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ITERATIVE NONDETERMINISTIC ALGEBRAS

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0. Introduction

Iterative algebras have been introduced by J. Tiuryn in [5] as models of Elgot's iterative algebraic theories [3]. They are proposed as semantic domains for programming languages when programs are syntactically described by systems of equations. One of the main problem is the construction of free iterative algebras providing a possibility to reduce semantical questions to syntactical ones.

The aim of the paper is the construction of free iterative non-deterministic algebras.

1. Preliminary definitions and results

A *signature* Σ is a countable family $\Sigma = (\Sigma_n | \ n \in N)$ of disjoint sets Σ_n whose elements are called *n-ary operators symbols*. N denotes the set of natural numbers 0, 1, 2, ... As usual, a *Σ-algebra* A consists of a set A (carrier of A) and a family $(\sigma^A | \ \sigma \in \Sigma)$ of operations such that $\sigma^A\colon A^n \to A$ for each $\sigma \in \Sigma_n$, $n \in N$. (Notice that Σ is also used for the union over all Σ_n for simplicity.) A *Σ-homomorphism* from a Σ-algebra A into another Σ-algebra B is a mapping h from A into B subject to the following condition

$$h(\sigma^A(a_1, \ldots, a_n)) = \sigma^B(h(a_1), \ldots, h(a_n))$$

for all $\sigma \in \Sigma_n$, $n \in N$, and all $a_1, \ldots, a_n \in A$.

The class of all Σ-algebras will be denoted by Alg_Σ. Given a subclass $\mathscr{A}$ of Alg_Σ and a set X, a Σ-algebra F of $\mathscr{A}$ together with a mapping $f\colon X \to F$ is called *$\mathscr{A}$-free over* X if the following property holds. Every mapping h from X into an arbitrary Σ-algebra A of $\mathscr{A}$ admits a unique extension to a Σ-homomorphism h^* from F into A depicted by the following commutative diagram

$$\begin{tikzcd} F \arrow[rr, "\exists ! h^*"] & & A \\ & X \arrow[ul, "f"] \arrow[ur, "h"'] & \end{tikzcd} \qquad \text{(for all } h\text{)}$$

Although $\mathscr{A}$-free Σ-algebras are only uniquely determined up to isomorphism we will speak of the $\mathscr{A}$-free Σ-algebra over X and, if it exists, it will be denoted by $F_{\mathscr{A}}(X)$. In the case $\mathscr{A}$ equals Alg_Σ, the $\mathscr{A}$-free Σ-algebras are also called *absolutely* free. Let Σ be a signature and X be a disjoint set. The set of all *Σ-terms over X* is the least set of expressions $T_\Sigma(X)$ such that

(1) $X \cup \Sigma_0 \subseteq T_\Sigma(X)$;

(2) if $\sigma \in \Sigma_n$ and $t_1, \ldots, t_n \in T_\Sigma(X)$ for $n \geqslant 1$, then $\sigma t_1 \ldots t_n \in T_\Sigma(X)$.

$T_\Sigma(X)$ can be made into a Σ-algebra by setting

$$\sigma^{T_\Sigma(X)}(t_1, \ldots, t_n) = \sigma t_1 \ldots t_n$$

for all $\sigma \in \Sigma_n$, $t_1, \ldots, t_n \in T_\Sigma(X)$ and $n \geqslant 1$. The constants are the elements of Σ_0. It is well known that this so-called *Σ-term algebra* $T_\Sigma(X)$ is the absolutely free Σ-algebra over X. Therefore, every Σ-term t of $T_\Sigma(X)$ can be interpreted in any Σ-algebra A provided a mapping h from X into A is given. Observe that each t of $T_\Sigma(X)$ contains only a finite number of variables from X, say $x_0, \ldots, x_{n-1}$. If $h(x_i) = a_i$ for $i = 0, \ldots, n-1$, then we set

$$t^A(a_0, \ldots, a_{n-1}) = h^*(t)$$

and t^A can be regarded as an n-ary (derived) operation.

To avoid cumbersome notation with the variables occurring in Σ-terms we use the following representation of natural numbers by finite sets: 0 for $\emptyset$, 1 for $\{\emptyset\}$, 2 for $\{\emptyset, \{\emptyset\}\}$ and so on, or in other terms

$$0 = \emptyset \quad \text{and} \quad n = \{0, 1, \ldots, n-1\} \quad \text{for} \quad n \geqslant 1.$$

Instead of $T_\Sigma(\{x_0, \ldots, x_{n-1}\})$ we agree to write simply $T_\Sigma(n)$.

