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ON THE NUMERICAL SOLUTIONS OF CONVECTION-DIFFUSION EQUATIONS

O. AXELSSON

Department of Mathematics, University of Nijmegen, The Netherlands

Dedicated to Professor Dr. L. Collatz on the occasion of his 70th birthday

1. Introduction. On boundary layers for convection-diffusion equations

Consider the linear boundary value problem

$$\mathcal{L}_\varepsilon u = -\varepsilon \Delta u + \boldsymbol{b}\nabla u + cu = f \quad \forall \boldsymbol{x} \in \Omega \subset \boldsymbol{R}^n,\ 0 < \varepsilon \leqslant 1, \tag{1.1}$$

where u is prescribed on $\Gamma = \partial\Omega$. The term $\boldsymbol{b}\nabla u$ comes from convection driven by a velocity field $\boldsymbol{b}$, and it dominates over the diffusion term, $-\varepsilon\Delta u$ ($\varepsilon > 0$), when ε is small. (1.1) is then an example of a singularly perturbed differential equation.

We assume that Γ is smooth, $\boldsymbol{b} \in [C^1(\bar{\Omega})]^n$, $c \in L^\infty(\bar{\Omega})$, $f \in L^2(\Omega)$. For notational simplicity we also assume that

$$c - \tfrac{1}{2}\operatorname{div}(\boldsymbol{b}) \geqslant c_0 > 0 \quad \forall \boldsymbol{x} \in \Omega. \tag{1.2}$$

As we shall see, this condition implies coercivity (and the stability in L_2-norm) of the solution $u = u_\varepsilon$, uniformly in ε. More general cases than implied by (1.2) can be dealt with by a proper transformation of variable and are treated in [3].

Consider at first the *reduced equation*,

$$\boldsymbol{b}\nabla u_0 + cu_0 = f \quad \forall \boldsymbol{x} \in \Omega, \tag{1.3}$$

with proper initial conditions to be given below. This is a first order scalar hyperbolic equation, and we assume that the given data are such that its solution is smooth enough for the validity of the error estimates to follow.

The corresponding characteristic equations are, assuming $|\boldsymbol{b}| \neq 0$ $\forall \boldsymbol{x} \in \Omega$,

$$\frac{d\boldsymbol{z}}{ds}(s,t) = \boldsymbol{b}\big(\boldsymbol{z}(s,t)\big), \quad s > 0,$$

$$\boldsymbol{z}(0,t) = \boldsymbol{d}(t), \quad 0 \leqslant t \leqslant 1,$$

t a parameter, where the vectorial function $\boldsymbol{d}$ will be defined below. This is a system of ordinary differential equations and from the assumptions made it follows, as is well known, that there exists a differentiable solution which is uniquely defined by $\boldsymbol{d}$ and t. Hence $\boldsymbol{z}(s,t)$, $s > 0$, $0 \leqslant t \leqslant 1$, defines a set of *characteristic curves* and, by unicity, there are no closed arcs (see Figure 1).

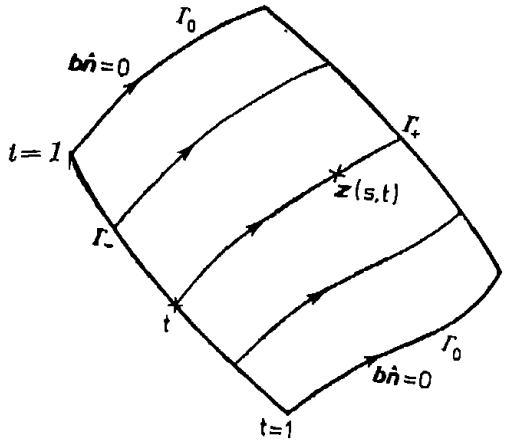

Fig. 1

We let $\hat{\boldsymbol{n}}$ be the outward pointing normal to Γ and let

$$\Gamma_- = \{\boldsymbol{x} \in \Gamma;\ \boldsymbol{b} \cdot \hat{\boldsymbol{n}} < 0\},$$

$$\Gamma_+ = \{\boldsymbol{x} \in \Gamma;\ \boldsymbol{b} \cdot \hat{\boldsymbol{n}} > 0\},$$

$$\Gamma_0 = \{\boldsymbol{x} \in \Gamma;\ \boldsymbol{b} \cdot \hat{\boldsymbol{n}} = 0\}.$$

Γ_- is the inflow part and Γ_+ is the outflow part of the boundary Γ. If Γ is piecewise smooth and $\boldsymbol{b} \cdot \hat{\boldsymbol{n}}$ is continuous on Γ_- and Γ_+, then we typically have a situation as in Figure 1.

By differentiation (the chain rule) we get

$$\frac{du_0}{ds}\big(\boldsymbol{z}(s,t)\big) = \nabla u_0\big(\boldsymbol{z}(s,t)\big)\frac{d\boldsymbol{z}}{ds} = \nabla u_0 \cdot \boldsymbol{b}$$

along the characteristic curves. Hence, along these, the partial differential

equation is reduced to an ordinary differential equation. Let

$$U = U(s,t) = u_0(z(s,t)),$$
$$F = F(s,t) = f(z(s,t)), \quad t \text{ fixed}, \ 0 \leqslant t \leqslant 1.$$

Then

$$\frac{dU}{ds} + cU = F, \quad s > 0. \tag{1.4}$$

The proper initial condition is the following. Let $\boldsymbol{d}(t)$, $0 \leqslant t \leqslant 1$, be the curve defining Γ_-, i.e.,

$$\Gamma_- = \{x = \boldsymbol{d}(t), \ 0 \leqslant t \leqslant 1\}.$$

Then we choose

$$U(0,t) = u(\boldsymbol{d}(t))$$

as the initial condition for the reduced equation, and the solution of (1.4) is

$$U(s,t) = \Big[U(0,t) + \int_0^s \exp(\varrho(\tau,t)) f(z(\tau,t))\, d\tau\Big] \exp(-\varrho(s,t)),$$

where

$$\varrho(s,t) = \int_0^s c(z(\tau,t))\, d\tau.$$

The remaining boundary conditions (on $\Gamma_0 \cup \Gamma_+$) are lost in the reduced equation.

Hence, along this part of the boundary there may occur layers. Usually, in practice, the layer along Γ_0 is less severe than that along Γ_+ and may hence be more easily resolved by a fine enough discretization. (In some problems, the set Γ_0 is even empty.) In this study we assume that there is only a layer along Γ_+.

EXAMPLE 1.1. Let $u_x + u_y = f(x,y)$, $0 \leqslant x, \ y \leqslant 1$. Here $\boldsymbol{b}^t = [1,1]$, so the characteristic curves may be represented by $\begin{cases} x = s \\ y = t+s \end{cases}$ in Ω' (similarly in Ω'').

