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# CURVES AND SURFACES IN REAL PROJECTIVE SPACES: AN APPROACH TO GENERIC PROJECTIONS

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## 0. Contents of the paper

Our purpose is to give an introduction, with examples and an intuitive discussion of them, to the problems that arise in the study of surfaces via the apparent contour of their generic projections. The exposition of these problems is preceded by a presentation of some very general properties of a class of curves that includes the contours of generic mappings from surfaces to Euclidean or projective planes (it is the class of "curves with cusps and double points", as they are formally defined in the opening part of the paper).

In Section 1 we recall some results present in the literature, concerning curves in the Euclidean plane. It seems natural to try to generalize these results in two main directions.

The first direction refers to curves in a real projective plane and is quickly sketched in Section 2.

The second is about apparent contours of surfaces, and it is the main subject of this paper, developed in the last two sections. Of these, Section 3 is mainly a survey of known facts about mappings between orientable surfaces. In Section 4, we turn to a new situation: we investigate the case in which the target manifold is a projective plane and present some results that regard the geometry of surfaces embedded in $\boldsymbol{RP}^n$.

I would like to express my gratitude to Professor Levine, Professor Martinet, Professor Siersma and Professor Trotman for the useful conversations we had during the Semester. Finally, I thank Professor Lazzeri for raising my interest into this subject.

## 1. Curves in $R^2$

Let us consider a particularly simple object: a curve in the Euclidean plane $R^2$.

More precisely, we define a "curve with cusps and double points" to be a compact connected subset $\gamma \subset R^2$, characterized in this way: $\forall x \in \gamma\ \exists U$ open in $R^2$, with $x \in U$, and a diffeomorphism $\varphi$: $U \to R^2$, such that $\varphi(U \cap \gamma)$ is

1) either the set $\{(x, y) \in R^2 \mid y = 0\}$;
2) or the set $\{(x, y) \in R^2 \mid x \cdot y = 0\}$;
3) or the set $\{(x, y) \in R^2 \mid x^2 - y^3 = 0\}$.

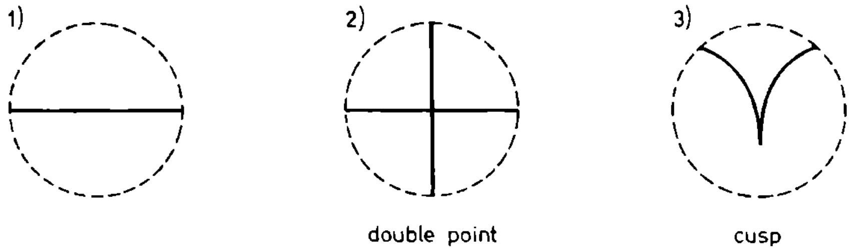



Fig. 1

In the complex algebraic case (curves in $CP^2$) similar objects give rise to a well developed theory (Plücker formulas, etc.). We pose to ourselves the following question: are there results of the same kind in the real case, concerning curves with cusps and double points?

We shall start by supposing that the curve is "irreducible", that is, it can be parametrized by one smooth ($\mathscr{C}^\infty$) map $\varphi$: $S^1 \to R^2$, which is singular only when the image is a cusp point. If $\gamma$ is an irreducible curve in $R^2$, with cusps and double points, it is known that the following relation holds:

$$(*) \qquad f + 2n + 2c = 2(d^+ - d^- + d_s^+ - d_s^-).$$

Here, $f$ is the number of inflection points of $\gamma$, $n$ that of double points, and $c$ enumerates the cusps; $d^+$ represents the number of positive double tangents (see Fig. 2), $d^-$ counts the negative double tangents, $d_s^+$ (respectively $d_s^-$) the lines of positive (respectively negative) type passing through two cusps, or tangent to $\gamma$ and passing through one cusp.