Now we are going to introduce *systems of Σ-equations*. A system S of n Σ-equations with m parameters $x_0, \ldots, x_{m-1}$ is given by

$$z_i = t_i, \quad i = 0, 1, \ldots, n-1,$$

where $t_i \in T_\Sigma(\{x_0, \ldots, x_{m-1}, z_0, \ldots, z_{n-1}\})$. Since S is fully determined by $t_0, \ldots, t_{n-1}$, we shall also write $S = (t_0, \ldots, t_{n-1})$ which means

$$S \in T_\Sigma(m+n)^n \quad \text{with } m \geqslant 0 \text{ and } n \geqslant 1.$$

A set $\mathfrak{S}$ of systems of Σ-equations is a subset of $\bigcup \{T_\Sigma(m+n)^n \mid m \geqslant 0$ and

$n \geqslant 1\}$. For a given set $\mathfrak{S}$ of systems of Σ-equations, $\mathfrak{S}_m$ denotes the subset of all systems of Σ-equations with m parameters.

Let A be a Σ-algebra. Every system $S = (t_0, \ldots, t_{n-1})$ of $T_\Sigma(m+n)^n$ induces a mapping

$$S^A\colon A^{m+n} \to A^n$$

by

$$S^A(a_0, \ldots, a_{m+n-1}) = (t_0^A(a_0, \ldots, a_{m+n-1}), \ldots, t_{n-1}^A(a_0, \ldots, a_{m+n-1}))$$

for all $a_0, \ldots, a_{m+n-1} \in A$.

We say that $S \in T_\Sigma(m+n)^n$ is *uniquely solvable* in A if, for all

$$\bar{a} = (a_0, \ldots, a_{m-1}) \quad \text{of } A^m,$$

there exists uniquely $\bar{s} = (s_0, \ldots, s_{n-1})$ of A^n such that

$$\bar{s} = S^A(\bar{a}, \bar{s}).$$

If S has a unique solution in A, then it will be denoted by $|S^A|$.

Note $|S^A|\colon A^m \to A^n$ for $S \in T_\Sigma(m+n)^n$.

DEFINITION. Let $\mathfrak{S}$ be a set of systems of Σ-equations. A Σ-algebra A is called *$\mathfrak{S}$-iterative* if each S of $\mathfrak{S}$ is uniquely solvable in A. ■

The class of all $\mathfrak{S}$-iterative Σ-algebras is denoted by $\mathfrak{S}\text{-Alg}_\Sigma$. Consider a set $\mathfrak{S}$ of systems of Σ-equations and assume that a given subclass $\mathscr{A}$ of $\mathfrak{S}\text{-Alg}_\Sigma$ has free Σ-algebras. Let $F_{\mathscr{A}}(m)$ be the $\mathscr{A}$-free Σ-algebra over m. By definition, every mapping h from m into any Σ-algebra A of $\mathscr{A}$ admits a unique extension to a Σ-homomorphism $h^*\colon F_{\mathscr{A}}(m) \to A$.

For any $S \in T_\Sigma(m+n)^n$ belonging to $\mathfrak{S}$ we conclude that the diagram

$$\begin{tikzcd}
F_{\mathscr{A}}(m)^m \arrow[r, "(h^*)^m"] \arrow[d, "|S|"'] & A^m \arrow[d, "|S^A|"] \\
F_{\mathscr{A}}(m)^n \arrow[r, "(h^*)^n"'] & A^n
\end{tikzcd}$$

commutes, i.e., $(h^*)^m \cdot |S^A| = |S| \cdot (h^*)$ for all h. Thereby, $|S|$ denotes the unique solution of S in $F_{\mathscr{A}}(m)$. $|S|$ is called the *symbolic solution.*

DEFINITION. Let $\mathfrak{S}$ be a set of systems of Σ-equations and let $\mathscr{A}$ be a subclass of $\mathfrak{S}\text{-Alg}_\Sigma$. Two systems S_1 and S_2 of $\mathfrak{S}$ with the same number of parameters are called *$\mathscr{A}$-equivalent*, in symbols $S_1 \sim_{\mathscr{A}} S_2$, if $|S_1^A| = |S_2^A|$ for all A of $\mathscr{A}$. ■