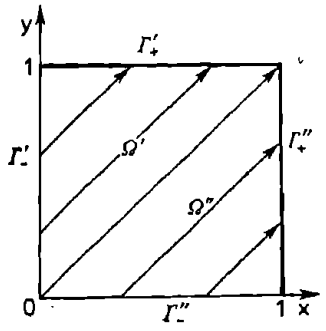

Fig. 2

The solution with $u(x, y)|_{\Gamma_-} = g(x-y)$ is

$$u(x, y) = g(x-y) + \int_0^x f(\tau, y-x+\tau)\,d\tau, \quad 0 \leqslant x \leqslant 1,\ 0 \leqslant y-x. \ \blacksquare$$

One may prove the existence of an asymptotic expansion

$$u(x, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k u_k(x),$$

where

$$(\boldsymbol{b}\cdot\nabla + c)u_0 = f, \qquad u_0 = u \text{ on } \Gamma_-,$$
$$(\boldsymbol{b}\cdot\nabla + c)u_k = \Delta u_{k-1}, \qquad u_k = 0 \text{ on } \Gamma_-, \qquad k = 1, 2, \ldots$$

This is an analytical function of ε. The layer term $\tilde{u}_\varepsilon$ representing the layer of the solution u_ε of (1.1) may be written

$$\tilde{u}_\varepsilon = u_\varepsilon - u(x, \varepsilon) = e^{-g/\varepsilon} v(x, \varepsilon),$$

where $g = 0$ on Γ_+ but $g > 0$ in Ω and v is regular in ε (see [12]).

EXAMPLE 1.2. Consider

$$\begin{cases} -\varepsilon u'' + bu' = 0, \quad b \geqslant b_0 > 0,\ b_0 \text{ a constant},\ 0 < x < 1 \text{ and} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad b \in C^1(0, 1], \\ u(0) = \alpha, \quad u(1) = \beta. \end{cases}$$

Let $z = u'$; then $z' = bz/\varepsilon$, i.e., $z = c\exp\left\{-\int_x^1 b(s)/\varepsilon\,ds\right\}$ so

$$u(x) = \alpha + (\beta-\alpha)\int_0^x e^{-\int_s^1 b/\varepsilon}\,ds \Big/ \int_0^1 e^{-\int_s^1 b/\varepsilon}\,ds$$
$$= \alpha + (\beta-\alpha)\,e^{-\int_x^1 b/\varepsilon\,ds}\,\frac{b(1)}{b(x)}\left(1 + O(\varepsilon)\right).$$

The reduced equation is $u' = 0$ with $u(0) = \alpha$ as the initial condition, and the boundary layer appears at the right boundary point.

EXAMPLE 1.3. Consider

$$\begin{cases} -\varepsilon u'' + axu' = 0, \quad -1 < x < 1, \\ u(-1) = \alpha, \quad u(1) = \beta. \end{cases}$$

It is easy to see that if $a < 0$ we get a layer at $x = 0$.

This is an example of the *turning point* problem with an interior layer.

2. Finite difference approximations

In order to illustrate the behaviour of different finite difference approximations on (1.1) we consider at first the two-point boundary value problem

$$\begin{aligned} &\mathscr{L}u = -\varepsilon u'' + b(x)u' + c(x)u = f, \quad 0 < x < 1, \\ &u(0) = \alpha, \quad u(1) = \beta. \end{aligned} \tag{2.1}$$

If we apply central differences at all interior mesh points, we get a $O(h^2)$ local truncation error for every fixed ε. However, in general the solution is not smooth uniformly in ε, so the discretization errors are $O(1)$, $h \to 0$, unless h is so small that the so called local Peclét number $\frac{1}{2\varepsilon} h \max_{0<x<1} |b|$ is not greater than 1. If h is too large, spurious oscillations appear.

From a practical point of view, when $\varepsilon \ll 1$, satisfying the Peclét number condition would imply far too small step sizes. However, it has also been observed that the spurious oscillations encountered above do not occur if the solution is smooth on $0 \leqslant x \leqslant 1$, i.e., if it does not have a layer behaviour. For a numerical test, see Figure 3. This is also the case if for instance we have a Neuman boundary condition $u' = 0$ at boundary layer points. This phenomena is most easily explained when we consider finite element Galerkin approximations (Section 4). As we

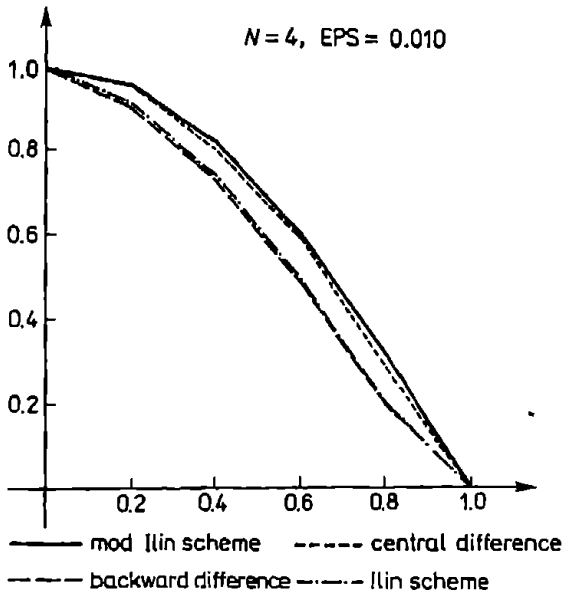

Fig. 3. Numerical solutions of

$$\begin{cases} -\varepsilon u'' + u' = f(x, \varepsilon), \ 0 < x < 1, \\ u(0) = 1, \ u(1) = 0, \end{cases}$$

where $u(x) = \cos\frac{\pi}{2}x$

shall see, the classical finite element methods (and hence the central difference method) are *stable*, but when we have a layer term present in the solution, the solution is not smooth enough in order to be approximated well by piecewise polynomials.

Difference schemes of positive type. In order to get difference schemes for which spurious oscillations do not appear, one has constructed positive schemes of different types, some of which will now be described. Let

$$(\mathscr{L}_h u)_i = r(x_i, h, \varepsilon)u_{i+1} + s(x_i, h, \varepsilon)u_i + t(x_i, h, \varepsilon)u_{i-1}. \tag{2.2}$$

We recapitulate: A difference operator is said to be of *positive type* if $s > 0$, $r < 0$, $t < 0$ and $s \geqslant -(r+t)$.

For such schemes we have monotonicity, that is $\mathscr{L}_h u \geqslant 0$ and $|u|_{\partial\Omega} = 0 \Rightarrow u \geqslant 0$ (and a discrete maximum principle), and, as is well known, discretization error estimates can be proved from the following lemma.