Formula $(*)$ is due to Fabricius-Bjerre: it was initially given for immersions of the circle in the plane ([11]) and successively for the case with cusps ([12]). The later paper deals in addition with curves presenting "beaks" or "cusps of the second kind". If we call $b$ the number of such singularities, relation $(*)$ becomes

$$f + b + 2n + 2c = 2d^+ - 2d^- + 2d_s^+ - 2d_s^-$$

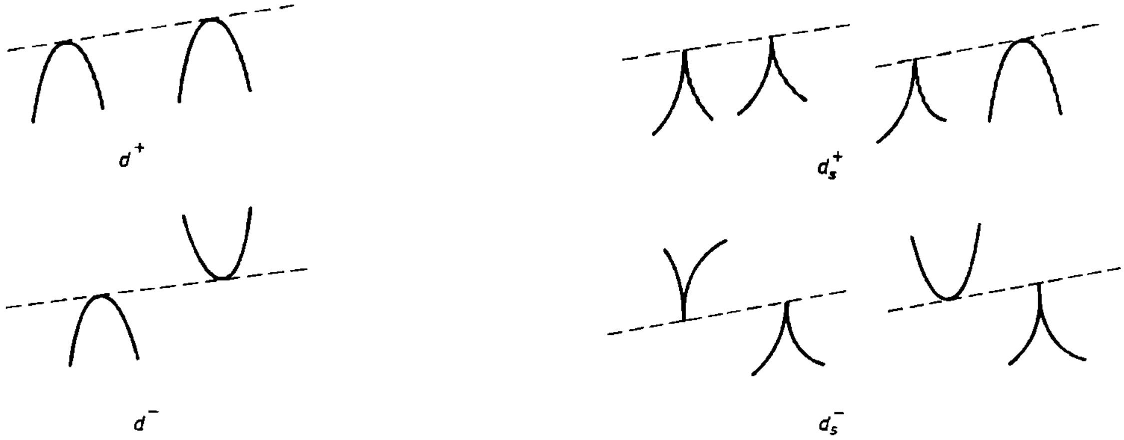



Fig. 2

where $d_s^+$, $d_s^-$ now include also tangents that pass through a beak, etc. These facts were discovered again by Halpern [15], who proved as well that (*) continues to hold for curves with any finite number of irreducible components. Further studies on the subject were undertaken by Banchoff [4], [5].

Relation (*) holds under the following hypothesis on $\gamma$:

(a) there are a finite number of inflection points (and of double tangents) so that all terms in the equality are well defined;

(b) there are no triple tangencies of any kind (Fig. 3);

(c) no tangent to an inflection point or to a cusp is a double tangent.