THEOREM 1. *Let $\mathfrak{S}$ be a set of systems of Σ-equations and let $\mathscr{A}$ be a subclass of $\mathfrak{S}\text{-Alg}_\Sigma$. Assume that $\mathscr{A}$ has $\mathscr{A}$-free Σ-algebras, then*

$$S_1 \sim_{\mathscr{A}} S_2 \quad \textit{iff} \quad |S_1| = |S_2|$$

for S_1, S_2 of $\mathfrak{S}_m$, $m \in N$.

Proof. By definition, $S_1 \sim_{\mathscr{A}} S_2$ implies $|S_1| = |S_2|$. Therefore, it remains to prove the opposite implication. But this follows easily from our considerations above. ∎

Theorem 1 shows the importance of the existence of free Σ-algebras. We know from universal algebra that the existence of free Σ-algebras is guaranteed for classes which are closed under subalgebras and products. This closure properties are fulfilled for any class of Σ-algebras which are defined by identities and implications. Such classes are called *quasivarieties.*

THEOREM 2 [4]. *For any quasivariety $\mathscr{A}$ of Σ-algebras and any set $\mathfrak{S}$ of systems of Σ-equations, the class of all $\mathfrak{S}$-iterative Σ-algebras in $\mathscr{A}$ has free Σ-algebras.* ∎

The proof of this theorem is quite simple but shows an interesting algebraic fact. A class of Σ-algebras is transformed into a quasivariety by enlargement the signature Σ. For each S of $\mathfrak{S}$, n new m-ary operator symbols are adjoined to Σ_m provided S has n Σ-equations with m parameters. Let $\Sigma(\mathfrak{S})$ be the resulting signature. Evidently, every $\mathfrak{S}$-iterative Σ-algebra can be regarded as a $\Sigma(\mathfrak{S})$-algebra and, moreover, every Σ-homomorphism can be regarded as a $\Sigma(\mathfrak{S})$-homomorphism. Then the class of all $\mathfrak{S}$-iterative Σ-algebras in $\mathscr{A}$ defines a quasivariety of $\Sigma(\mathfrak{S})$-algebras since the requirements of the existence and uniqueness of solutions are described by identities and implications. Hence free $\Sigma(\mathfrak{S})$-algebras exist. But this are the free $\mathfrak{S}$-iterative Σ-algebras in $\mathscr{A}$.

2. Nondeterministic Σ-algebras

To deal with nondeterministic computations we need two distinguished operator symbols: a binary operator symbol $+$ describing nondeterministic choice and a nullary operator symbol 0 describing abort. Let Σ be a signature not containing $+$ and 0. By $\bar{\Sigma}$ we denote the new signature obtained from Σ by adjoining $+$ and 0, i.e., $\bar{\Sigma}_0 = \Sigma_0 \cup \{0\}$, $\bar{\Sigma}_2 = \Sigma_2 \cup \{+\}$ and $\bar{\Sigma}_n = \Sigma_n$ otherwise.

DEFINITION. A Σ-algebra A is called *nondeterministic* if

(1) $(A, +, 0)$ is a commutative monoid,

(2) for every $\sigma \in \Sigma_n$, $n \geqslant 1$, the associated operation σ^A is n-fold linear, i.e.,

$$\sigma^A(a_1, \ldots, 0, \ldots, a_n) = 0,$$

$$\sigma^A(a_1, \ldots, a_i + a_i', \ldots, a_n) = \sigma^A(a_1, \ldots, a_i, \ldots, a_n) + \sigma^A(a_1, \ldots, a_i', \ldots, a_n). \quad ∎$$

The class of all nondeterministic Σ-algebras is denoted by $\mathrm{Alg}_\Sigma^{\mathrm{nd}}$. Let A be a Σ-algebra. Define

$$N\langle A\rangle = \{p \mid p\colon A \to N \text{ and } \operatorname{supp}(p) \text{ is finite}\},$$

where $\operatorname{supp}(p) = \{a \in A \mid p(a) \neq 0\}$ is the support of p. The elements of $N\langle A\rangle$ are said to be *polynomials over A* with coefficients in N. As usual, we write

$$p = \sum_{a \in A} (p, a)a \quad \text{with}\,(p, a) = p(a) \text{ for } a \in A.$$