LEMMA 2.1. *Assume that there exists a function w* (called *barrier function*) *defined on $\Omega_h \cup \partial\Omega_h$, which satisfies $\mathscr{L}_h w > 0$, $w \geqslant 0$ where $\mathscr{L}_h$ is a difference operator of positive type. Then any function v defined on $\Omega_h \cup \partial\Omega_h$ satisfies*

$$|v|_{\Omega_h \cup \partial\Omega_h} \leqslant |v|_{\partial\Omega_h} + \frac{\max\limits_{\Omega_h} w}{\min\limits_{\Omega_h} \mathscr{L}_h w} |\mathscr{L}_h v|_{\Omega_h}. \tag{2.3}$$

Here $|v|_{\Omega_h} = \max\limits_{x \in \Omega_h} |v|$ etc. and $\Omega_h \cup \partial\Omega_h$ is the set of mesh points. If we let $v = u - u_h$, the discretization error, then $\mathscr{L}_h v = \mathscr{L}_h u - f$, the truncation error, and the lemma provides an estimate of the maximal discretization error.

The simplest difference approximation of positive type uniformly in ε is called the *upwind scheme*:

Let

$$u''(x) \simeq \delta_0^2 u(x) = h^{-1}[u(x+h) - 2u(x) + u(x-h)],$$

and

$$b(x)u'(x) \simeq \begin{cases} b(x)[u(x+h) - u(x)]/h, & \text{if} \quad b < 0, \\ b(x)[u(x) - u(x-h)]/h, & \text{if} \quad b > 0, \end{cases}$$

i.e., use central differences for the second order derivatives but "upwind" (forward and backward differences, respectively) for the first order derivative.

Obviously, this gives a positive scheme but the local truncation error is only $O(h)$.

Modified upwind schemes of $O(h^2)$ for fixed ε have appeared, e.g., in [13], [8] and [2].

If $b(x_i) > 0$ they take the following (or a similar) form:

$$(\mathscr{L}_h u)_i = -\tilde{\nu}_i \delta_0^2 u(x_i) + b(x_{i-1/2}) \frac{u(x_i) - u(x_{i-1})}{x_i - x_{i-1}} + c(x_i) u(x_i) = f(x_i)$$

where

$$\tilde{\nu}_i = \varepsilon \Big/ \left[1 + b(x_i) \frac{x_i - x_{i-1}}{2\varepsilon}\right], \quad i = 1, 2, \ldots, N,$$

and similarly if $b(x_i) < 0$.

In [1] and later in [9], the following scheme, referred to as Il'in scheme, was derived. Let $z_i = b(x_i) h/(2\varepsilon)$. Then

$$(\mathscr{L}_h u)_i = -\tilde{\tilde{\nu}}_i \delta_0^2 u(x_i) + b(x_i) \frac{u(x_{i+1}) - u(x_{i-1})}{x_{i+1} - x_{i-1}} + c(x_i) u(x_i) = f(x_i)$$

where

$$\tilde{\tilde{\nu}}_i = \varepsilon z_i \coth z_i, \quad i = 1, 2, \ldots, N.$$

Since

$$\coth z_i \begin{cases} > 1, & \text{if} \quad b_i > 0, \\ < -1, & \text{if} \quad b_i < 0, \end{cases}$$

we easily see that this is a positive scheme. It shifts automatically, so that it is always upwind, since

$$\coth(z) \to \begin{cases} 1, & \text{as} \quad z \to \infty, \\ -1, & \text{as} \quad z \to -\infty. \end{cases}$$

It is also of second order, because, for fixed ε,

$$-\varepsilon z_i \coth(z_i) \delta_0^2 u(x_i) = -\varepsilon[1 + O(z_i) \delta_0^2 u(x_i)] = -\varepsilon\big(1 + O(h^2)\big) \delta_0^2 u(x_i).$$

Hence, the truncation error, $(\mathscr{L}_h u)_i - f_i$, satisfies

$$\begin{aligned} (2.4) \qquad (\mathscr{L}_h u)_i - f_i &= -\varepsilon\big(z_i \coth(z_i) - 1\big) \delta_0^2 u(x_i) - \\ &\quad - \varepsilon \delta_0^2 u(x_i) + b_i \delta_0 u(x_i) + c_i u(x_i) - (-\varepsilon u'' + b u' + c u)_i \\ &= O(h^2), \quad h \to 0, \end{aligned}$$

for ε fixed.

As follows from Lemma 2.1, the discretization error in the modified upwind scheme and the Il'in scheme is $O(h^2)$ for ε fixed. But these estimates are not uniform in ε! In fact, both schemes reduce to the upwind scheme as $\varepsilon \to 0$.

By a Taylor expansion, assuming that $u \in C^4(0,1)$, we get for the modified upwind scheme

$$\mathscr{L}_h u - f = -(\tilde{\nu} - \varepsilon)u'' - \frac{bh}{2}u'' + O(h^2)$$

$$= -\left(\frac{bh}{2}\right)^2 \Big/ \left(\varepsilon + \frac{bh}{2}\right)u'' + O(h^2) \sim -\frac{bh}{2}u'' + O(h^2)$$

if $\varepsilon \ll h$. For the Il'in scheme we get the truncation error (see (2.4))

$$\mathscr{L}_h u - f = -\varepsilon(z\coth z - 1)u'' + O(h^2),$$

and as

$$c\frac{z^2}{z+1} \leqslant z\coth z - 1 \leqslant C\frac{z^2}{z+1}, \quad 0 < z \quad \text{where } 0 < c < C,$$

we lose an order of accuracy for small ε in both cases.

In addition, schemes of upwind type suffer from the undesirable property of smearing out sharp fronts.

This is due to the fact that in these schemes the leading term in the truncation error $((\sim bh/2)u'')$ introduces an artificial viscosity of $\sim bh/2$ in the upwind scheme and somewhat less in the other positive schemes.

By considering the regular term and the layer term separately one may now derive error estimates for $u = u_\varepsilon$ in (2.1); for details see [10]. For a one-dimensional problem without turning points, that is, where $b(x) \geqslant b_0 > 0$, $0 < x < 1$, one gets:

Modified upwind scheme:

$$|u(x_i) - u_{h,i}| \leqslant \left[\frac{h^2}{h+\varepsilon} + \exp\left(\frac{b_0(1-x_i)}{-b_0 h + \varepsilon}\right)\right], \quad \varepsilon \leqslant h.$$

If $x_l = 1 - lh$ we get

$$|u(x_l) - u_{h,l}| \leqslant C\left[\frac{h^2}{h+\varepsilon} + \exp\left(-\frac{b_0 lh}{b_0 h + \varepsilon}\right)\right] \simeq O\big(\exp(-l)\big), \quad \varepsilon \ll h,$$

that is, close to the layer we have an error $O(1)$, $h \to 0$. (As usual, C indicates a constant, not necessarily the same at different occurrences.)