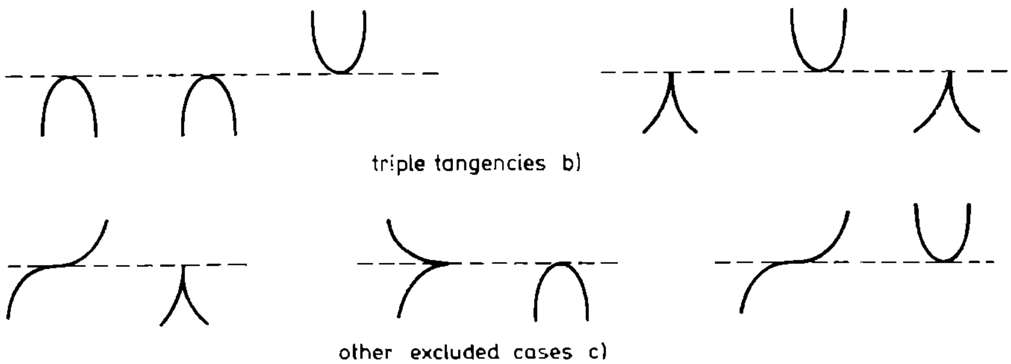



Fig. 3

These conditions will be verified by almost any curve with cusps and double points. Let $\mathscr{C}$ be the class of the "good" curves: given $\gamma \in \mathscr{C}$, one can find a neighbourhood $W$ of the identity in the space of all diffeomorphisms from $\boldsymbol{R}^2$ to $\boldsymbol{R}^2$, such that $\forall \varphi \in W\ \varphi(\gamma) \in \mathscr{C}$. Furthermore, given any curve $\gamma$ with cusps and double points, $\forall$ neighbourhood $W$ of the identity in the same space, $\exists\, \varphi \in W$ such that $\varphi(\gamma) \in \mathscr{C}$ (in the most elementary situation, when $\gamma$ is the image of an *embedding* $\varphi$: $S^1 \to \boldsymbol{R}^2$, and so has no cusps or double points, the generic character of conditions (a), (b), (c) can be proved in a nice simple way by studying $\gamma$ and its dual ([7], [8])).

The case where $\gamma$ is the image of an *immersion* $\varphi\colon S^1 \to \boldsymbol{R}^2$ has been studied in detail by Ozawa [27]. We have seen that the formula of Fabricius-Bjerre provides a necessary condition on the numbers of flexes $f$, of double points $n$, of double tangents $d^+$, $d^-$ presented by a curve $\gamma$ such that $\gamma = \operatorname{Im}\varphi$, with $\varphi$ a smooth immersion of the circle in the Euclidean plane. On the other hand, if we take any 4 nonnegative integers $f$, $n$, $d^+$, $d^-$ satisfying the relation

$$f+2n = 2d^+ - 2d^-$$

(which implies $f$ even), does this mean that there is at least one curve $\gamma$ displaying exactly these numbers of "singularities"?

If the curve $\gamma$ presents inflections (hence, at least two of them) the question was answered positively by Halpern [16]. However, when $f = 0$ the condition $n = d^+ - d^-$ is no more sufficient in order to find a curve presenting such values for $n$, $d^+$, $d^-$: for example, if $\gamma$ has no self-intersections, there can be no double tangents, and so the four values $f = 0$, $n = 0$, $d^+ = 1$, $d^- = 1$ satisfy the preceding relation but cannot be realized by a plane curve.

Halpern showed that, in this case, another requirement is sufficient: namely that, besides being $n = d^+ - d^-$, $d^-$ is even and satisfies $d^- \leqslant n^2 - n$. Ozawa's paper establishes that this condition must be verified by any curve without inflections, that is, it is also a necessary condition for such curves. Hence we have

THEOREM (Fabricius-Bjerre, Halpern, Ozawa). *The necessary and sufficient conditions for* 4 *nonnegative integers* $d^+$, $d^-$, $n$, $f$ ($f$ *even*) *to be attained by a curve* $\gamma$, $\gamma = \operatorname{Im}\varphi$ *with* $\varphi\colon S^1 \to \boldsymbol{R}^2$ *an immersion, as the numbers of its positive and negative double tangents, double points and inflections, are the following*:

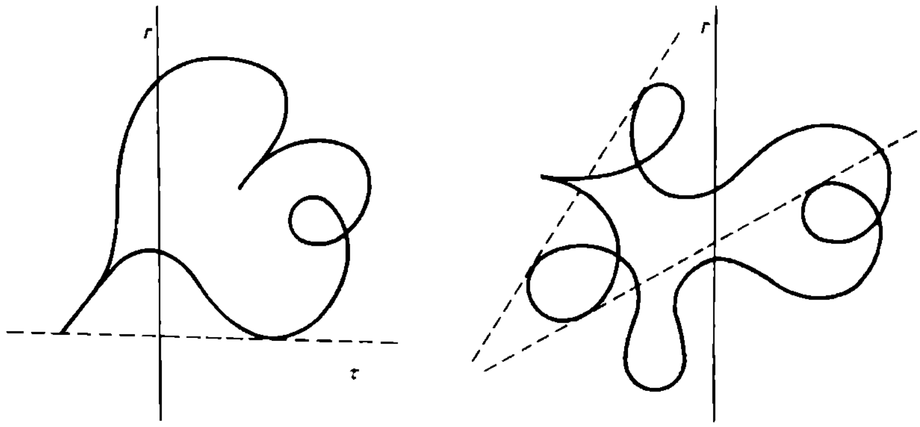
1) $f+2n = 2d^+ - 2d^-$;
2) *if* $f = 0$, $d^-$ *is even and* $d^- \leqslant n^2 - n$.