$N\langle A\rangle$ can be made into a Σ-algebra by setting
(1) 0 is defined by $(0, a) = 0$ for all $a \in A$,

$$(2) \qquad p_1 + p_2 = \sum_{a \in A} \big((p_1, a) + (p_2, a)\big)a,$$

$$(3) \qquad \sigma(p_1, \ldots, p_n) = \sum_{a \in A} \Big(\sum_{\substack{a_1, \ldots, a_n \in A \\ a = \sigma^A(a_1, \ldots, a_n)}} (p_1, a) \ldots (p_n, a)\Big)a$$

for any $\sigma \in \Sigma_n$, $n \geqslant 1$.

For simplicity the operations in $N\langle A\rangle$ are denoted by the operation symbols.

Without proof we state

LEMMA 1. *If A is a Σ-algebra, then $N\langle A\rangle$ is a nondeterministic $\bar{\Sigma}$-algebra.* ∎

This leads immediately to

THEOREM 3. *$N\langle T_\Sigma(X)\rangle$ is the free nondeterministic Σ-algebra over X.*

Proof. Let A be an arbitrary nondeterministic $\bar{\Sigma}$-algebra. Since A can be considered as a Σ-algebra, every mapping h from X into A admits a unique extension to a Σ-homomorphism h^* from $T_\Sigma(X)$ into A. Define a mapping

$$h^{\#}\colon N\langle T_\Sigma(X)\rangle \to A$$

by

$$h^{\#}(p) = \sum_{t \in T_\Sigma(X)} (p, t)\, h^*(t) \qquad \text{for all } p \text{ of } N\langle T_\Sigma(X)\rangle.$$

It may easily be shown that $h^{\#}$ is the unique extension of h to a $\bar{\Sigma}$-homomorphism. ∎

Consider as an example a system S of two $\bar{\Sigma}$-equations with one parameter x given by

$$z_0 = g\big(f(z_0) + x, f(z_1)\big), \qquad z_1 = f\big(g(z_0, f(z_1) + z_0)\big),$$

using infix notation for $+$ and parantheses for better readability. Σ contains the unary operation symbol f and the binary operation symbol g. Since we are only interested to solve S in any nondeterministic $\bar{\Sigma}$-algebra, S may be transformed according to the requirements that f and g are interpreted as linear mappings as follows:

$$z_0 = g\big(f(z_0), f(z_1)\big) + g\big(x, f(z_1)\big), \qquad z_1 = f\big(g(z_0, f(z_1))\big) + f\big(g(z_0, z_0)\big).$$

More formally, to any system $S \in T_{\bar{\Sigma}}(m+n)^n$ of $\bar{\Sigma}$-equations we find a system $\bar{S} \in N\langle T_\Sigma(m+n)\rangle^n$ such that S and $\bar{S}$ are equivalent modulo the class of all nondeterministic $\bar{\Sigma}$-algebras. $\bar{S}$ will be called a *polynomial system of Σ-equations.*

3. Formal power series over $T_\Sigma(X)$

Formal power series over $T_\Sigma(X)$ with coefficients in a field were first studied by Berstel and Reutenauer in [2]. Here, we are going to introduce them by a metric completion of the set $N\langle T_\Sigma(X)\rangle$ of all polynomials.

LEMMA 2. *$N\langle T_\Sigma(X)\rangle$ is a metric space with respect to the distance function*

$$d(p, p') = \begin{cases} 0 & \text{if } p = p', \\ 2^{-\min\{\delta(t) \mid (p,t) \neq (p',t)\}} & \text{if } p \neq p', \end{cases}$$

where $\delta(t)$ is the depth of $t \in T_\Sigma(X)$. It is recursively defined by

$$\delta(t) = \begin{cases} 0 & \text{if } t \in X \subset \Sigma_0, \\ 1 + \max\{\delta(t_i) \mid i = 1, \ldots, n\} & \text{if } t = \sigma t_1 \ldots t_n. \end{cases} \quad \blacksquare$$

The metric completion of $N\langle T_\Sigma(X)\rangle$ is the set $N\langle\langle T_\Sigma(X)\rangle\rangle$ of all mappings p from $T_\Sigma(X)$ into N called *formal power series over* $T_\Sigma(X)$ with coefficients in N. We also write

$$p = \sum_{t \in T_\Sigma(X)} (p, t)t$$

for the mapping p: $T_\Sigma(X) \to N$, where $(p, t) = p(t)$ for $t \in T_\Sigma(X)$.

