SINGULARITIES
BANACH CENTER PUBLICATIONS, VOLUME 20
PWN-POLISH SCIENTIFIC PUBLISHERS
WARSAW 1988

STABLE MAPPINGS OF 3-MANIFOLDS INTO THE PLANE

HAROLD LEVINE

Department of Mathematics, Brandeis University,
Waltham, MA, U.S.A.

1

My main objective in this talk is to describe the gross geometric anatomy of smooth stable maps of 3-manifolds into the plane. As are all stable maps, these maps are characterized by certain transversality conditions which, in this low dimensional case, can be given concretely in terms of germs and multi-germs. We begin with this characterizing fine geometric anatomy.

For $f\colon M \to \boldsymbol{R}^2$, M a 3-dimensional compact manifold, we let $S(f)$ be the singular set of f, $\{P \in M \mid rk T_P f < 2\}$. f is stable if:

(S_1) For each $P \in M$ there are coordinates (u, x, y) centered at P and (U, X) centered at $f(P)$ such that:

$$\big(U(f(u, x, y)), X(f(u, x, y))\big)$$
$$= \begin{cases} (u, x), & P \text{ a regular point}, \\ (u, x^2+y^2), & P \text{ a definite fold point}, \\ (u, x^2-y^2), & P \text{ an indefinite fold point}, \\ (u, y^2+xu-x^3/3), & P \text{ a cusp point}, \end{cases} \quad P \in S(f).$$

In these coordinate neighborhoods in M, $S(f)$ consists of smooth arcs of definite or indefinite fold points or one of each meeting smoothly at a cusp point. Thus $S(f)$ is a union of embedded circles and $f \mid S(f)$ is an immersion except at cusps where $f \mid S(f)$ is singular with image $\{(x^2, 2x^3/3)\}$.

(S_2) If P is a cusp point, $f^{-1}(P) \cap S(f) = \{P\}$ and $f \mid (S(f) - \{\text{cusps}\})$ is an immersion with normal crossings.

2

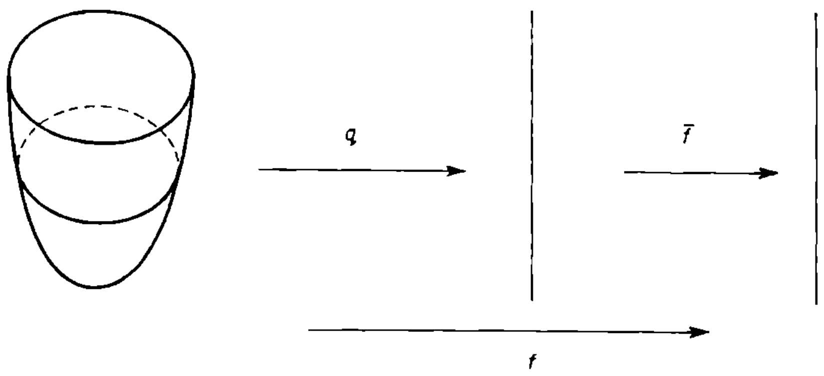
In order to study such maps systematically, we introduce an equivalence relation in M. We say that $P \sim P^1$ if $f(P) = f(P^1)$ and P and P^1 belong to the same connected component of $f^{-1}f(P)$. We let $W = M/\sim$ and factor $f\colon M \to \boldsymbol{R}^2$ through this quotient.

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "q"'] & & \boldsymbol{R}^2 \\ & W \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

We give W the quotient topology. As a consequence of the stability of f, W is easy to describe and is a 2-dimensional simplicial complex with very simple singularities. We call a subset of M *saturated*, if it is the union of connected components of f-fibres.