Il'in scheme:

$$|u(x_i) - u_{h,i}| \leqslant C\left[\frac{h^2}{h+\varepsilon} + \frac{h^2}{\varepsilon}\exp\left(-\frac{b_0}{\varepsilon}(1-x_i)\right)\right], \quad i = 1, 2, \ldots, h^{-1} - 1.$$

In this case the boundary layer term does not "pollute" the error away from the layer, since already at $x = 1 - h$ the second term is negligible if $\varepsilon \ll h$. Hence the above pointwise error estimate is uniform in ε.

Note however that in Il'in's scheme we have used the knowledge of the asymptotic behaviour of the layer term in the construction of the difference scheme.

A simple proof of the fact that, in most difference schemes of positive type, one order of accuracy is lost for ε small enough was given in [10]:

THEOREM 2.1. *Assume that r, s, t in (2.2) are continuous functions of h, ε for $h > 0$, $\varepsilon \leqslant 1$, and suppose that, for any smooth function $u(x)$, the truncation error satisfies $|\tau(x, h, \varepsilon)| \leqslant Ch^{\delta}$, $\delta > 1$, where C does not depend on x, h or ε. Then the difference operator $\mathscr{L}_h$ in (2.2) is not of positive type for ε, h sufficiently smooth.*

In [13] a scheme is presented where r, s, t are not continuous in (ε, h) and there is no loss of one order of accuracy. See also [4] for a generalization of the results in [10].

A generalized Il'in scheme. Let

$$\mathscr{L}^* v = (-\varepsilon v')' - (bv)' + cv, \qquad 0 < x < 1$$

be the adjoint operator to $\mathscr{L}$ in (2.1), that is,

$$\int_0^1 \mathscr{L} uv\, dx = \int_0^1 u\mathscr{L}^* v\, dx + \text{boundary terms}.$$

(Here ε may be a function x. Without limitation, we may assume that $u(0) = u(1) = 0$.)

Let $\{x\}_0^{N+1}$ be a subdivision of $[a, b]$ and let g_i be the local Green's function of $\mathscr{L}^*$ and x_i, that is,

$$\mathscr{L}^* g_i(x) = 0, \qquad x \in (x_{i-1}, x_i) \cup (x_i, x_{i+1}),$$

$$g_i(x) = 0, \qquad x \in [a, x_{i-1}] \cup [x_{i+1}, b],$$

$$\varepsilon(x_i)[g_i'(x_i -) - g_i'(x_i +)] = 1.$$

Then

$$\int_a^b \mathscr{L} u g_i\, dx = \int_{x_{i-1}}^{x_{i+1}} \mathscr{L} u g_i\, dx = \int_{x_{i-1}}^{x_{i+1}} f g_i\, dx,$$

and by partial integration we get the difference scheme

$$-\varepsilon_{i-1} g_i'(x_{i-1}) u_{i-1} + u_i + \varepsilon_{i+1} g_i'(x_{i+1}) u_{i+1} = \int_{x_{i-1}}^{x_{i+1}} f g_i\, dx, \qquad i = 1, 2, \ldots, N. \tag{2.5}$$

Note that if we are able to evaluate g_i' and $\int f g_i\, dx$ exactly, this scheme gives the exact solution.

For

$$\mathcal{L}u = -\varepsilon u'' + bu' = f, \quad 0 < x < 1,$$
$$u(0) = u(1) = 0,$$

where ε is constant, $x_i - x_{i-1} = h$, $i = 1, 2, \ldots$, we get by letting $b = b(x_i)$ be constant over $(x_{i-1}, x_{i+1}]$ when evaluating g_i

$$-\frac{e^{\frac{b_i}{\varepsilon}h} - 1}{e^{\frac{b_i}{\varepsilon}h} - e^{-\frac{b_i}{\varepsilon}h}} u_{i-1} + u_i - \frac{1 - e^{-\frac{b_i}{\varepsilon}h}}{e^{\frac{b_i}{\varepsilon}h} - e^{-\frac{b_i}{\varepsilon}h}} u_{i+1} = \int_{x_{i-1}}^{x_{i+1}} f g_i \, dx,$$

$i = 1, 2, \ldots, N$. This reduces to the Il'in scheme if we approximate

$$\int_{x_{i-1}}^{x_{i+1}} f g_i \, dx \simeq f_i \int_{x_{i-1}}^{x_{i+1}} g_i \, dx.$$

However, it is better to approximate f by a polynomial of at least first degree and then evaluate the integral over the corresponding forms exactly (cf. Figure 3). We call (2.5) the generalized Il'in scheme. Finally, we shortly comment on positive schemes for higher dimensional problems.

Assume then that the first component b_1 of $\boldsymbol{b}$ satisfies $b_1 \geqslant b_0 \ \forall \boldsymbol{x} \in \Omega$, where b_0 is a positive constant. Then it is easy to see that

$$w(x) = |x_1|_\Omega^2 - x_1^2 + 3R(|x_1|_\Omega + x_1),$$

where R is the radius in a circle, with centre at the origin, which circumscribes Ω, is a barrier function. In fact, $w \geqslant 0$ and for the upwind scheme we get

$$\mathcal{L}_h w \geqslant 2\varepsilon + b_0 R$$

(similarly for the other positive schemes).

Hence for such problems, by use the barrier function of Lemma 2.1, we get similar error estimates as for the two-point boundary value problems. Since the accuracy of difference schemes is so low, we shall now consider instead the Galerkin finite element methods, where we are able to get any order of accuracy for (1.1) when ε is fixed and also for the reduced problem.

3. Stability of variational formulation

We consider now a problem with more general boundary conditions than (1.1), namely

$$\text{(3.1)} \qquad \begin{cases} u = \gamma_1 & \forall \boldsymbol{x} \in \Gamma_1, \\ \varepsilon \nabla \hat{u} \cdot \boldsymbol{n} + \sigma(u - \gamma_2) = 0, \quad \sigma \geqslant 0 & \forall \boldsymbol{x} \in \Gamma_2, \end{cases}$$

where $\Gamma_2 = \Gamma \setminus \Gamma_1$ and $\Gamma_1 \cap \Gamma_0 = \emptyset$. We assume that $\sigma \in L^\infty(\Gamma)$ and that $\sigma \equiv 0$ on Γ_0. For the purpose of analyzing the Galerkin finite element method of this problem we introduce the norm

$$\|v\|_{1,s} = \Big\{\varepsilon |v|_1^2 + \|v\|_0^2 + \oint_{\Gamma \setminus \Gamma_0} v^2 d\Gamma\Big\}^{1/2}, \qquad v \in H^1(\Omega),$$

where $|v|_1 = \big\{\int_\Omega |\nabla v|^2 d\Omega\big\}^{1/2}$ is the seminorm. We let

$$V = \{v \in H^1(\Omega);\ v = \gamma_1\ \forall \boldsymbol{x} \in \Gamma_1\}$$

and we let $\overset{\circ}{V}$ be the corresponding space of functions, satisfying homogeneous essential boundary conditions, e.g.,

$$\overset{\circ}{V} = \{v \in H^1(\Omega);\ v = 0\ \forall \boldsymbol{x} \in \Gamma_1\}.$$

