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MATHEMATICAL MODELS OF PHENOMENOLOGICAL PIEZOELECTRICITY

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1. Introduction

In the present lecture we will consider some mathematical models of phenomenological piezoelectricity. We begin our survey with the model of W. Voigt [1], now ranked among the classical models, in which the quasi-static electric field is coupled with dynamic mechanical motion.

A further step in our considerations will be a departure from the quasi-static electric field; we will concern ourselves with a more general case, in which the dynamic electro-magnetic field and the temperature field are coupled with the deformation field ([2], [4]). Finally, we will examine the most general model, designed by R. D. Mindlin [3], in which the influence of the electric polarization gradient upon the electromechanical field is taken into account.

Certain crystals, such as quartz, tourmaline, Seignette salt, when subject to a stress, become electrically polarized (J. and P. Currie, 1880). This is the simple piezoelectric effect. Besides the simple piezoelectric effect there occurs an inverse effect, in which the electric potential produces a deformation. This effect was foreseen by Lippmann [5] in 1881 on thermodynamic grounds and confirmed by the Curie brothers [6] also in 1881.

The practical applications of piezoelectric effects are widely known; first of all in the generation of ultrasonic waves and also in the conversion of electromagnetic energy into mechanical energy and conversely, in prospecting solids with piezoelectric properties, etc. [7].

2. Electromagnetism

We begin our consideration from the electromagnetic foundation of the problem.

The Maxwell equations in the MKS system have the form [8]

$$\operatorname{curl}\mathbf{H} = \mathbf{I}+\dot{\mathbf{D}}, \tag{2.1}$$

$$\operatorname{curl}\mathbf{E} = -\dot{\mathbf{B}}, \tag{2.2}$$

where $\mathbf{H}$ is the vector of the magnetic field, $\mathbf{E}$ the vector of the electric field, $\mathbf{B}$ the vector of magnetic induction and $\mathbf{I}$ the vector of the electric conduction current. In a solid we have the following material relations for the field vectors $\mathbf{D}$, $\mathbf{B}$:

$$\mathbf{D} = \varepsilon_0\mathbf{E}+\mathbf{P}, \tag{2.3}$$

$$\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M}). \tag{2.4}$$

Here, $\mathbf{P}$ is the vector of electric polarization and $\mathbf{M}$ the magnetization vector; ε_0, μ_0 denote the electric and the magnetic permeabilities in a vacuum.

Equations (2.1) and (2.2) should be completed by the Gauss equation

$$\operatorname{div}\mathbf{D} = \varrho_e, \tag{2.5}$$

and an equation following from equation (2.2), namely

$$\operatorname{div}\mathbf{B} = 0. \tag{2.6}$$

Changing over from the Maxwell equations (2.1)–(2.6) to the quasi-static approximation is rather inconvenient. Much more useful will be the discussion of the Maxwell equations expressed by the potentials: the scalar electric potential φ, and the vectorial magnetic potential $\mathbf{A}$.

These equations have the following form [8]:

$$\operatorname{curl}\mathbf{H} = \mathbf{I}+\mathbf{D}, \tag{2.7}$$

$$\mathbf{B} = \operatorname{curl}\mathbf{A}, \tag{2.8}$$

$$\mathbf{E} = -\operatorname{grad}\varphi-\mathbf{A}, \tag{2.9}$$

$$\operatorname{div}\mathbf{D} = \varrho_e. \tag{2.10}$$

In virtue of (2.1) and (2.2) we obtain the Poynting theorem [8]

$$\frac{\partial}{\partial t}\int_V U_l\,dV = -\int_A \mathbf{n}\cdot\mathbf{h}\,dA-\int_V \mathbf{E}\cdot\mathbf{I}\,dV, \tag{2.11}$$

where

$$U_e = \mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}, \qquad \mathbf{h} = \mathbf{E}\times\mathbf{H}.$$

Equations (2.11), the mathematical consequence of the Maxwell equations, can be physically interpreted as the balance of electromagnetic energy. Thus the scalar $\mathbf{n}\cdot\mathbf{h}$ represents the flux of electromagnetic energy through the surface A of the body into the surrounding medium. The expression U_e

$= \mathbf{E}\cdot\dot{\mathbf{D}}+\mathbf{H}\cdot\dot{\mathbf{B}}$ is identified with the time increment of the electromagnetic energy. Finally, $\mathbf{E}\cdot\mathbf{I}$ represents Joule's heat.