## 2. Curves in $\boldsymbol{RP}^2$

The first possible generalization of the results that we have just recalled concerns the case in which the curve $\gamma$ lies in a real projective plane, $\gamma \subset \boldsymbol{RP}^2$.

The definition of "curve with cusps and double points" is trivially extended to the new setting. But, of course, the definition of positive and negative double tangents does not make any sense in the present situation. In fact, when $\tau$ is a double tangent for $\gamma \subset \boldsymbol{RP}^2$, since $\{\boldsymbol{RP}^2 \setminus \tau\}$ is connected we cannot distinguish the positive from the negative case as we did before, when we checked whether $\gamma$ belonged, in a neighbourhood of the two tangency points, to the same or to different components of $\{\boldsymbol{R}^2 \backslash \tau\}$.

However, it is still possible to define positive and negative double tangents if we fix a line $r \subset \boldsymbol{RP}^2$ in a generic way ($r$ will cut $\gamma$ transversally outside of its singular points, inflection points, etc.). $\{\boldsymbol{RP}^2 \backslash r\}$ is now an affine plane and we have a natural notion of positive or negative double tangency, *with respect to* $r$ (Fig. 4).

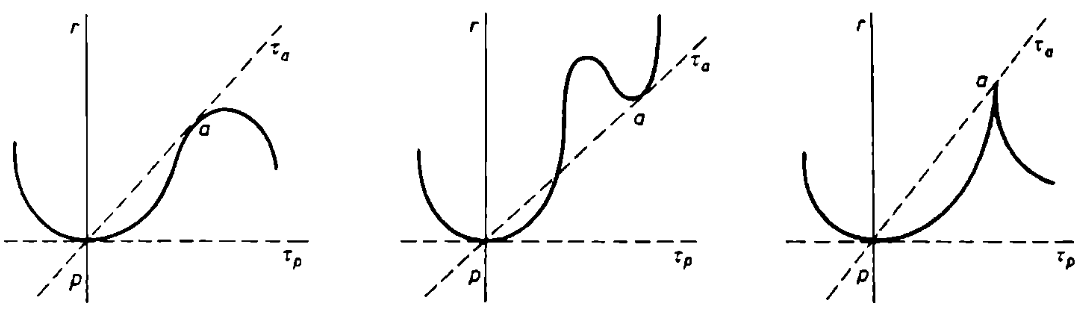



negative tangency (a)

positive tangencies (b)

Fig. 4

We can define: $m_r$ = number of intersection points of $\gamma$ and $r$; $f$, $n$, $c$ as before; $d_r^+$, $d_r^-$, $d_{rs}^+$, $d_{rs}^-$ as before but with respect to $r$. Furthermore, we need to take into account the following numbers: $d_{r\bullet}^+$, $d_{r\bullet}^-$: they represent the numbers of "positive" (respectively "negative") tangents of $\gamma$ passing through an intersection point $p$ of $\gamma$ and $r$; the negative case is that in which $\gamma$, in the neighbourhood of a tangency point $a \in \gamma$, belongs to the same connected domain of $\boldsymbol{RP}^2 \backslash \{\tau_a \cup \tau_p\}$ to which belongs $r$ – if this does not happen, we have the positive case (Fig. 5); $d_{rs\bullet}^+$, $d_{rs\bullet}^-$: here we count the lines which pass through an intersection point of $r$ and $\gamma$ and also through a cusp of $\gamma$; positive, and negative cases are distinguished as in the preceding situation.