At first we prove the stability of solutions of (1.1), (3.1). In these notes we only prove it for the case

$$c - \tfrac{1}{2}\operatorname{div}(\boldsymbol{b}) \geqslant c_0 \qquad \forall \boldsymbol{x} \in \Omega, \tag{3.2a}$$

$$\sigma + \tfrac{1}{2}\boldsymbol{b}\cdot\hat{\boldsymbol{n}} \geqslant \sigma_0 \qquad \forall \boldsymbol{x} \in \Gamma_+ \cup \Gamma_-, \tag{3.2b}$$

where c_0, σ_0 are positive constants. More general problems can often be transformed to a problem where (3.2a, b) is satisfied, see for instance [3]. At first we note that by Green's formula

$$\begin{aligned}\int_\Omega \boldsymbol{b}\nabla u v\, d\Omega &= \int_\Omega -\operatorname{div}(\boldsymbol{b}v) u\, d\Omega + \oint_\Gamma \boldsymbol{b}\hat{\boldsymbol{n}} u v\, d\Gamma \\ &= \int_\Omega -[\boldsymbol{b}\nabla v u + \operatorname{div}(\boldsymbol{b}) uv]\, d\Omega + \oint_\Gamma \boldsymbol{b}\hat{\boldsymbol{n}} u v\, d\Gamma \\ &\qquad\qquad \forall u \in H^1(\Omega),\ v \in \overset{\circ}{V}.\end{aligned}$$

In particular

$$\int_\Omega \operatorname{div}(\boldsymbol{b}v) v\, d\Omega = \int_\Omega \tfrac{1}{2}\operatorname{div}(\boldsymbol{b}) v^2\, d\Omega + \oint_\Gamma \tfrac{1}{2}\boldsymbol{b}\hat{\boldsymbol{n}} v^2\, d\Gamma.$$

Let $(.,.)$ be the inner product in L_2 and let the bilinear form $a_\varepsilon(.,.)$ be defined by

$$a_\varepsilon(u, v) = \int_\Omega \big(\varepsilon \nabla u \nabla v - \operatorname{div}(\boldsymbol{b}v) u + cuv\big) d\Omega + \oint_{\Gamma_2} (\sigma + \boldsymbol{b}\hat{\boldsymbol{n}}) uv\, d\Gamma \tag{3.3}$$

$$\forall u \in H^1(\Omega),\ v \in \overset{\circ}{V},$$

where $\Gamma_2 = \Gamma \setminus (\Gamma_1 \cup \Gamma_0)$. We have

$$\begin{aligned}(\mathscr{L}_\varepsilon u, v) &= \int_\Omega \mathscr{L}_\varepsilon u v\, d\Omega \\ &= \int_\Omega \big(\varepsilon \nabla u \nabla v - \operatorname{div}(\boldsymbol{b}v) u + cuv\big) d\Omega + \oint_\Gamma [-\varepsilon \nabla u \hat{\boldsymbol{n}} v + b\hat{\boldsymbol{n}} uv]\, d\Gamma \\ &\qquad\qquad \forall u \in H^1(\Omega),\ v \in \overset{\circ}{V}.\end{aligned} \tag{3.4}$$

Hence the variational formulation of (1.1), (3.1) is:

Find $u_\varepsilon \in V$ such that

$$a_\varepsilon(u_\varepsilon, v) = (f, v) + \oint_{\Gamma_2} \sigma\gamma_2 v\, d\Gamma \quad \forall v \in \overset{\circ}{V}. \tag{3.5}$$

In order to prove the coercivity of the bilinear form we let $\tilde{u} \in H^1(\Omega)$ satisfy the boundary conditions (3.1). For instance, u may be chosen as the (weak) solution of $\Delta\tilde{u} = 0$ $\forall \boldsymbol{x} \in \Omega$, satisfying these boundary conditions. Then $\tilde{u}_\varepsilon - \tilde{u} \in \overset{\circ}{V}$ and by (3.3), (3.6),

$$a_\varepsilon(u_\varepsilon - \tilde{u}, u_\varepsilon - \tilde{u}) = \int_\Omega \{\varepsilon |\nabla(u_\varepsilon - \tilde{u})|^2 + [c - \tfrac{1}{2}\operatorname{div}(\boldsymbol{b})](u_\varepsilon - \tilde{u})^2\} d\Omega + \oint_{\Gamma_2} (\sigma + \tfrac{1}{2}\boldsymbol{b}\cdot\hat{\boldsymbol{n}})(u_\varepsilon - \tilde{u})^2 d\Gamma. \tag{3.7}$$

Hence, by (3.2a, b),

$$a_\varepsilon(u_\varepsilon - \tilde{u}, u_\varepsilon - \tilde{u}) \geqslant \varrho \|u_\varepsilon - \tilde{u}\|^2_{1,\varepsilon} \quad \forall u_\varepsilon - \tilde{u} \in \overset{\circ}{V}, \tag{3.8}$$

where $\varrho = \min(1, c_0, \sigma_0) > 0$.

The boundedness of $a_\varepsilon(.,.)$ follows by the C–B–S inequality, and a trace inequality, see e.g. [11],

$$|a_\varepsilon(u, v)| \leqslant C \|u\|_{1,\varepsilon} \|v\|_1 \tag{3.9a}$$

or

$$|a_\varepsilon(u, v)| \leqslant C \|u\|_1 \|v\|_{1,\varepsilon} \quad \forall u \in H^1(\Omega),\ v \in \overset{\circ}{V}. \tag{3.9b}$$

By (3.5) we have

$$a_\varepsilon(u_\varepsilon, u_\varepsilon - \tilde{u}) = (f, u_\varepsilon - \tilde{u}) + \oint_{\Gamma_2} \sigma\gamma_2 (u_\varepsilon - \tilde{u})^2 d\Gamma$$

and hence by (3.8), (3.6),

$$\begin{aligned} \varrho \|u_\varepsilon - \tilde{u}\|^2_{1,\varepsilon} &\leqslant a_\varepsilon(u_\varepsilon - \tilde{u}, u_\varepsilon - \tilde{u}) = a_\varepsilon(u_\varepsilon, u_\varepsilon - \tilde{u}) - a_\varepsilon(\tilde{u}, u_\varepsilon - \tilde{u}) \\ &= (f, u_\varepsilon - \tilde{u}) + \oint_{\Gamma_2} \sigma\gamma_2 (u_\varepsilon - \tilde{u}) d\Gamma - a_\varepsilon(\tilde{u}, u_\varepsilon - \tilde{u}). \end{aligned}$$

Hence, by the C–B–S inequality and by (3.9b), we get

$$\tfrac{1}{2}\varrho \|u_\varepsilon - \tilde{u}\|^2_{1,\varepsilon} \leqslant C[\|f\|^2 + \|\sigma\gamma_2\|^2_{L^2(\Gamma)} + \|\tilde{u}\|^2_1]$$

or

$$\|u_\varepsilon\|_{1,\varepsilon} \leqslant \|u_\varepsilon - \tilde{u}\|_{1,\varepsilon} + \|\tilde{u}\|_{1,\varepsilon} \leqslant C[\|f\| + \|\sigma\gamma_2\|^2_{L^2(\Gamma)} + \|\tilde{u}\|_1], \quad 0 < \varepsilon \leqslant 1, \tag{3.10}$$

where C does not depend on ε, which proves the stability of the solution in the norm $\|\cdot\|_{1,\varepsilon}$.