Extending the operations for polynomials to power series in a straightforward manner, we get

LEMMA 3. *$N\langle\langle T_\Sigma(X)\rangle\rangle$ can be made into a nondeterministic $\bar{\Sigma}$-algebra.* $\blacksquare$

Notice that, for $\sigma \in \Sigma_n$, $n \geq 1$, and $p_1, \ldots, p_n \in N\langle\langle T_\Sigma(X)\rangle\rangle$,

$$\sigma(p_1, \ldots, p_n) = \sum_t \Big(\sum_{\substack{t_1, \ldots, t_n \\ t = \sigma t_1 \ldots t_n}} (p_1, t_1) \cdot \ldots \cdot (p_n, t_n) \Big) t$$

is well-defined.

Since $N\langle\langle T_\Sigma(X)\rangle\rangle$ belongs to $\mathrm{Alg}_\Sigma^{\mathrm{nd}}$ and $N\langle T_\Sigma(X)\rangle$ is the free nondeterministic $\bar{\Sigma}$-algebra over X, each polynomial p of $N\langle T_\Sigma(X)\rangle$ with n variables induces a derived n-ary operation $\bar{p}$ on $N\langle\langle T_\Sigma(X)\rangle\rangle$.

By an easy calculation we may prove

LEMMA 4. *Let $p \in N\langle T_\Sigma(X)\rangle$. If $\mathrm{supp}(p) \cap X = \emptyset$, then $\bar{p}$ is a contractive mapping.* $\blacksquare$

Furthermore, $N\langle\langle T_\Sigma(X)\rangle\rangle$ is a complete metric space. If a polynomial system of Σ-equations induces a contractive mapping, then it has a unique solution by the Banach Fixpoint Theorem. This forces the following

DEFINITION. A polynomial system $S = (p_0, \ldots, p_{n-1})$ of $\bar{\Sigma}$-equations

$$z_i = p_i, \quad i = 0, \ldots, n-1,$$

is called *proper* if the support of each p_i does not contain the unknowns, that is

$$\operatorname{supp}(p_i) \cap \{z_0, \ldots, z_{n-1}\} = \emptyset \quad \text{for} \quad i = 0, \ldots, n-1. \blacksquare$$

As a conclusion of Lemma 4 we get

THEOREM 4. *If $\mathfrak{S}$ is the set of all proper polynomial systems of $\bar{\Sigma}$-equations with parameters in X, then $N\langle\langle T_\Sigma(X)\rangle\rangle$ is a $\mathfrak{S}$-iterative nondeterministic $\bar{\Sigma}$-algebra.* ■

4. Regular formal power series over $T_\Sigma(X)$

We intend to construct free nondeterministic $\bar{\Sigma}$-algebras by means of regular formal power series over $T_\Sigma(X)$ in the next section. In order to introduce them here some preparations are necessary.

Any commutative monoid $(A, +, 0)$ can be regarded as a semimodule over N if scalar product na is defined to be the n-fold sum of a for all $n \in N$ and $a \in A$. In analogy to modules tensor product $\otimes$ of semimodules may be defined. By definition, the elements $a \otimes b$ of the tensor product $A \otimes B$ of two semimodules satisfy the following conditions

$$(a + a') \otimes b = a \otimes b + a' \otimes b, \tag{1}$$

$$a \otimes (b + b') = a \otimes b + a \otimes b', \tag{2}$$

$$na \otimes b = a \otimes nb = n(a \otimes b) \tag{3}$$

for all $a, a' \in A$, $b, b' \in B$ and $n \in N$. By $A^{\otimes n}$ we denote the n-fold tensor product of A. Notice that any n-fold linear mapping from A^n into A can be considered as a linear mapping from $A^{\otimes n}$ into A.