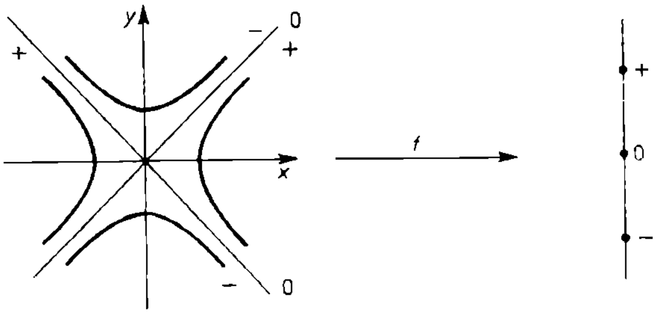
2.0. At any regular point $P \in M - S(f)$, f is locally a projection, so a neighborhood of $q(P)$ in W can be identified with a neighborhood of $f(P)$ in $\boldsymbol{R}^2$.

2.1. If P is a definite fold point, the map f in coordinates is a product $(u, x, y) \to (u, x^2+y^2)$ so it suffices to consider the map $(x, y) \to (x^2+y^2)$ on each (u = constant)-plane. The f-fibres in this plane are just concentric circles through the regular points in a neighborhood of P and single points at the definite fold points. Thus the neighborhood $\{|u| < \varepsilon,\ x^2+y^2 < \delta\}$ for $\varepsilon > 0$, $\delta > 0$ is saturated.



Thus the q-image of a neighborhood of P can be identified with its f-image. In this case the germ of this image at $q(p)$ or $f(p)$ is the germ of a closed half plane, the boundary of which is the image of the arc of definite fold points.

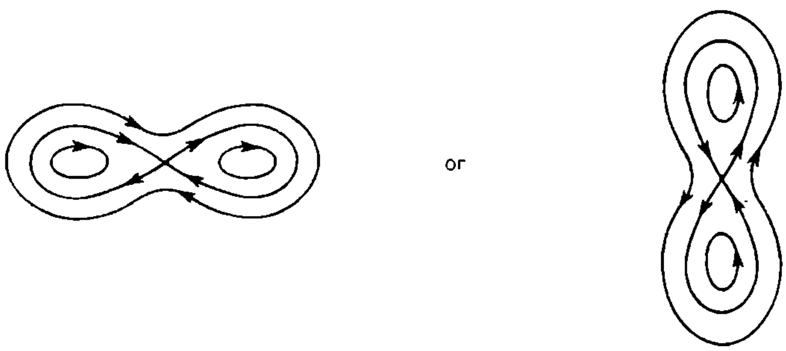
2.2. If P is an indefinite fold point, the map f is a product $(u, x, y) \to (u, x^2-y^2)$ and on each (u = constant)-plane we have:



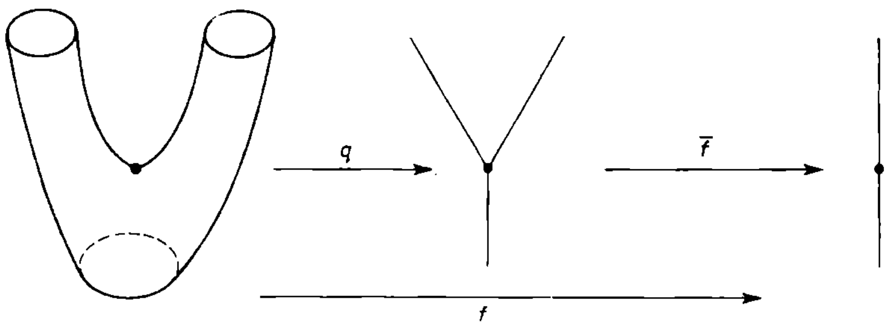
In this coordinate neighborhood the components of the f-fibres are not closed in M; that is they are not whole q-fibres. Thus we must leave the coordinates neighborhood to obtain a saturated set. The possibilities for completing these q-fibres will depend on the orientability of M.

Notice that the fibres of $f \mid M-S(f)$ can be oriented iff $M-S(f)$ is orientable. In fact, orienting the fibres of $f \mid M-S(f)$ is the same as orienting the line bundle $K = (\text{kernel}\, Tf \mid M-S(f))$. The quotient bundle, N, of $T(M-S(f))$ by K being isomorphic via Tf to $f^*(TR^2) \mid M-S(f)$ is always orientable. Thus K is orientable iff $M-S(f)$ is.