4. Galerkin discretization error estimate in L_2-norm

Let $V_h \subset V$ and $\overset{\circ}{V}_h \subset \overset{\circ}{V}$ be finite dimensional subspaces spanned by the usual Lagrangian finite element basis functions and let $u_{\varepsilon,h}$ be the corresponding Galerkin approximation of u_ε, that is,

$$a_\varepsilon(u_{\varepsilon,h}, v_h) = (f, v_h) + \oint_{\Gamma_2} \sigma\gamma_2 v_h d\Gamma \quad \forall v_h \in \overset{\circ}{V}_h. \tag{4.1}$$

Then, by (3.5) and (4.1), we get

$$a_\varepsilon(u_\varepsilon - u_{\varepsilon,h}, v_h) = 0 \quad \forall v_h \in \overset{\circ}{V}_h. \tag{4.2}$$

Hence, by (3.8), (3.9a) and (4.2), we have

$$\begin{aligned} \varrho\|u_\varepsilon - u_{\varepsilon,h}\|_{1,\varepsilon}^2 &\leqslant a_\varepsilon(u_\varepsilon - u_{\varepsilon,h}, u_\varepsilon - u_{\varepsilon,h}) = a_\varepsilon(u_\varepsilon - u_{\varepsilon,h}, u_\varepsilon - u_{\varepsilon,h} - v_h) \\ &\leqslant C\|u_\varepsilon - u_{\varepsilon,h}\|_{1,\varepsilon}\|u_\varepsilon - u_{\varepsilon,h} - v_h\|_1 \quad \forall v_h \in \overset{\circ}{V}_h, \end{aligned}$$

and so, in particular,

$$\|u_\varepsilon - u_{\varepsilon,h}\|_{1,\varepsilon} \leqslant C \inf_{w_h \in V_h} \|u_\varepsilon - w_h\|_1. \tag{4.3}$$

We consider now four different cases of smoothness of u_ε.

Case 1. Assume that u_ε is smooth uniformly in ε or, more precisely, let $u_\varepsilon \in H^{k+1}(\Omega) \cap V$ and $\|u_\varepsilon\|_{k+1} \leqslant C_1$, $0 < \varepsilon \leqslant 1$, where C_1 does not depend on ε. Then, as is well known,

$$\inf_{w_h \in V_h} \|u_\varepsilon - w_h\|_1 = |O(h^k)|, \quad h \to 0,$$

uniformly in ε, where k is the degree of the piecewise polynomial basis functions. Hence, by (4.3) we have

$$\|u_\varepsilon - u_{\varepsilon,h}\|_{1,\varepsilon} = O(h^k),$$

and in particular

$$\left\{\int_\Omega (u_\varepsilon - u_{\varepsilon,h})^2 d\Omega + \int_{\Gamma_+ \cup \Gamma_-} (u_\varepsilon - u_{\varepsilon,h})^2 d\Gamma\right\}^{1/2} = O(h^k), \quad h \to 0.$$

Note that this error estimate, which is uniform in ε, is not quasioptimal (however, see the Remark below).

Case 2. Assume now that u_ε has a layer (of width ε) at Γ_+ (see Figure 1). Then

$$\|u_\varepsilon\|_1 = O(\varepsilon^{-1/2}), \quad \varepsilon \to 0,$$

and the bound in (4.3) is in general not even bounded uniformly in ε.

However, because of the stability bound (3.10), we have

$$\|u_\varepsilon - u_{\varepsilon,h}\|_1 \leqslant \|u_\varepsilon\|_{1,\varepsilon} + \|u_{\varepsilon,h}\|_{1,\varepsilon} \leqslant O(1), \qquad h \to 0$$

and this bound is independent on ε.

Case 3. Assume that $\partial u_\varepsilon/\partial \boldsymbol{n} = 0 \ \forall \boldsymbol{x} \in \Gamma_+$ (i.e., $\sigma \equiv 0$ at Γ_+). Then $\partial u_\varepsilon/\partial \boldsymbol{n}$ (but not u_ε) may have a layer at Γ_+, and so

$$\inf_{w_h \in V_h} \|u_\varepsilon - w_h\|_1 \leqslant \inf_{w_h \in V_h} \|u_0 - w_h\|_1 + \|u_\varepsilon - u_0\|_1 = \inf_{w_h \in V_h} \|u_0 - w_h\|_1 + O(\varepsilon^{1/2}).$$

Here u_0 is the solution of the reduced problem and is smooth up to the boundary, i.e.,

$$u_0 \in H^{k+1}(\Omega).$$

Hence

$$\inf_{w_h \in V_h} \|u_\varepsilon - w_h\|_1 = O(h^k) + O(\varepsilon^{1/2}), \qquad h \to 0.$$

Case 4. Assume now

(i) $\boldsymbol{b}$ is parallell to $\boldsymbol{n}$ at Γ_+,

(ii) $c \geqslant 0 \quad \forall \boldsymbol{x} \in \Gamma_+$,

(iii) $\boldsymbol{b} \nabla u_\varepsilon + c u_\varepsilon = f \quad \forall \boldsymbol{x} \in \Gamma_+$.

Then $\Delta u_\varepsilon = 0$ at Γ_+ and $\partial^2 u_\varepsilon/\partial \boldsymbol{n}^2$ may have a layer.

Let

$$\mathscr{L}_0 u_0 = f \qquad \forall \boldsymbol{x} \in \Omega$$

and let u_0 satisfy the boundary conditions at the inflow boundary Γ. Also let

$$\mathscr{L}_0 u_1 = \Delta u_0 \qquad \forall \boldsymbol{x} \in \Omega.$$

Then

$$u_\varepsilon = u_0 + \varepsilon u_1 + \tilde{u}_\varepsilon,$$

where $\tilde{u}_\varepsilon$ is the layer term.