$d_{r\bullet}^+$ $\qquad$ $d_{r\bullet}^-$ $\qquad$ $d_{rs\bullet}^+$

Fig. 5

Finally, we set

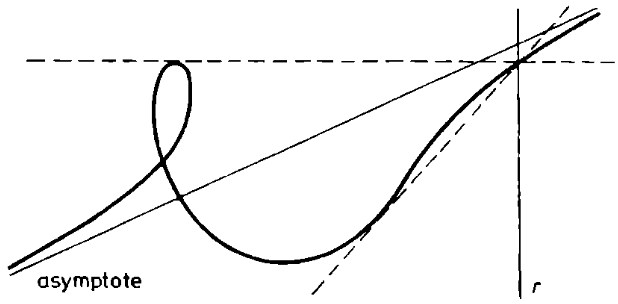
$$D_r = d_r^+ - d_r^- + d_{r*}^+ - d_{r*}^- + d_{rs}^+ - d_{rs}^- + d_{rs*}^+ - d_{rs*}^- .$$

We see that, under reasonable (and "generic") regularity assumptions patterned on those of 1), *the following relation holds* ([24], [25, Th.1]):

$$(**) \qquad f + 2n + 2c = m_r^2 - 2m_r + 2D_r .$$

As in the Euclidean case, when beaks are present one replaces $f$ by $f+b$ (number of flexes and of beaks), introducing obvious adjustments in the definitions of $d_{rs}^+$, $d_{rs}^-$, $d_{rs*}^+$, $d_{rs*}^-$.

In the projective plane, $f$ need not be even (Fig. 6).




$f=1, \quad n=1, \quad m_r=1, \quad d_{r*}^+=2$

Fig. 6

In fact, since $f$ and $m_r$ have, by the (**), the same parity, $f$ is even $\Leftrightarrow$ $[\gamma] = 0$ (where $[\gamma]$ denotes the homology class induced by $\gamma$ in $H_1(\boldsymbol{RP}^2; \boldsymbol{Z}_2)$).

One can observe that (**) actually applies to a larger class of curves than the one we have described ([25, Remarks 1 and 4]). Furthermore, by duality arguments, one can show ([24], [25, Th.2]) that *curves with cusps and double points satisfy also the relation*

$$c + 2d + 2f = \alpha_p^2 - 2\alpha_p + 2N_p$$

which relies on the choice of a generic point $p \subset \boldsymbol{RP}^2$: $\alpha_p$ is the number of tangents which can be drawn from $p$ to $\gamma$, $d$ is the total number of double tangents of $\gamma$, and $N_p$ takes into account all the geometrical configurations ("positive" and "negative" double points, etc.) which are dual to those represented by $D_r$ in (**).

## 3. Apparent contours of surfaces

The second direction in which the ideas of Section 1 can be further developed consists in taking into account the case of curves, in $\boldsymbol{R}^2$ and $\boldsymbol{RP}^2$, which arise as components of apparent contours of generic projections of surfaces.

Given a generic mapping $f\colon S \to N$, where $S$ is a compact surface without boundary and $N$ is a connected surface, the set of critical points of $f$ is given by a finite number of circles embedded in $S$, which are called the *fold curve* $C$ of $f$, while the set of critical values $\gamma = f(C) \subset N$ is a curve with cusps and double points, called the *apparent contour* of $f$ ([29], [13]).

Of special interest are the cases of the linear projection, with image in a 2-dimensional vector subspace, of a surface embedded in $\boldsymbol{R}^n$ ([22] and [1] for $n = 3$), and of the central projection, with image in a 2-dimensional hyperplane, of a surface embedded in $\boldsymbol{RP}^n$ (for $n = 3$, a thorough study carried on by Landis, Platonova, Shcherback, Goryunov has yielded a complete classification of all germs of projections for generic surfaces [2], [3]).

The results we have just quoted are essentially of a local character while, in the rest of this paper, we shall be concerned with a different question: which statements can be made, from a *global* point of view, about apparent contours of surfaces?

In the following, we are going to consider the case when $N = \boldsymbol{R}^2$ or $N = \boldsymbol{RP}^2$.