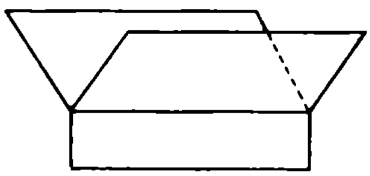
For the moment, we assume that $q^{-1}q(P) \cap S(f) = \{P\}$. We will return to the possibility that the component of the f-fibre through P has a second point P^1 in $S(f)$. If M is oriented, we orient the f-fibres in our coordinate neighborhood. Compatible with this orientation there are only two ways to complete these local fibres to obtain whole q-fibres



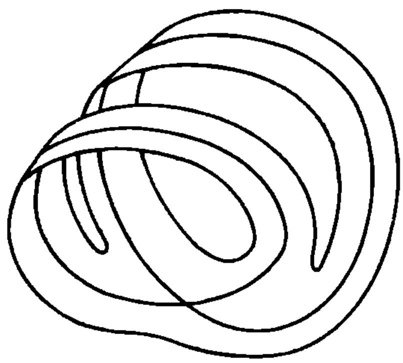
Thus a saturated neighborhood of P is the product of a disc with two holes and an interval. The map on each of these punctured discs looks like:

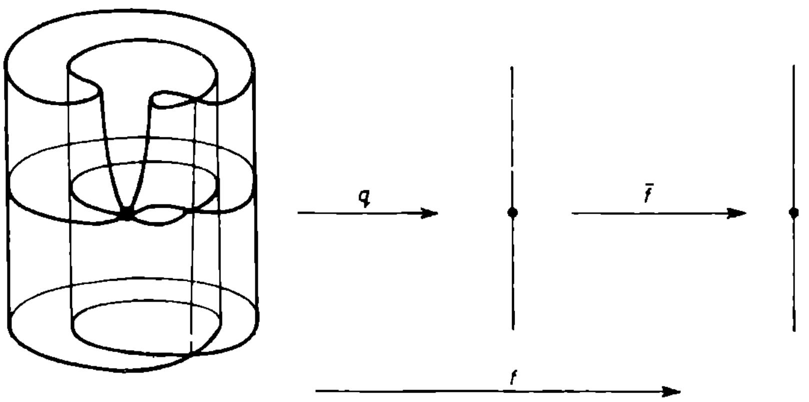


Thus a neighborhood of $q(P)$ in W is the product of a "Y" with an interval and the segment at the branching of the Y's is the image of the arc of indefinite points. The q-fibre through each indefinite point is a figure eight.



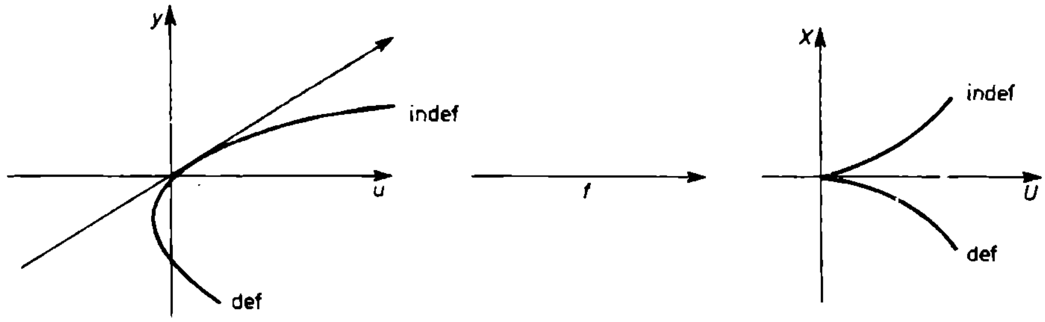
If M is not oriented, there is one other possibility. Namely, a saturated neighborhood of P is the product of a Möbius band with one hole and an interval, and a neighborhood of $q(P)$ and $\bar{f}(P)$ is a product of two intervals, $[-1, 1] \times I$. The image of the arc of indefinite points is $\{0\} \times I$. The maps f, q and $\bar{f}$ on this punctured Möbius band may be visualized as:



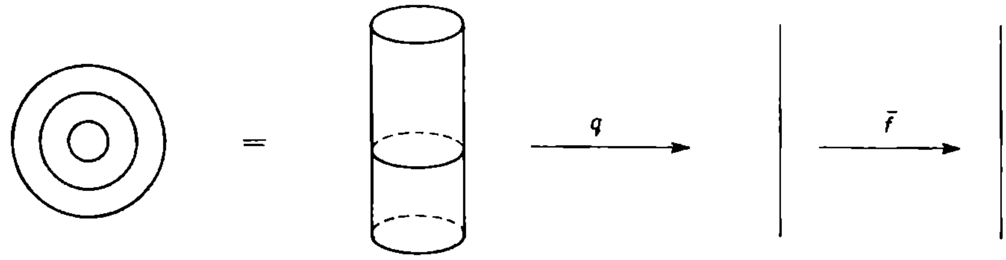


Notice that in all cases the f-fibres above two points on opposite sides of the image of a curve of indefinite points differ by a single surgery (oriented surgery if M is oriented).

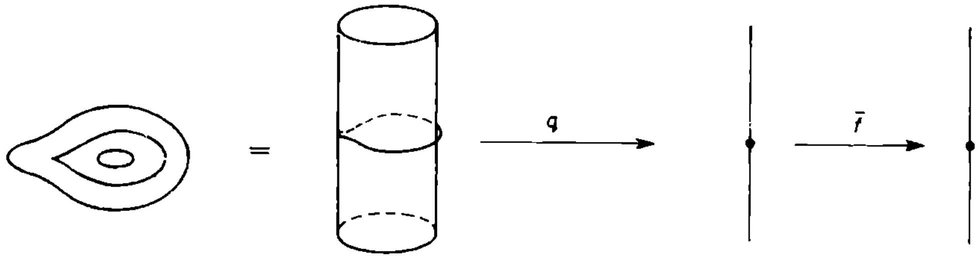
2.3. If P is a cusp, f in coordinates is $(u, x, y) \to (u, y^2 + xu - x^3/3)$, no longer a product. The singular set of this map is $\{(x^2, x, 0)\}$ with image $\{(x^2, 2x^3/3)\}$.



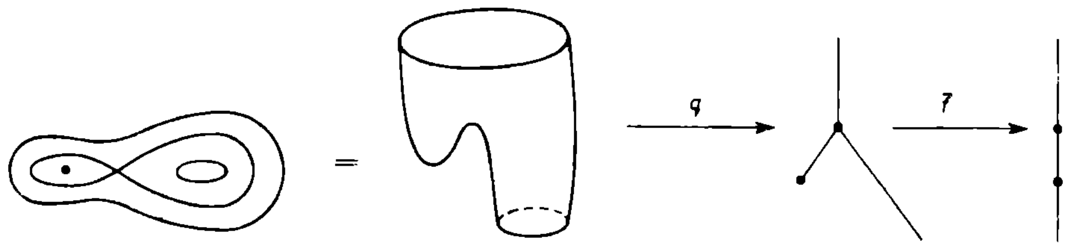
We describe a saturated neighborhood of P in M and its image in W by describing the f-fibre components in the f-pre-image of (U = constant)-lines whose union gives the saturated neighborhood.



Above any (U = negative)-line.

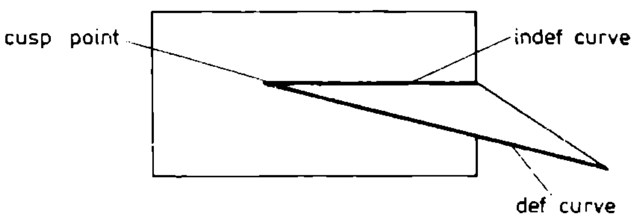


Above to (U = 0)-line.



Above any (U = positive)-line.

