1}, \ldots, y_{k+l}$ of other sorts such that $x_{k+i}$ and $y_{k+i}$ are of the same sort $(1 \leqslant i \leqslant l)$, and we have the axiom

$$(\{x_1 \ldots x_{k+l}\}\colon \varphi'(\varphi_0(x_1), \ldots, \varphi_0(x_k), x_{k+1}, \ldots, x_{k+l}) \stackrel{e}{=} \varphi_0(\varphi(x_1, \ldots, x_{k+l}))).$$

EXAMPLE 3. Finally, let us briefly sketch a simple application of the Proposition in Section 5: We use the same signature as in Example 1, and as only axiom we use

$$(\{x, x', x'''\}\colon \varphi''(x, x') \stackrel{e}{=} \varphi'''(x, x') \Rightarrow \varphi'(x''') \stackrel{e}{=} \varphi).$$

The situation is here relatively simple, since premise and conclusion have different variables, and therefore we get: Whenever a set $P$ of equational pairs on some sorted set $M$ of variables – or the congruence relation on $T_M$ generated by $P$ – contains for some terms $t$ and $t'$ of sorts $s$ and $s'$, respectively, a realization of the premise, i.e., an equational pair of the form $(\varphi''^{T_M}(t, t'), \varphi'''^{T_M}(t, t'))$, then the closure of $P$ contains all pairs of the form $(\varphi^{T_M}, \varphi^{T_M'}(t'''))$ and $(\varphi^{T_M'}(t'''), \varphi^{T_M})$ for any term $t'''$ of sort $s'''$ in $T_M$. The closure of $P$ is then the congruence relation on $T_M$ generated by $P$ and all these pairs. And this congruence relation is the closed set on $T_M \times T_M$ generated by $P$ and the other pairs listed above with respect to the total operations defined in 5.

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