The Poynting vector $\mathbf{h}$ can be written in terms of potentials φ and $\mathbf{A}$

$$\mathbf{h} = \mathbf{E}\times\mathbf{H} = \varphi(\mathbf{I}+\mathbf{D})-\mathbf{A}\times\mathbf{H}. \tag{2.12}$$

Consider now piezoelectric bodies (which are dielectrics). In general they are electrically neutral, contain the same amount of positive and negative charges and do not conduct electric curremt. The introduction of a dielectric into an electromagnetic field changes the latter. Consequently the vectors $\mathbf{E}$ and $\mathbf{D}$ are not parallel and differ by the polarization $\mathbf{P}$.

For piezoelectrics we introduce the same simplifications as for non-magnetizable dielectrics

$$\mathbf{I} = 0, \qquad \mathbf{M} = 0, \qquad \varrho_e = 0. \tag{2.13}$$

A further simplification consists in neglecting the magnetic therm $\mathbf{A}$ in (2.9). From the energy balance (2.11) we obtain

$$\frac{\partial}{\partial t}\int_V U_e\, dV = -\int_A \mathbf{n}\cdot\varphi\dot{\mathbf{D}}\, dA = \int_V \mathbf{E}\cdot\dot{\mathbf{D}}\, dV \tag{2.14}$$

and from the system of equations (2.7)–(2.10) and (2.3)

$$\mathbf{E} = -\operatorname{grad}\varphi, \qquad \operatorname{div}\mathbf{D} = 0, \qquad \mathbf{D} = \varepsilon_0\mathbf{E}+\mathbf{P}, \qquad \mathbf{A} = 0, \qquad \mathbf{B} = 0. \tag{2.15}$$

Introducing $(2.15)_3$ into $(2.15)_2$ we obtain the following equation for the function φ:

$$-\varepsilon_0 \nabla^2\varphi+\operatorname{div}\mathbf{P} = 0. \tag{2.16}$$

The simplification $\mathbf{A} \simeq 0$ or $|\mathbf{A}| \ll |\operatorname{grad}\varphi|$ is valid when the electromagnetic waves essentialy are uncoupled from the elastic waves and we are considering wavelengths near those of the elastic waves, which are much shorter than the electromagnetic wavelength of the same frequency.

The justification of the above (experimentally confirmed) simplification was presented in an interesting paper by H. F. Tiersten [9].

3. Energy balance. Constitutive and differential equations of piezoelectricity

Assume that the body under consideration undergoes a deformation due to an external loading electromagnetic field, which may vary with time. Assume also that there are no heat sources in the body and no heat conduction (i.e., that the process is adiabatic). Apply to an arbitrary region V of the body bounded by surface A the principle of energy conservation [10]

$$\frac{\partial}{\partial t}\int_V (\tfrac{1}{2}\varrho\dot{v}_i\dot{v}_i+U)\, dV = \int_V X_i\dot{v}_i\, dV+\int_A p_i\dot{v}_i\, dA+\int_V E_i\dot{D}_i\, dV. \tag{3.1}$$

Here ϱ is the density, $v_i = \dot{u}_i$ the time derivative of displacement, X_i the components of body forces, $p_i = \sigma_{ji} n_j$ the contact forces, and U the internal (mechanical and electromagnetic) energy.

The principle of energy conservation states that the time increment of the kinetic and internal energies is equal to the power of the externai forces and the electromagnetic energy.

Transforming equation (3.1) and bearing in mind the equation of motion

$$\sigma_{ji,j} + X_i = \varrho \dot{v}_i, \tag{3.2}$$

we obtain the local form of energy balance

$$H = \sigma_{ij} \dot{\varepsilon}_{ij} - \dot{E}_i D_i \tag{3.3}$$

where

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \qquad H = U - E_i D_i,$$

and H is the electric entalpy. It is evident that $H = H(\varepsilon_{ij}, E_i)$. Hence we obtain

$$\left(\sigma_{ij} - \frac{\partial H}{\partial \varepsilon_{ij}}\right)\dot{\varepsilon}_{ij} - \left(D_i + \frac{\partial H}{\partial E_i}\right)\dot{E}_i = 0. \tag{3.4}$$

This equation should hold for arbitrary values of $\dot{\varepsilon}_{ij}$, $\dot{E}_i$, whence

$$\sigma_{ij} = \frac{\partial H}{\partial \varepsilon_{ij}}, \qquad D_i = -\frac{\partial H}{\partial E_i}. \tag{3.5}$$

This relation will be employed in deriving the constitutive relations.