Next, any $\sigma \in \Sigma_n$, $n \geqslant 1$, a so-called derivation σ^{-1} is associated as a linear mapping from $N\langle\langle T_\Sigma(X)\rangle\rangle$ into the n-fold tensor product $N\langle\langle T_\Sigma(X)\rangle\rangle^{\otimes n}$ by setting

$$\sigma^{-1}(p) = \sum_{t_1, \ldots, t_n} (p, \sigma t_1 \ldots t_n) t_1 \otimes \ldots \otimes t_n.$$

Evidently, σ^{-1} is linear, i.e., $\sigma^{-1}(0) = 0$ and $\sigma^{-1}(p + p') = \sigma^{-1}(p) + \sigma^{-1}(p')$. We extend σ^{-1} to subsets A of $N\langle\langle T_\Sigma(X)\rangle\rangle$ by

$$\sigma^{-1}(A) = \{\sigma^{-1}(p) \mid p \in A\}.$$

Without proof we state the following lemmata.

LEMMA 5. *For each p of $N\langle\langle T_\Sigma(X)\rangle\rangle$ the equality*

$$p = \sum_{t\in X\cup\Sigma_0} (p, t) + \sum_{n\geqslant 1}\sum_{\sigma\in\Sigma_n} \sigma(\sigma^{-1}(p)).$$

holds. ∎

LEMMA 6. *Let $p_1, \ldots, p_n \in N\langle\langle T_\Sigma(X)\rangle\rangle$. For each $\sigma\in\Sigma_n$, $n\geqslant 1$, we get*

$$\sigma^{-1}(\sigma(p_1, \ldots, p_n)) = p_1\otimes\ldots\otimes p_n. \;\blacksquare$$

A subsemimodule A of $N\langle\langle T_\Sigma(X)\rangle\rangle$ is called *stable* whenever $\sigma^{-1}(A)$ is included in the n-fold tensor product of A for any $\sigma\in\Sigma_n$, $n\geqslant 1$.

DEFINITION. A formal power series p of $N\langle\langle T_\Sigma(X)\rangle\rangle$ is called *regular* if p belongs to a finitely generated stable subsemimodule of $N\langle\langle T_\Sigma(X)\rangle\rangle$. ∎

LEMMA 7. *Let $p\in N\langle\langle T_\Sigma(X)\rangle\rangle$. If p is regular, then there is a proper polynomial system S of Σ-equations such that p is the first component of the solution of S.*

Proof. Let $p\in N\langle\langle T_\Sigma(X)\rangle\rangle$ be regular. By definition, there exists a finitely generated stable subsemimodule A of $N\langle\langle T_\Sigma(X)\rangle\rangle$ such that $p\in A$. Let $p_1, \ldots, p_{n-1}$ be the generators of A. Without loss of generality we can put p to this set of generators and assume $\{p_0, p_1, \ldots, p_{n-1}\}$ is a set of generators for A, where $p_0 = p$. To describe the generators of the k-fold tensor product of A in a convenient way we introduce the following notation. Let w be a word of length k over the alphabet $\{0, 1, \ldots, n-1\}$, that is, $w\in n^k$. If $w = i_1\ldots i_k$, then we put $w = p_{i_1}\otimes\ldots\otimes p_{i_k}$. Clearly,

$$A^{\otimes k} = \{\sum_{w\in n^k} r_w w \mid r_w\in N\}.$$

Take $\sigma\in\Sigma_k$. Since A is stable

$$\sigma^{-1}(p_i)\in A^{\otimes k} \quad \text{for} \quad i = 0, 1, \ldots, n-1.$$

Hence

$$(*) \qquad \sigma^{-1}(p_i) = \sum_{w\in n^k} r(\sigma)_{i,w}\, w \quad \text{for} \quad i = 0, 1, \ldots, n-1.$$

Now define a polynomial system of Σ-equations as follows

$$z_i = \sum_{t\in X\cup\Sigma_0} (p_i, t)t + \sum_{k\geqslant 1}\sum_{\sigma\in\Sigma_k}\sum_{w\in n^k} r(\sigma)_{i,w}\,\sigma(w), \qquad i = 0, \ldots, n-1.$$

Obviously, this system is proper and has as unique solution $(p_0, \ldots, p_{n-1})$ since, by Lemma 5,

$$p_i = \sum_{t\in X\cup\Sigma_0} (p_i, t)t + \sum_{k\geqslant 1}\sum_{\sigma\in\Sigma_k} \sigma(\sigma^{-1}(p_i)), \qquad i = 0, \ldots, n-1,$$

and because of $(*)$ we conclude that $(p_0, \ldots, p_{n-1})$ is a solution.