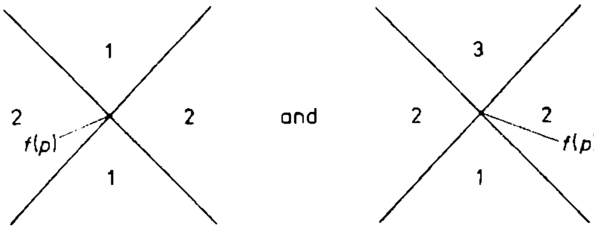
Thus a neighborhood of the q-image of a cusp is:



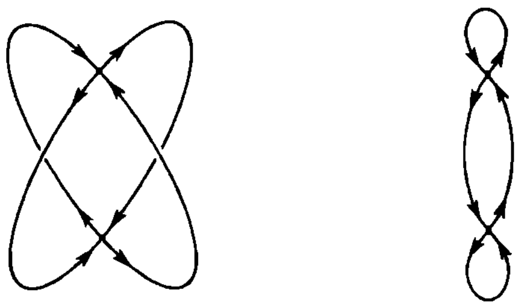
Remark. By the stability assumption (S_2), we know P is the only singular point on $f^{-1}f(P)$ for P a cusp. Hence we know that $q^{-1}q(P)$ is just an embedded circle except for one cusp singularity at P.

2.4. We now deal with the situation that we deferred in 2.2, namely that of P an indefinite fold point such that $q^{-1}q(P) \cap S(f) = \{P, P^1\}$. Obviously, P^1 is also an indefinite fold point and the f-images of the arcs of indefinite fold points through P and P^1 cross transversally. *To simplify the exposition we complete the local description of W in the M-oriented case only.*

We notice first that above a little neighborhood of $f(P)$, there are only two possible arrangements of the number of $\bar{f}$-preimages in a connected neighborhood of W containing $q(P)$.

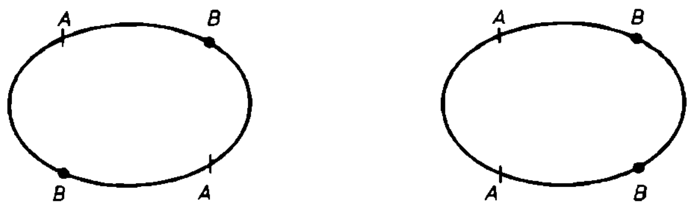


These two possibilities actually occur and the q-fibres above $q(P)$ correspond to the two connected graphs with two transversal saddle nodes.



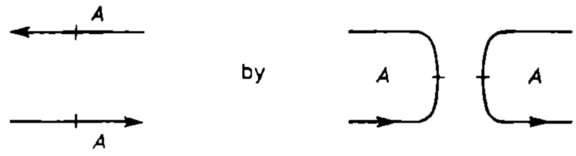
without self-loop   with self-loops

To see how these two cases arise, we follow the q-fibre as we move about the neighborhood $f(P)$. As we cross each of the image arcs of the indefinite fold point curves the f-fibres change as a result of an oriented surgery. Note that there are only two ways to perform two oriented surgeries on a circle, namely:



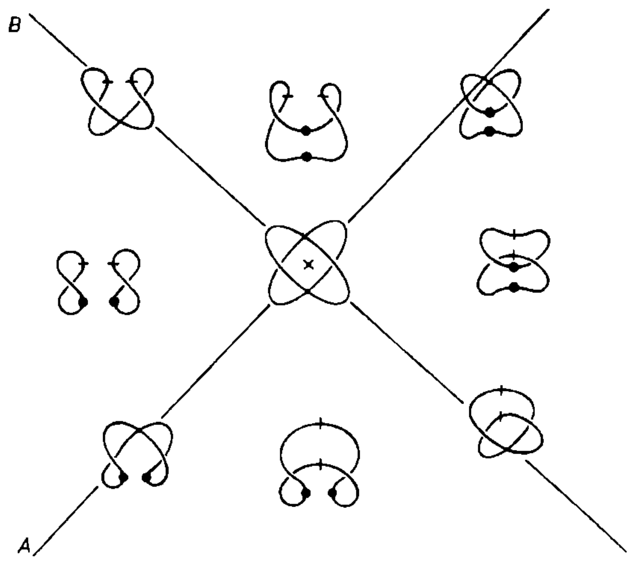
alternating   parallel

where the A-surgery replaces

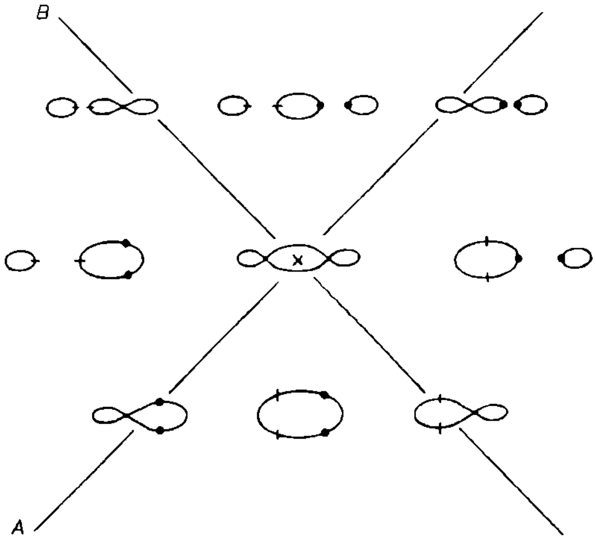


(similarly for B-surgery). We draw the fibres in a saturated neighborhood containing P and P^1 above a neighborhood of $f(P)$. We perform the surgeries A and B as we cross image of the indefinite fold curves labelled A and B respectively.

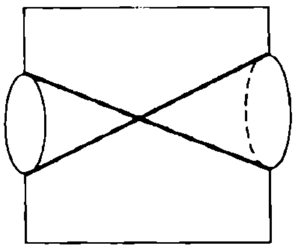
In the alternating case:

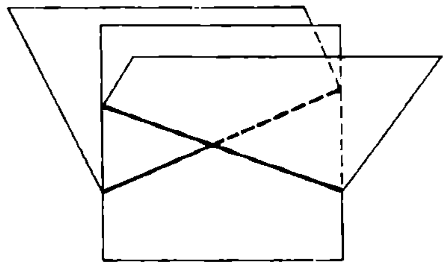


In the parallel case:



Thus on W' neighborhoods are in these two cases:





In the alternating case we call $q(P)$ a $(1\cdot 2\cdot 2\cdot 1)$-point and in the parallel case we call $q(P)$ a $(1\cdot 2\cdot 2\cdot 3)$-point. We have now completed the catalog of local descriptions of W.

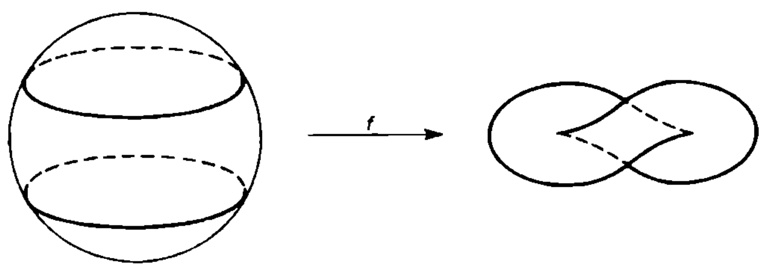
3

The following is an old problem addressed in [2], [3] using the auxiliary space W.

Given f: $M \to \mathbf{R}^2$, M a compact, oriented 3-manifold and f stable, when does there exist an h: $M \to \mathbf{R}^2$ such that (f, h): $M \to \mathbf{R}^2 \times \mathbf{R}^2$ is an immersion and how many different such h are there.

Haefliger in [1] solved the analogous problem of the existence of an immersion in $\mathbf{R}^3$ over a stable map of a surface into $\mathbf{R}^2$. *The necessary and sufficient condition for the existence of such an immersion is that for each component curve of $S(g)$, a neighborhood of that component is orientable iff the number of cusp points on that component is even.*

He gave the following beautiful example of a stable map of S^2 into $\mathbf{R}^2$ that cannot be lifted.



Here the polar caps are embedded as the tear drop shaped regions and the equatorial cylinder is mapped to the degenerate fattened figure eight.

In the rest of this talk I will give two examples of stable maps of 3-manifolds into $\mathbf{R}^2$ that cannot be lifted to immersions in $\mathbf{R}^4$. To show that no such immersions exist will require a few of the ideas used in the classification of the regular homotopy classes of such lifts.