Let us expand the electric entalpy $H(\varepsilon_{ij}, E_i)$ into the MacLaurin series in the vicinity of the natural state ($\varepsilon_{ij} = 0$, $E_i = 0$), neglecting terms higher than of the second order. For a homogeneous anisotropic body we obtain the following expression:

$$H(\varepsilon_{ij}, E_i) = \tfrac{1}{2} c^E_{ijkl} \varepsilon_{ij} \varepsilon_{kl} - e_{kij} \varepsilon_{ij} E_k - \tfrac{1}{2} \epsilon^{\varepsilon}_{ij} E_i E_j. \tag{3.6}$$

Here c^E_{ijkl} is the elastic stiffnes for $E_i = \text{const}$, e_{kij} are the piezoelectric constants and ϵ_{ij} the dielectric permittivity (dielectric constant) for $\varepsilon_{kl} = \text{const}$.

From the thermodynamic consideration and symmetry of stress tensor σ_{ij} and strain tensor ε_{ij}, we have

$$c_{ijkl} = c_{klij}, \quad c_{ijkl} = c_{jikl}, \quad c_{ijkl} = c_{ijkl}, \quad e_{kij} = e_{kji}, \quad \epsilon_{ij} = \epsilon_{ji}. \tag{3.7}$$

In the most general case of triclinic crystal (without a centre of symmetry) we have 21 independent elastic constants, 18 independent piezoelectric constants and 6 independent dielectric constants.

We see from W. Voigt's theory that the piezoelectric effect can occur in materials which do not have a centre of symmetry. In bodies which do have a centre of symmetry the polar tensor e_{kij} vanishes.

For non-centrosymmetric crystals relations (3.5) lead to the constitutive equations

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} - e_{kij}E_k, \tag{3.8}$$

$$D_i = e_{ikl}\varepsilon_{kl} + \epsilon_{ik}E_k. \tag{3.9}$$

The above-mentioned relations present the material law and the mathematical model of quasi-static piezoelectricity in the classical Voigt theory. Stresses σ_{ij} and electric displacements D_i are linear functions of strains ε_{ij} and components of vector E_i.

For centrosymmetric crystals we have the following formulae:

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}, \qquad D_i = \epsilon_{ik}E_k. \tag{3.10}$$

Let us now introduce the constitutive relations (3.8), (3.9) into equations of motion and Gauss' law (div $\mathbf{D} = 0$). Accordingly, we arrive at a four-equation system in which the displacements u_i and the electric potential φ appear as unknown functions:

$$c_{ijkl}u_{k,lj} + e_{kij}\varphi_{,kj} + X_i = \varrho\ddot{u}_i, \tag{3.11}$$

$$e_{ikl}u_{k,li} - \epsilon_{ik}\varphi_{,ki} = 0. \tag{3.12}$$

In the case of centrosymmetric crystals equations (3.11), (3.12) will take the simple form

$$c_{ijkl}u_{kl,j} + X_i = \varrho\ddot{u}_i, \tag{3.13'}$$

$$\epsilon_{ik}\varphi_{,ki} = 0. \tag{3.13''}$$

Voigt's piezoelectricity theory at present can be said to be developed to the full. Not only the general theorems, such as variational theorems, the fundamental energetic theorem, the theorem on uniqueness of solutions, the theorem of reciprocity of work, the Hamilton principle have been derived [10], but also a series of particular problems, concerning mainly vibrations of thin plates and the propagation of Rayleigh and Love surface waves, have been deal with and worked out for various classes of crystals [11]–[15].

4. Coupling of elastic and electromagnetic waves

In the preceding considerations we dealt with the coupling between the quasistatic electric field and the motion of the elastic body. We now proceed to a more general problem, namely the dynamic elastic and dynamic electromagnetic problem of piezoelectricity.

Consider the complete set of Maxwell equations, assuming that for the piezoelectric medium we have $\varrho_e = 0$, $\mathbf{I} = 0$, $\mathbf{M} = 0$:

$$\operatorname{curl}\mathbf{H} = \dot{\mathbf{D}}, \tag{4.1}$$

$$\operatorname{curl}\mathbf{E} = -\dot{\mathbf{B}}, \tag{4.2}$$

$$\operatorname{div}\mathbf{B} = 0, \tag{4.3}$$

$$\operatorname{div}\mathbf{D} = 0. \tag{4.4}$$

This set of equations is completed by the material relations

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}, \tag{4.5}$$

$$\mathbf{B} = \mu_0\mathbf{H}. \tag{4.6}$$

Performing the operation of rotation over equation (4.2) and making use of equation (4.1) and relation (4.6), we arrive at the wave equation

$$\operatorname{curl}\operatorname{curl}\mathbf{E} = -\mu_0\ddot{\mathbf{D}}. \tag{4.7}$$

We can represent the components of the vector $\mathbf{D}$ by means of the constitutive relations for the quasistatic problem (Eq. (3.9)). We obtain

$$E_{i,jj} - E_{j,ji} = \mu_0(e_{ikl}\ddot{\varepsilon}_{kl} + \epsilon_{ik}\ddot{E}_k), \tag{4.8}$$

when

$$\varepsilon_{kl} = \tfrac{1}{2}(u_{k,l} + u_{l,k}).$$

In these three equations the unknown functions are u_i and E_i. The remaining three equations can be deduced from the equations of motion. Making use of equation (3.2) and (3.8), we obtain

$$c_{ijkl}u_{k,lj} - e_{kij}E_{k,j} + X_i = \varrho\ddot{u}_i. \tag{4.9}$$

Thus, we have derived six differential equations in which six functions appear: three components of displacement u_i and three components of the vector E_i.