Therefore, $p = p_0$ is the first component which proves the statement. ∎

To prove the opposite implication, that each first component of the solution of a given proper polynomial system S of Σ-equations is regular, we have to transform S into a certain normal form due to

LEMMA 8 [2]. *Let $p \in N\langle\langle T_\Sigma(X)\rangle\rangle$. If p is the first component of the solution of some proper polynomial system of Σ-equations, then p is also the first component of the solution of a proper polynomial system*

$$z_i = p_i, \quad i = 0, 1, \ldots, n-1,$$

with $\operatorname{supp}(p_i) = (X \cup \Sigma_0) \cup \Sigma(Z^*)$, *where* $Z = \{z_0, z_1, \ldots, z_{n-1}\}$ *and* $\Sigma(Z^*) = \bigcup_{k \geqslant 1} \Sigma(Z^k)$ *with* $\Sigma(Z^k) = \{\sigma z_{i_1} \ldots z_{i_k} | \; z_{i_1}, \ldots, z_{i_k} \in Z\}$. ∎

LEMMA 9. *Let $p \in N\langle\langle T_\Sigma(X)\rangle\rangle$. If p is the first component of the solution of a proper polynomial system of Σ-equations, then p is regular.*

Proof. Let S be a proper polynomial system of Σ-equations as described in Lemma 8. Assume $(p_0, \ldots, p_{n-1})$ is the unique solution of S. Let A be the subsemimodule generated by the components $p_0, \ldots, p_{n-1}$. To prove that the first component p_0 is regular it remains to show that A is stable. Because of the special required form of S we derive

$$\sigma^{-1}(p_i) = \sum_{w \in n^k} (p_i, \sigma(w)) \mathbf{w}, \quad i = 0, \ldots, n-1,$$

for any $\sigma \in \Sigma_k$, $k \geqslant 1$, where $\mathbf{w} = p_{i_1} \otimes \ldots \otimes p_{i_k}$ for $w = i_1 \ldots i_k$. Since σ^{-1} is linear, $\sigma^{-1}(p)$ belongs to $A^{\otimes k}$ for all p of A. Hence A is stable. ∎

Lemma 7 and Lemma 9 show that the set of all regular formal power series over $T_\Sigma(X)$, denoted by $N^{\text{reg}}\langle\langle T_\Sigma(X)\rangle\rangle$, coincides with the set of all first components of solutions of proper polynomial systems of Σ-equations. The latter set is denoted by $N^{\text{rat}}\langle\langle T_\Sigma(X)\rangle\rangle$. Its elements are called *rational* formal power series over $T_\Sigma(X)$.

THEOREM 5. $N^{\text{reg}}\langle\langle T_\Sigma(X)\rangle\rangle = N^{\text{rat}}\langle\langle T_\Sigma(X)\rangle\rangle$. ∎

5. Free iterative nondeterministic $\bar{\Sigma}$-algebras

Throughout this section, $\mathfrak{S}$ denotes the set of all proper polynomial systems of Σ-equations whose parameters belong to a fixed set X. Instead of $\mathfrak{S}$-iterative we shall simply speak of iterative nondeterministic $\bar{\Sigma}$-algebras. As already mentioned, the free iterative nondeterministic $\bar{\Sigma}$-algebras will be constructed by means of regular formal power series. For that purpose we have first to show that they form an iterative nondeterministic $\bar{\Sigma}$-algebra.

LEMMA 10. *$N^{\text{reg}}\langle\langle T_\Sigma(X)\rangle\rangle$ forms an iterative subalgebra of $N\langle\langle T_\Sigma(X)\rangle\rangle$ containing $N\langle T_\Sigma(X)\rangle$.*