Along the characteristic lines we then have

$$\int_\Omega (u'_\varepsilon - u'_0 - \varepsilon u'_1)^2 d\Omega = \varepsilon O(\varepsilon^2), \qquad \varepsilon \to 0,$$

and

$$\inf_{w_h \in V_h} \|u_\varepsilon - w_h\|_1 \leqslant \inf_{w_h \in \Gamma_h} \|u_\varepsilon - \tilde{u}_\varepsilon - w_h\|_1 + \|\tilde{u}_\varepsilon\|_1 = O(h^k) + O(\varepsilon^{3/2}), \qquad h \to 0.$$

Hence, for small values of ε, the influence of the layer term is further decreased.

In particular, from the above we see that (for small values of ε) the central difference scheme, corresponding to certain piecewise linear basis functions ($k = 1$), works well for the problem

$$\mathscr{L}_\varepsilon u = f, \quad u = \gamma \quad \forall x \in \Gamma_-, \quad \frac{\partial u}{\partial n} = 0 \quad \forall x \in \Gamma_+, \quad 0 < \varepsilon \ll 1.$$

This explains why we do not get any spurious oscillations even for large values of h, for certain boundary conditions, when we use the central difference scheme (cf. Section 2).

However, in general, the bound on $\|u_\varepsilon\|_{k+1}$ depends on ε. In that case we may use the following technique. Let

$$(u, v)_{g_\varepsilon} = \int_\Omega uv \exp(-g_\varepsilon)\, d\Omega,$$

$$a_{g_\varepsilon}(u, v) = \int_\Omega [\varepsilon \nabla u \nabla v - \operatorname{div}(\boldsymbol{b}v)u + cuv] \exp(-g_\varepsilon)\, d\Omega + \oint_{\Gamma_2} (\gamma + \boldsymbol{b}\hat{n}) uv \exp(-g_\varepsilon)\, d\Gamma,$$

etc.

One can prove (see [3]) that it is possible to choose g_ε such that the coercivity is valid for this weighted bilinear form and an error estimate

$$\|u_\varepsilon - u_{\varepsilon,h}\|_{1,\varepsilon,g_\varepsilon} \leqslant C \inf_{w_h \in V_h} \|u_\varepsilon - w_h\|_{1,g_\varepsilon}$$

can be derived. In fact, in the limit case ($\varepsilon \ll h$) the exponential weight function is very close to the Heaviside step function. The effect of the latter weight function is to weight away any influence on the solution of the layer elements. In a one-dimensional problem with a layer at the right boundary point, the step function is given in Figure 4. (For higher dimensional problems we have a corresponding weight function along the characteristic lines.)

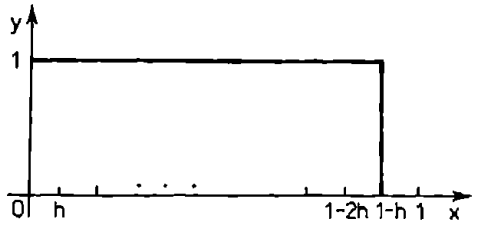

Fig. 4

In particular, the boundary conditions on the outflow boundary Γ_+ do not influence the solution.

It is easy to see that the use of the step-function is equivalent to using the boundary condition $\nabla u \cdot \hat{n} = 0$ $\forall x \in \Gamma_+$, that is, the correspond-

ing bilinear form is (if $\Gamma_1 = \Gamma_-$)

$$(4.4)\quad a_{\Omega_0}(u, v) = \int_{\Omega} \left(\varepsilon \nabla u \nabla v - \operatorname{div}(\boldsymbol{b}v)u + cuv\right) d\Omega \qquad \forall u \in V,\ v \in \overset{\circ}{V},$$

and the variational formulation is

$$a_{\Omega_0}(u_\varepsilon, v) = (f, v) \qquad \forall v \in \overset{\circ}{V}.$$

The approximation of the boundary conditions at Γ_+ by Neuman boundary conditions causes a layer in $\partial u/\partial \boldsymbol{n}$ (case 2) and hence an error $O(\varepsilon^{1/2})$ in the L_2-norm.

As follows from what has been said, the method of exponential weighting along the characteristic lines, however, is more general and gives smaller errors uniformly in ε. It is also applicable to some turning point problems. For details, see [3].

Remark. The reduced equation (1.3) is a first order hyperbolic equation with the boundary conditions on Γ_- as initial condition and, as is well known, in general we do not get a quasioptimal rate of convergence of the discretization error in L_2-norm. In [4] it is shown however, that for polynomial basis functions of odd degree one gets in fact a quasioptimal error estimate for this problem, defined on a region which can be mapped onto a union of rectangles.

5. Error estimates in maximum norm

Consider now the singularly perturbed problem (1.1) but with no essential boundary conditions. Instead, let

$$(5.1)\qquad \varepsilon \nabla u \cdot \hat{\boldsymbol{n}} + \sigma(u - \gamma) = 0 \qquad \forall \boldsymbol{x} \in \Gamma,$$

where $\sigma = -b\hat{\boldsymbol{n}} \geqslant 2\sigma_0 > 0$ on Γ_- and $\sigma = 0$ on Γ_0.

In the same way as in Section 4, we realize that away from the layer the solution of this problem differs from that where $u = \gamma$ on Γ_- at most by $O(\varepsilon)$, even in maximum norm. We have

$$(5.2)\quad (\mathscr{L}_\varepsilon u, v)$$

$$= \int_{\Omega} [-\varepsilon \Delta u + \boldsymbol{b} \nabla u + cu] v\, d\Omega$$

$$= \int_{\Omega} [\varepsilon \nabla u \nabla v - \operatorname{div}(\boldsymbol{b}v) u + cuv] d\Omega + \oint_{\Gamma} (-\varepsilon \nabla u \cdot \hat{\boldsymbol{n}} + \boldsymbol{b} \cdot \hat{\boldsymbol{n}} u) v\, d\Gamma$$

$$= (u, \mathscr{L}_\varepsilon^* v) + \oint_{\Gamma} \left[\varepsilon \left(\frac{\partial v}{\partial \boldsymbol{n}} u - \frac{\partial u}{\partial \boldsymbol{n}} v\right) + \boldsymbol{b} \cdot \hat{\boldsymbol{n}} uv\right] d\Gamma$$

where $\mathscr{L}_\varepsilon^*$ is the adjoint operator,

$$\mathscr{L}_\varepsilon^* v = -\varepsilon \Delta v - \operatorname{div}(\boldsymbol{b} v) + cv.$$

Let $u_\varepsilon \in V = H^1(\Omega)$ be the solution of (1.1) with the natural boundary conditions (5.1). Its variational formulation is

$$(\mathscr{L}_\varepsilon u_\varepsilon, v) = (f, v) \quad \forall v \in V$$

or

$$(5.3) \quad a_\varepsilon(u_\varepsilon, v) = \int_\Omega [\varepsilon \nabla u_\varepsilon \nabla v - \operatorname{div}(\boldsymbol{b} v) u_\varepsilon + c u_\varepsilon v] d\Omega + \oint_{\Gamma_+} (\sigma + \boldsymbol{b}\hat{\boldsymbol{n}}) u_\varepsilon v \, d\Gamma$$
$$= (f, v) + \oint_\Gamma \sigma \gamma v \, d\Gamma \quad \forall v \in V.$$

We assume that (3.2a, b) is satisfied and since

$$a_\varepsilon(u, u) = \int_\Omega \{\varepsilon |\nabla u|^2 + [c - \tfrac{1}{2}\operatorname{div}(\boldsymbol{b})] u^2\} + \oint_\Gamma (\sigma + \tfrac{1}{2}\boldsymbol{b}\hat{\boldsymbol{n}}) u^2 d\Gamma \geqslant \varrho \|u\|_{1,\varepsilon}^2,$$
$$\varrho = \min(1, c_0, \sigma_0) \quad \forall u \in V,$$

$a_\varepsilon(.,.)$ is coercive on $V \times V$.