J. Kayme [16] investigated the propagation of elastic plane waves and electromagnetic waves in ammonium dihydrogen phosphate (ADF). This crystal belongs to the tetragonal system (of class 42 m). It appeared that for this case the influence of the electromagnetic field on the velocity of propagation of an elastic wave is insignificant. The waves propagate with constant velocity and do not undergo dispersion. In another case investigated by J. Kayme [16], in which it was assumed that $\mathbf{I} \neq 0$, attenuated waves, undergoing dispersion, were obtained.

5. Thermopiezoelectricity

In the preceding considerations we assumed that the thermodynamic process is adiabatic. Now we discard this restriction. Thus, the heat flow through surface elements is denoted by $\mathbf{q}$ (referring to a unit area and a unit time.) In the interior of the body acts a heat source W (referring to unit volume of the body and unit time). Consequently there arises in the body a temperature increment θ, equal to the temperature difference $\theta = T - T_0$, where T is the absolute temperature and T_0 the temperature of the natural state, in which there are no strains or stresses.

We shall deal with the energy balance, taking into account the thermal terms [4].

$$\frac{\partial}{\partial t}\int_V (\tfrac{1}{2}\varrho v_i v_i + U)\, dV = \int_V (X_i v_i + W)\, dV + \int_A (p_i v_i - q_i n_i)\, dA, \tag{5.1}$$

and the Clausius-Duhem inequality

$$S - \left(\frac{q_i}{T}\right)_{,i} - \frac{W}{T} > 0. \tag{5.2}$$

The energy balance contains now the non-mechanical power, the flux of heat through the surface of the body and the energy generated by heat in the interior of the body. In the Clausius-Duhem inequality (5.2), S is the entropy with respect to unit volume and unit time.

From the transformation of (5.1) by introducing the free energy $F = U - ST$ and electric entalpy $H = F - D_i E_i$, we obtain the following relations:

$$\sigma_{ij} = \frac{\partial H}{\partial \varepsilon_{ij}}, \quad D_i = -\frac{\partial H}{\partial E_i}, \quad S = -\frac{\partial H}{\partial T}, \quad -\frac{q_i T_{,i}}{T} > 0. \tag{5.3}$$

The inequality $\Omega = -q_i T_{,i}/T > 0$ is satisfied by assuming that

$$q_i = -k_{ij} T_{,j}. \tag{5.4}$$

Equation (5.4) is the Fourier law for an anisotropic body. The quantity $\Omega > 0$ should be a positive definite quadratic form. Inequality (5.3) leads, in view Silvester's theorem, to restrictions on the symmetric coefficients of heat conductivity k_{ij}. Expanding the entalpy H into a Taylor series in the vinicity of the natural state

$$H = \tfrac{1}{2} c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - e_{kij}\varepsilon_{ij}E_k - \tfrac{1}{2}\epsilon_{ij}E_i E_j - \gamma_{ij}\varepsilon_{ij}\theta - g_i E_i \theta - \frac{c_\varepsilon}{2T_0}\theta^2, \tag{5.5}$$

and making use of relations (5.3), we arrive at the constitutive equations

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} - \gamma_{ij}\theta \quad - e_{kij}E_k, \tag{5.6}$$

$$S = \gamma_{ij}\varepsilon_{ij} + \frac{c_\varepsilon}{T_0}\theta \quad + g_i E_i, \tag{5.7}$$

$$D_i = e_{ikl}\varepsilon_{kl} + g_i\theta \quad + \epsilon_{ik}E_k. \tag{5.8}$$

Equations (5.6), (5.7), (5.8) are the Duhamel–Neumann equations generalized to piezoelectricity; the second of them is an expression for entropy and the last is an expression for electric displacement. This is the mathematical model for linear thermopiezoelectricity. We have here ten new constants: γ_{ij}, g_i, c_ε. The constant c_ε has the meaning of the specific heat at constant strain ε_{kl}.