Proof. Clearly, every polynomial p of $N\langle T_\Sigma(X)\rangle$ is regular since, by Theorem 5, p is (the first component of) the unique solution of the trivial proper polynomial system $z = p$. Therefore, all constants associated to elements of $\bar{\Sigma}_0$ belong to $N^{\text{reg}}\langle\langle T_\Sigma(X)\rangle\rangle$. To show that $N^{\text{reg}}\langle\langle T_\Sigma(X)\rangle\rangle$ forms a subalgebra it remains to prove that $N^{\text{reg}}\langle\langle T_\Sigma(X)\rangle\rangle$ is closed under all n-ary operations for $n \geqslant 1$. Let p_1 and p_2 be regular. By Theorem 5, there are proper polynomial systems S_1 and S_2 such that p_i is the first component of the solution of S_i, $i = 1, 2$. Now, one may easily define a proper polynomial system S such that $p_1 + p_2$ is the first component of the solution of S. Hence, the sum of two regular formal power series is always regular. Let $\sigma \in \Sigma_n$, $n \geqslant 1$. If $p_1, \ldots, p_n \in N^{\text{reg}}\langle\langle T_\Sigma(X)\rangle\rangle$, then there are proper polynomial systems $S_1, \ldots, S_n$ such that p_i is the first component of the solution of S_i, $i = 1, \ldots, n$. Put all these systems together with a new first equation $z_0 = \sigma z_{1,0} \ldots z_{n,0}$, where $z_{i,0}$ are the unknowns of the first equation of each S_i. Obviously, the new system has as first component of its solution $\sigma(p_1, \ldots, p_n)$. Thus, $N^{\text{reg}}\langle\langle T_\Sigma(X)\rangle\rangle$ is closed under all operations. By definition, $N^{\text{reg}}\langle\langle T_\Sigma(X)\rangle\rangle$ is iterative. ∎

Iterative nondeterministic $\bar{\Sigma}$-algebras can be thought of as models of iterative matrix theories. In analogy to [1] we will prove that all regular formal power series over $T_\Sigma(X)$ form the free iterative nondeterministic $\bar{\Sigma}$-algebra over X. The main idea is based on the fact that, by Theorem 5, every regular formal power series p is the first component of the solution of some proper polynomial system S of Σ-equations. Then p can be mapped onto the first component of the solution of S in any iterative nondeterministic $\bar{\Sigma}$-algebra A provided X is interpreted in A. In order to show that this mapping is well-defined we have to make clear that, for any two proper polynomial systems of Σ-equations whose first component of their solutions are equal, the first component of the solutions in A are also equal. Let S and S' be two proper polynomial systems of Σ-equations. From Lemma 9 we know that the components of their solutions generate stable subsemimodules in $N^{\text{reg}}\langle\langle T_\Sigma(X)\rangle\rangle$. Without loss of generality we may assume that they generate the same stable subsemimodule. But, then all components of one solution are linear combinations of the other ones. The same holds true in any iterative nondeterministic $\bar{\Sigma}$-algebra, which implies the above required property.

THEOREM 6. *$N^{\text{reg}}\langle\langle T_\Sigma(X)\rangle\rangle$ is the free iterative nondeterministic $\bar{\Sigma}$-algebra over X.*

Proof. We have to show that every mapping h from X into an arbitrary iterative nondeterministic $\bar{\Sigma}$-algebra A admits a unique extension to a $\bar{\Sigma}$-homomorphism h^*: $N^{\text{reg}}\langle\langle T_\Sigma(X)\rangle\rangle \to A$. Define h^* by sending any regular formal power series p to the first component of the solution of S in A provided p is the first component of the solution of S in $N^{\text{reg}}\langle\langle T_\Sigma(X)\rangle\rangle$. Evidently, h^* extends h. To show that h^* is a $\bar{\Sigma}$-homomorphism may be done by constructing appropriate systems of Σ-equations similarly to the proof of Lemma 9. Moreover, h^* is unique since it extends also the $\bar{\Sigma}$-homomorphism $h^{\#}$ from $N\langle T_\Sigma(X)\rangle$ into A defined in Theorem 3. Assume g is another $\bar{\Sigma}$-homomorphism from $N^{\mathrm{reg}}\langle\langle T_\Sigma(X)\rangle\rangle$ into A extending h. Then, g restricted to $N\langle T_\Sigma(X)\rangle$ equals $h^{\#}$. Let S be a proper polynomial system of Σ-equations. Obviously, its interpretation in A under g is the same as under h^*. Therefore, the first component p of the solution of S in $N^{\mathrm{reg}}\langle\langle T_\Sigma(X)\rangle\rangle$ will be mapped onto $h^*(p)$, i.e., $g(p) = h^*(p)$ for all $p \in N^{\mathrm{reg}}\langle\langle T_\Sigma(X)\rangle\rangle$. ■

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