The corresponding Galerkin solution $u_{\varepsilon,h}$ is defined by

$$a_\varepsilon(u_{\varepsilon,h}, v_h) = (f, v_h) + \oint_\Gamma \sigma \gamma v_h d\Gamma \quad \forall v_h \in V_h,$$

and we have as before

$$(5.4) \quad a_\varepsilon(u_\varepsilon - u_{\varepsilon,h}, v_h) = 0 \quad \forall v_h \in V_h.$$

Let $G_\varepsilon(\boldsymbol{x}, \cdot)$ be the Green function at $\boldsymbol{x}$, satisfying

$$\mathscr{L}_\varepsilon^* G_\varepsilon(\boldsymbol{x}, \boldsymbol{y}) = \delta_{\boldsymbol{x}}(\boldsymbol{y}) \quad \forall \boldsymbol{y} \in \Omega,$$

$$(5.5) \quad \frac{\partial G_\varepsilon}{\partial \boldsymbol{n}}(\boldsymbol{x}, \boldsymbol{y}) = 0 \quad \forall \boldsymbol{y} \in \Gamma \setminus \Gamma_+,$$

$$\varepsilon \frac{\partial G_\varepsilon}{\partial \boldsymbol{n}}(\boldsymbol{x}, \boldsymbol{y}) + (\sigma + \boldsymbol{b}\hat{\boldsymbol{n}}) G_\varepsilon(\boldsymbol{x}, \boldsymbol{y}) = 0 \quad \forall \boldsymbol{y} \in \Gamma_+,$$

where $\delta_{\boldsymbol{x}}(\cdot)$ is the Dirac delta function at $\boldsymbol{x}$. Then it follows from (5.3) that

$$(u_\varepsilon - u_{\varepsilon,h})(\boldsymbol{x}) = \int_\Omega \mathscr{L}_\varepsilon^* G_\varepsilon(\boldsymbol{x}, \boldsymbol{y})(u_\varepsilon - u_{\varepsilon,h})(\boldsymbol{y}) d\boldsymbol{y}$$
$$= \int_\Omega \{\varepsilon \nabla G_\varepsilon(\boldsymbol{x}, \boldsymbol{y}) \nabla(u_\varepsilon - u_{\varepsilon,h})(\boldsymbol{y}) - \operatorname{div}[\boldsymbol{b} G_\varepsilon(\boldsymbol{x}, \boldsymbol{y})](u_\varepsilon - u_{\varepsilon,h})(\boldsymbol{y}) +$$
$$+ c G_\varepsilon(\boldsymbol{x}, \boldsymbol{y})(u_\varepsilon - u_{\varepsilon,h})(\boldsymbol{y})\} d\boldsymbol{y}$$
$$= a_\varepsilon(u_\varepsilon - u_{\varepsilon,h}, G_\varepsilon(\boldsymbol{x}, \cdot)).$$

Hence from (5.4)

$$(u_\varepsilon - u_{\varepsilon,h})(x) = a_\varepsilon\big(u_\varepsilon - u_{\varepsilon,h},\ G_\varepsilon(x,\cdot) - v_h\big) \qquad \forall v_h \in V_h$$

or by (3.9 a)

$$|(u_\varepsilon - u_{\varepsilon,h})(x)| \leqslant C\,\|u_\varepsilon - u_{\varepsilon,h}\|_{1,\varepsilon} \inf_{v_h \in V_h} \|G_\varepsilon(x,\cdot) - v_h\|_1 .$$

This means that in the discrete maximum norm of the error at nodal points we may get a higher order of accuracy than is achieved by the error in the norm $\|\cdot\|_{1,\varepsilon}$. The extra order of accuracy we may get depends on the smoothness of $G_\varepsilon(x,\cdot)$ and on the choice of basis functions in V_h.

In general, $G_\varepsilon(x,\cdot)$ is not smooth, at least not at x.

However, if V_h contains the local Green functions at the nodal points x then we get even the exact value of the solution at these points (compare the generalized Il'in scheme in Section 2). In general these functions are not known, but their singular behaviour may be known. Furthermore, they are smooth away from the points x. At $y \in \Gamma_-$ we have a layer in $\dfrac{\partial G_\varepsilon}{\partial n}(x,y)$, but thanks to the natural boundary condition (5.5) G_ε itself does not have a layer. Hence we may choose basis functions such that the singularity of $G_\varepsilon(x,\cdot)$ at a single nodal point x or at a set of nodal points is exactly described.

Then

$$\inf_{v_h \in V_h} \|G_\varepsilon(x,\cdot) - v_h\|_1 = O(h^k + \varepsilon^{1/2}), \qquad h \to 0,$$

and we get superconvergence

$$|(u_\varepsilon - u_{\varepsilon,h})(x)| = O(h^k + \varepsilon^{1/2})^2, \qquad h \to 0.$$

In a one-dimensional problem we do not have to extend the set V_h of basis functions because if $x \in \Gamma_+$ the Green functions are smooth everywhere in the open interval between Γ_-, Γ_+.

Hence at such a point we get superconvergence with the usual polynomial basis functions. For small ε, this can be applied in order to get superconvergence at any point. We then introduce an artificial boundary Γ' in Ω through the point in question and let Γ' act as Γ_+ above.

6. Conclusions

We have shown that for regular solutions the classical Galerkin approximation is stable without any kind of upwinding. (Depending on the velocity field $\boldsymbol{b}$, however, we may have to make a transformation of variable in order to have condition (1.2) satisfied.) For solutions with

a layer term, one has to apply exponential weighting or exponential upwinding only in the elements which contain a layer. We claim therefore that, with the method proposed in this paper, finite element methods will, both in theory and practice, be as much a success for non-self-adjoint problems, inclusive of singular perturbation problems, as for self-adjoint problems. In problems where the location of layers is not known beforehand one may locate them roughly, for instance by a simple difference upwinding.

The proposed method also provides a practical method for the solution of systems of first order hyperbolic problems with or without the use of artificial viscosity.

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