We know that for any stable map f, that $W-q(S(f))$ is a disjoint union of smooth connected surfaces. Let $\mathscr{R}$ denote the set of all such component surfaces. Obviously each $R\in\mathscr{R}$ is immersed by $\bar{f}$ in $\boldsymbol{R}^2$ and so each $R\in\mathscr{R}$ has a natural orientation via $\bar{f}$. Furthermore $q^{-1}(R)$ is a circle bundle—a trivial one if M is oriented. Thus if M is oriented the circle fibres of $q^{-1}(R)$ can be coherently oriented in a way compatible with the orientations of M and R.

Suppose there were an h: $M\to\boldsymbol{R}^2$ such that (f, h) immersed M in $\boldsymbol{R}^4$. Then every circle fibre of $q^{-1}(R)$ for all $R\in\mathscr{R}$ would be immersed in $\boldsymbol{R}^2$ by h. In the oriented case, if we oriented the circles as described above, the rotation number of h would be the constant, $r_h(R)$, for all the q-fibres over R. If M is not oriented, define $r_h(R)$ to be the absolute value of the rotation number of h on any q-fibre over R with any orientation.

We make two preliminary remarks:

(a) If $R\in\mathscr{R}$ has the q-image of an arc of definite fold points on its boundary then $|r_h(R)| = 1$ for (f, h) any immersion. This is clear since Th is injective on ker Tf. At any fold point therefore, h embeds a planar neighborhood transverse to the fold curve. If the fold curve is a definite fold curve, this planar neighborhood may be taken saturated. Thus h embeds the circle fibres above R close enough to the definite fold curve and hence $|r_h(R)| = 1$.

(b) Let K be the Klein bottle obtained from $I\times S^1$ by identifying $0\times S^1$ with $1\times S^1$ with opposite orientations. Let i: $I\times S^1\to K$ be the identification map. Let h: $K\to\boldsymbol{R}^2$ be such that $h\circ i\,|\,t\times S^1$ is an immersion for all t. Then since the rotation number, $\operatorname{rot}(h\circ i\,|\,t\times S^1)$ is independent of t and

$$\operatorname{rot}(h\circ i\,|\,0\times S^1) = -\operatorname{rot}(h\circ i\,|\,1\times S^1),$$

we see that $\operatorname{rot}(h\circ i\,|\,t\times S^1) = 0$ for all t.

Example 1. Let ϕ: $\boldsymbol{R}^3\to\boldsymbol{R}^3$: $(x, y, z)\to(x, -y, z)$ and π: $\boldsymbol{R}^3\to\boldsymbol{R}$: $(x, y, z)\to x$. Let $S^2 = \{(x, y, z)\in\boldsymbol{R}^3\,|\,x^2+y^2+z^2 = 1\}$. The diffeomorphism ϕ restricts to an orientation reversing diffeomorphism that commutes with π which maps S^2 onto $[-1, 1]$. Let M be the identification space of $[0, 1]\times S^2$ obtained by identifying $0\times S^2$ with $1\times S^2$ using ϕ. M is an S^2 bundle over S^1. We let g map M into $S^1\times[-1, 1]$ by mapping each fibre S^2 by π onto $[-1, 1]$. Finally we let f: $M\to\boldsymbol{R}^2$ be the composition of g with any embedding of $S^1\times[-1, 1]$ in $\boldsymbol{R}^2$. This f is obviously stable and makes the following diagram commute.

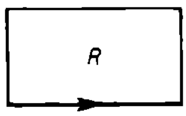
$$\begin{tikzcd}
I\times S^2 \arrow[r, "\bar{i}"] \arrow[d, "1\times\pi"'] & M \arrow[d, "g"] \arrow[dr, "f"] & \\
I\times[-1,1] \arrow[r, "i"] & S^1\times[-1,1] \arrow[r, hook] & R^2
\end{tikzcd}$$

Here $\bar{i}$ and i are the obvious identification maps. By construction $g^{-1}(S^1\times t) = K_t$ is a Klein bottle for any $|t|\neq 1$ and $S(f) = g^{-1}(S^1\times\{-1, 1\})$, two definite fold curves.