Introduce now the constitutive relations (5.6)–(5.8) into the equations of motion and the Gauss equation

$$c_{ijkl}u_{k,lj}+e_{kij}\varphi_{,kj}-\gamma_{ij}\theta_{,j}+X_i=\varrho\ddot{u}_i, \tag{5.9}$$

$$e_{ikl}u_{k,li}-\epsilon_{ik}\varphi_{,ki}+g_i\theta_{,i}=0. \tag{5.10}$$

The set of equations will be complete if the equation of heat conduction is supplemented. This equation is derived on the basis of the entropy balance

$$T\dot{S}=-q_{i,i}+W. \tag{5.11}$$

Taking into account the constitutive relation (5.8), the Fourier law (5.4) and assuming that $|\theta/T_0|\ll 1$, we arrive at the linear heat conduction equation

$$k_{ij}\theta_{,ij}-c_\varepsilon\dot{\theta}-T_0(\gamma_{ij}\dot{\varepsilon}_{ij}-g_i\dot{\varphi}_{,i})=-W. \tag{5.12}$$

Equations (5.9) (5.10) and (5.12) constitute the complete set of equations of thermopiezoelectricity.

For the centrosymmetric crystals we obtain only the coupling between the displacements and temperature

$$c_{ijkl}u_{k,lj}-\gamma_{ij}\theta_{,j}+X_i=\varrho\ddot{u}_i, \tag{5.13}$$

$$k_{ij}\theta_{,ij}-c_\varepsilon\dot{\theta}-T_0\gamma_{ij}\dot{\varepsilon}_{ij}=-W, \tag{5.14}$$

$$\epsilon_{ik}\varphi_{,ik}=0. \tag{5.15}$$

Within the domain of thermopiezoelectricity we can obtain a series of general theorems on solutions, principle of virtual work, [17], the Hamilton principle [18], the reciprocity theorem of work [19]. So far only a few particular problems, related to the propagation of Rayleigh waves have been solved [20].

The above considerations can be generalized to coupled dynamical problems of mechanical motion and electromagnetic dynamical field. In this case we are faced with a system of seven equations

$$E_{i,jj}-E_{j,ji}=\mu_0\ddot{D}_i, \tag{5.16}$$

$$\sigma_{ji,j}+X_i=\varrho\ddot{u}_i, \tag{5.17}$$

$$k_{ij}\theta_{,ij}-c_\varepsilon\dot{\theta}-T_0(\gamma_{ij}\dot{\varepsilon}_{ij}+g_i\dot{E}_i)=-W. \tag{5.18}$$

In these equations we introduce the constitutive relations (5.6), (5.8). Observe that in view of the heat coupling all waves are attenuated and idspersed.

6. Polarization gradient theory of piezoelectrics

In the derivation of the classical phenomenological theory, piezoelectricity is expressed as an interaction between strain and electric displacement or electric polarization.

The equations of classical piezoelectricity are based on the assumption that the internal energy is a function of only strain and polarization (or only strain and electric displacement).

R. D. Mindlin [3] has introduced the polarization gradient into the internal energy and examined the consequences of considering an additional linear electromechanic effect. The resulting mathematical theory has interesting novel properties. The generalized theory accomodates the mathematical representation of surface energy of deformation and polarization (measured in the laboratory).

It can account for an apparent anomaly observed in measurements of the electrical capacitance of thin dielectric films ([21], [22]). The additional electromechanical effects are not confined to non-centrosymmetric materials.

Let us decompose the internal energy U of the dielectric into the energy U^L related to the deformation of the body, the polarization and polarization gradient and the energy related to the electric field φ.

$$U = U^L(\varepsilon_{ij}, P_i, P_{j,i}) + \tfrac{1}{2}\epsilon_0 \varphi_{,i}\varphi_{,i}. \tag{6.1}$$

Introducing the entalpy $H = U - E_i D_i$, we obtain from (6.1)

$$H = U^L(\varepsilon_{ij}, P_i, P_{j,i}) - \tfrac{1}{2}\epsilon_0 \varphi_{,i}\varphi_{,i} + \varphi_{,i} P_i. \tag{6.2}$$

Consider a body V bounded by surface A separating the body from the vacuum V'. Toupin's form ([23], [24]) of the Hamilton principle is the following one:

$$\delta \int_{t_1}^{t_2} dt \int_{V^*} (K - H)\, dV + \int_{t_1}^{t_2} dt \Big[\int_V (X_i \delta u_i + E_i^0 \delta P_i)\, dV + \int_A p_i \delta u_i\, dA\Big] = 0. \tag{6.3}$$

Here $V^* = V + V'$ and E_i^0 is the external electric field; K is the kinetic energy. We have

$$\delta H = \frac{\partial U^L}{\partial \varepsilon_{ij}}\delta\varepsilon_{ij} + \frac{\partial U^L}{\partial P_i}\delta P_i + \frac{\partial U^L}{\partial P_{j,i}}\delta P_{j,i} + P_i\delta\varphi_{,i} + \varphi_{,i}\delta P_i - \epsilon_0 \varphi_{,i}\delta\varphi_{,i}. \tag{6.4}$$

We define the stress σ_{ij}, the local electric force E_i' and the tensor E_{ij} by the formulae

$$\sigma_{ij} = \frac{\partial U^L}{\partial \varepsilon_{ij}}, \qquad E_i^L = -\frac{\partial U^L}{\partial P_i}, \qquad E_{ij} = \frac{\partial U^L}{\partial P_{j,i}}. \tag{6.5}$$

Introducing (6.4) and (6.5) into the Hamilton principle (6.3), after simple transformation we arrive at the equation

$$(6.6)\quad \int_{t_1}^{t_2} dt \int_V [(\sigma_{ji,j}+X_i-\varrho\ddot{u}_i)\delta u_i+(E_{ji,j}+E_i^L-\varphi_{,i}+E_i^0)\delta P_i+ \\ +(-\epsilon_0\varphi_{,ii}+P_{i,i})\delta\varphi]\,dV-\int_{t_1}^{t_2} dt\int_{V'}\epsilon_0\varphi_{,ii}\delta\varphi\, dV+ \\ +\int_{t_1}^{t_2} dt\int_A [(p_i-\sigma_{ji}n_j)\delta u_i-E_{ji}n_j\delta P_i+n_i(\epsilon_0|\varphi_{,i}|-P_i)\delta\varphi]\,dA=0.$$

In view of the arbitrariness of the virtual increments we obtain Euler's equations

$$(6.7)\quad \sigma_{ji,j}+X_i=\varrho\ddot{u}_i,$$

$$(6.8)\quad E_{ji,j}+E_i^L-\varphi_{,i}+E_i^0=0,$$

$$(6.9)\quad -\epsilon_0\varphi_{,ii}+P_{i,i}=0 \quad \text{on} \quad V,$$

and

$$(6.9'')\quad \varphi_{,ii}=0 \quad \text{on} \quad V',$$

These equations should be completed by the natural boundary conditions following from (6.6):

$$(6.10)\quad \sigma_{ji}n_j=p_i,$$

$$(6.11)\quad E_{ji}n_j=0,$$

$$(6.12)\quad (-\epsilon_0|\varphi_{,i}|+P_i)n_i=0.$$

Besides condition (6.10) for traction we may assume the displacement condition (where $\delta u_i=0$). Similarly, besides $E_{ji}n_j=0$ we may take a condition for the polarization P_i. Finally, besides condition (6.12) prescribing the charge on the surface we may prescribe the potential φ.

Let us take the energy $U^L(P_i, P_{j,i}, \varepsilon_{ij})$ in the form

$$(6.13)\quad U^L=b_{ij}^0P_{j,i}+\tfrac{1}{2}a_{ij}^{\varepsilon,G}P_iP_j+\tfrac{1}{2}b_{ijkl}^{\varepsilon,P}P_{j,i}P_{l,k}+ \\ +\tfrac{1}{2}c_{ijkl}^{P,G}\varepsilon_{ij}\varepsilon_{kl}+d_{ijkl}^{P}P_{j,i}\varepsilon_{kl}+f_{ijkl}^{G}P_i\varepsilon_{jk}+j_{ijk}P_iP_{k,j}.$$

Here the indices P, G, ε denote a fixed polarization, the polarization gradient and the strain. In view of the constitutive relations (6.5) we obtain the constitutive equations

$$(6.14)\quad \sigma_{ij}=c_{ijkl}\varepsilon_{kl}+f_{kij}P_k \;\; \vert \;\; +d_{klij}P_{l,k},$$

$$(6.15)\quad -E_j^L=f_{jkl}\varepsilon_{kl}+a_{jk}P_k \;\; \vert \;\; +j_{jkl}P_{l,k},$$

$$(6.16)\quad E_{ij}=d_{ijkl}\varepsilon_{kl}+j_{kij}P_k \quad +b_{ijkl}P_{l,k}+b_{ij}^0\delta_{ij}.$$

Introducing relations (6.14)–(6.16) into equations (6.7)–(6.9) we arrive at a system of seven differential equations with the following unknows: the polarization P_i, the displacement u_i and the electric potential φ. Observe that the introduction of the polarization gradient does not raise the order of the differential equations.