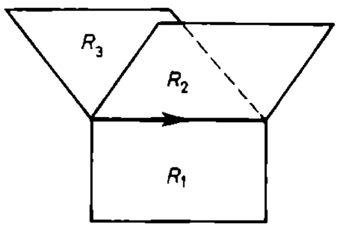
Suppose (f, h): $M \to \boldsymbol{R}^2 \times \boldsymbol{R}^2$ is an immersion, then h: $K_t \to \boldsymbol{R}^2$ immerses the circle f-fibre over each point of $S^1 \times t$, with rotation number zero by (b) above. On the other hand since $W = S^1 \times [-1, 1]$ and R consists of a single element namely, $S^1 \times (-1, 1)$, whose boundary consists of the images of the two definite fold curves, $|r_h(R)| = 1$ by (a) above. Contradiction. Thus no h exists which makes (f, h) an immersion.

EXAMPLE 2. In this example we do not describe M directly but describe W and $f(M)$. For this example we need the following easily proven statement: *Let M be orientable. If there is an h: $M \to \boldsymbol{R}^2$ making (f, h) an immersion, then $S(f)$ can be oriented so that the rotation number function $r = r_h$: $\mathscr{R} \to \boldsymbol{Z}$ satisfies*:

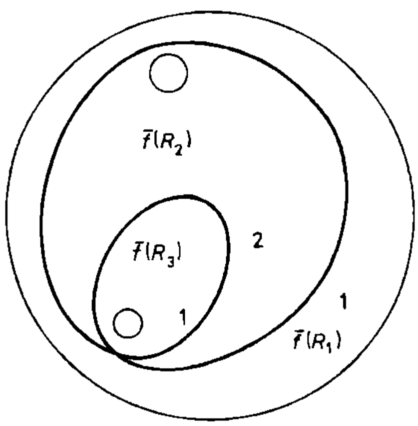
For a neighborhood of a point in the q-image of an arc of definite fold points, $r(R) = 1$ $(= -1)$, if the $\bar{f}$-image of R lies to the left (right) of the image of the oriented definite fold curve.



For a neighborhood of a point on the q-image of an arc of indefinite fold points, $r(R_2)+r(R_3)-r(R_1) = 1$ $(= -1)$, if the $\bar{f}$-images R_2 and R_3 lie to the left (right) of the $\bar{f}$-image of the oriented indefinite fold curve and the $\bar{f}$-image of R_1 lies to the right (left).



Now suppose that $f(M) \subseteq \boldsymbol{R}^2$ looks like:



The integers in the regions indicate the number of times W covers that region. We assume that the $\bar{f}$-pre-image of the doubly-covered region is connected. The single element $R_2 \in \mathscr{R}$ that double covers that region is an annulus whose boundary circles double-cover both boundary components of $\bar{f}(R_2)$. The embedded circles are all images of definite fold curves and the immersed circle is the image of an indefinite fold curve. The self intersection point of this immersed circle is the image of a point of type $(1 \cdot 2 \cdot 2 \cdot 1)$.

Suppose there were an h: $M \to \boldsymbol{R}^2$ such that (f, h) immerses M in $\boldsymbol{R}^4$. Take an orientation of $S(f)$ as described above, and let $r_h(R_i) = r_i$. By (a) since every region R_i has the image of a definite fold curve on its boundary, $|r_i| = 1$. Further by the unproven statement above,

$$2r_2 - r_1 = r_3 - 2r_2.$$

There is no solution satisfying both of these conditions.

References

[1] A. Haefliger, *Quelques remarques sur les applications d'une surface dans le plan*, Ann. Inst. Fourier (Grenoble) 10, (1960), 47–60.

[2] L. Kushner, H. Levine and P. Porto, *Mapping three manifolds into the plane I*, Bol. Soc. Mat. Mexicana (2) 29 (1984), 11–33.

[3] H. Levine, *Classifying Immersions into $\boldsymbol{R}^4$ over Stable Maps of 3-Manifolds into $\boldsymbol{R}^2$*, Lecture Notes in Math. 1157, Springer-Verlag, Berlin–New York 1985.

Presented to the semester
Singularities
15 February–15 June, 1985