It is noteworthy that electromechanical coupling appear also in a body with central symmetry. Although in this particular case we have $f_{ijk} = 0$, $j_{ijk} = 0$ (since odd tensors do not appear in bodies with central symmetry), the constants d_{ijkl} do not vanish. It follows from (6.14), (6.16) that these constants play the role of coupling between the mechanical and the electric fields.

The tensor E_{ij} is asymmetric. For $\varepsilon_{ij} = 0$, $P_i = 0$, $P_{i,j} = 0$ the stresses σ_{ij} and the local electric field E_i^L vanish. But in this case the tensor E_{ij} does not vanish and is equal to $b_{ij}^0 = \text{const}$. It is evident that the magnitude b_{ij}^0 will be introduced in the solution of the set of differential equations through the boundary conditions. In consequence we obtain a residual state of strains and stresses in the medium.

In this case of the isotropic body the constitutive equations will take the form

$$(6.17)\qquad \sigma_{ij} = c_{12} u_{k,k}\delta_{ij} + c_{44}(u_{i,j}+u_{j,i}) + d_{12} P_{k,k}\delta_{ij} + d_{44}(P_{i,j}+P_{j,i}),$$

$$(6.18)\qquad E_{ij} = d_{12} u_{k,k}\delta_{ij} + d_{44}(u_{i,j}+u_{j,i}) + b_{12} P_{k,k}\delta_{ij} + b_{44}(P_{i,j}+P_{j,i}) + \\ + b_{77}(P_{j,i}-P_{i,j}) + b_{ij}^0\delta_{ij},$$

$$(6.19)\qquad E_i^L = -aP_i.$$

Substitution of relations (6.17)–(6.10) into the differential equations of the problem leads to the three-coupled equations system

$$(6.20)\quad c_{44}\nabla^2\mathbf{u} + (c_{12}+c_{44})\nabla\nabla\cdot\mathbf{u} + d_{44}\nabla^2\mathbf{P} + (d_{12}+d_{44})\nabla\nabla\cdot\mathbf{P} + \mathbf{X} = \varrho\ddot{\mathbf{u}},$$

$$(6.21)\quad d_{44}\nabla^2\mathbf{u} + (d_{12}+d_{44})\nabla\nabla\cdot\mathbf{u} + (b_{44}+b_{77})\nabla^2\mathbf{P} + (b_{12}+b_{44}-b_{77})\nabla\nabla\cdot\mathbf{P} - \\ -a\mathbf{P} - \nabla\varphi + \mathbf{E}^0 = 0,$$

$$(6.22)\qquad \nabla\cdot P - \epsilon_0\nabla^2\varphi = \varrho_e \quad \text{for} \quad \mathbf{x}\in V,$$

$$(6.23)\qquad \nabla^2\varphi = 0 \quad \text{for} \quad \mathbf{x}\in V'.$$

Into the differential equations we introduce in addition the boundary conditions

$$(6.24)\qquad \sigma_{ij}n_j = p_i, \quad E_{ji}n_j = 0, \quad [-\epsilon_0|\varphi_{,i}| + P_i]n_i = 0 \quad \text{on} \quad A.$$

Let us return to system (6.20)–(6.23). It is a system of coupled equations, the system being very complex and very difficult to solve in this form. It is evident that the functioning of the sources: body forces **X**, external electric field **E**, and electric charges ϱ_e, produces in the body a state of strain, attended by polarization and electric potential as well. The above mentioned fields are also induced by boundary conditions and initial conditions.

The gradient theory developed and flourished particularly in the period of the last ten years. Especially the static problems engaged a good deal of attention in the investigations. Green functions of elastostatics were obtained; stress functions were designed, the reaction of electric charge in elastic half-space was examined. Much attention was devoted to the problem of surface energy ([25]–[29]).

Within the range of dynamics we should notice some works concerning the propagation of plane wave and Rayleigh and Love's surface waves ([30], [31]).

In recent yers, the fundamentals of thermo-piezoelectricity, both stationary and dynamic, have been worked out ([33]–[37]).

7. Coupling of elastic and electromagnetic waves in the gradient theory of dielectrics

In our considerations referring to the gradient theory of dielectrics we understood the electromagnetic field as the quasi-static electric field. At present, we give up this assumption, and we will consider the full system of Maxwell's equations for dielectrics (assuming $\mathbf{I} = 0$, $\mathbf{M} = 0$):

$$\operatorname{curl}\mathbf{H} = \dot{\mathbf{D}}, \tag{7.1}$$

$$\mathbf{B} = \operatorname{curl}\mathbf{A}, \tag{7.2}$$

$$\mathbf{E} = -\operatorname{grad}\varphi - \dot{\mathbf{A}}, \tag{7.3}$$

$$\operatorname{div}\mathbf{D} = \varrho_e. \tag{7.4}$$

Equations (7.1), (7.2) together with the relations:

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}, \qquad \mathbf{B} = \mu_0\mathbf{H}, \tag{7.5}$$

are the point of departure for further considerations. Carrying out an elimination, we obtain the wave equations:

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right)\varphi - \epsilon_0^{-1}\operatorname{div}\mathbf{P} + \epsilon_0^{-1}\varrho_e = 0, \tag{7.6}$$

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right)\mathbf{A} + c^{-2}\epsilon_0^{-1}\dot{\mathbf{P}} = 0, \quad \text{for } \mathbf{x} \in V. \tag{7.7}$$

Here c is the speed of light.

Let us notice that the balance equilibrium equation of intermolecular forces

$$E_{ji,j}+E_i^L-\varphi_{,i}+E_i^0=0 \tag{7.8}$$

is coupled with equation (7.3), owing to the relation $E_i=-\varphi_{,i}-\dot{A}_i$.

If we have to do with an isotropic body, we will obtain the following system of equations

$$c_{44}\nabla^2\mathbf{u}+(c_{12}+c_{44})\nabla\nabla\cdot\mathbf{u}+d_{44}\nabla^2\mathbf{P}+(d_{12}+d_{44})\nabla\nabla\cdot\mathbf{P}+\mathbf{X}=\varrho\ddot{\mathbf{u}}, \tag{7.9}$$

$$\begin{aligned}d_{44}\nabla^2\mathbf{u}+(d_{12}+d_{44})\nabla\nabla\cdot\mathbf{u}+(b_{44}+b_{77})\nabla^2\mathbf{P}+(b_{12}+b_{44}-b_{77})\nabla\nabla\cdot\mathbf{P}+\\+a\mathbf{P}-\nabla\varphi-\dot{\mathbf{A}}+\mathbf{E}^0=0,\end{aligned} \tag{7.10}$$

$$\left(\nabla^2-\frac{1}{c^2}\partial_t^2\right)\varphi-\epsilon_0^{-1}\nabla\cdot\mathbf{P}+\epsilon_0^{-1}\varrho_e=0, \tag{7.11}$$

$$\left(\nabla^2-\frac{1}{c^2}\partial_t^2\right)\mathbf{A}-\epsilon_0^{-1}c^{-2}\dot{\mathbf{P}}=0. \tag{7.12}$$

The above differential equations were given by R. D. Mindlin [32]. Moreover we can generalize our mathematical model to thermopiezoelectricity. Hence we obtain the following system of equations:

$$c_{44}\nabla^2\mathbf{u}+(c_{12}+c_{44})\nabla\nabla\cdot\mathbf{u}+d_{44}\nabla^2\mathbf{P}+(d_{12}+d_{44})\nabla\nabla\cdot\mathbf{P}+\mathbf{X}=\varrho\ddot{\mathbf{u}}+\gamma\nabla\theta, \tag{7.13}$$

$$\begin{aligned}d_{44}\nabla^2\mathbf{u}+(d_{12}+d_{44})\nabla\nabla\cdot\mathbf{u}+(b_{44}+b_{77})\nabla^2\mathbf{P}+(b_{12}+b_{44}-b_{77})\nabla\nabla\cdot\mathbf{P}-\\-a\mathbf{P}-\nabla\varphi+\mathbf{E}^0=\eta\nabla\theta,\end{aligned} \tag{7.14}$$

$$\left(\nabla^2-\frac{1}{c^2}\partial_t^2\right)\varphi-\epsilon_0^{-1}\nabla\cdot\mathbf{P}+\epsilon_0^{-1}\varrho_e=0, \tag{7.15}$$

$$\left(\nabla^2-\frac{1}{c^2}\partial_t^2\right)\mathbf{A}-\epsilon_0^{-1}c^{-2}\dot{\mathbf{P}}=0, \tag{7.16}$$

$$k\nabla^2\theta-c_\varepsilon\dot{\theta}-T_0(\gamma\dot{u}_{k,k}+\eta\dot{P}_{k,k})=-W. \tag{7.17}$$

In the present report we have revised the mathematical models of piezoelectricity and thermopiezoelectricity from the simplest Voigt model up to the very complex model of the dynamic thermopiezoelectricity. Together with generalization of the model arises the expansion of mathematical difficulties. Simultaneously, the developed (and intricate) models explain a series of anomalies and allow us to discover new phenomena. The investigation of coupled fields leads to the creation of new interdisciplinary branches of science, since there arise new phenomena and new effects. We encounter here a collaboration of scientist representing various fields, namely mechanics, acoustics, thermodynamics and electrodynamics.

The development of the theory of coupled fields is a characteristic trend in the modern mechanics of solids.

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