It is obvious that the components of apparent contours form a very special subset of the class of curves that we have examined in Sections 1 and 2, and so one expects them to satisfy more restrictive conditions. For instance, the projective curve of Figure 6 cannot belong to any apparent contour, since each component of a contour must display an even number of inflection points ([24], [26]). Furthermore, in a contour we will never see a portrait like that of Figure 4a) since it is fairly obvious that there must always be an even number of inflection points lying between two cusps.

We give now a naive example of a typical global question.

EXAMPLE 1. Can any one of the following pictures represent the apparent contour of a sphere immersed in $\boldsymbol{R}^3$ and projected to a plane $H \subset \boldsymbol{R}^3$?

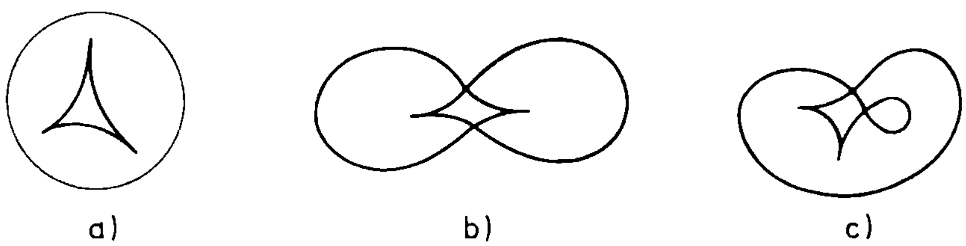


Fig. 7

We shall collect some facts, dealt with in the literature, that provide us with an answer.

*Figure 7 a).* The parity of the cusps in the apparent contour gives a first important information about the surface. The early paper by Whitney

[29], mostly dedicated to an analysis of the local properties of differentiable mappings between surfaces, contained already a global result of this form: *for any excellent mapping* $f\colon \boldsymbol{RP}^2 \to \boldsymbol{R}^2$ *the number of cusp points is odd* [29, Theorem 30A, p. 409].

A sharper result immediately followed:

THEOREM ([28, Th.9, p. 84]). *The number of cusp points presented in the generic case by the apparent contour of a compact manifold $M$ of any dimension $n \geqslant 2$, projected to a Euclidean plane, is congruent* (mod 2) *to the Euler–Poincaré characteristic of $M$, $\chi(M)$.*

Knowing this, the curve represented in Figure 7a) can be excluded at once. It has an odd number of cusps, and surfaces of odd characteristic are not orientable. This curve, in fact, is the contour of a projective plane (it is not difficult to construct an embedding of $\boldsymbol{RP}^2$ in $\boldsymbol{R}^4$ which gives rise to a projection with such critical values). Furthermore, it is easy to realize how $\boldsymbol{RP}^2$ is the only surface $S$ which can give rise to that profile: if $\varphi\colon \boldsymbol{R}^2 \to \boldsymbol{R}$ is a linear function of rank 1, by composing $\varphi$ with the projection map we get a Morse function on $S$ having exactly three critical points. By the way, all manifolds of dimension $n \geqslant 2$ admitting such a function were described in a classical paper [9].

A deeper result about the problems that had been raised by Whitney and Thom is due to Levine:

THEOREM ([18]). *Let $f\colon M \to N$ be a generic map between two compact oriented manifolds,* $\dim M \geqslant 2$ *and* $\dim N = 2$. *Then*:

(1) *if* $\dim M$ *is odd, $f$ is homotopic to a generic $g$ such that $g$ has no cusps*;

(2) *if* $\dim M$ *is even and* $> 2$, *$f$ is homotopic to a generic $g$ such that*:

(a) *if $\chi(M)$ is even, $g$ has no cusps,*

(b) *if $\chi(M)$ is odd, $g$ has one cusp*;

(3) *if* $\dim M = 2$, *then $f$ is homotopic to a generic $g$ such that on each fold curve of $g$ there are at most two cusps.*

Point (3), which concerns surfaces, has been made more precise in [10]: *if $M$ is any closed surface and $N$ is an orientable surface, and $\varphi\colon M \to N$ is a continuous map, then one can find an excellent map $f\colon M \to N$, homotopic to $\varphi$, which has no cusp points on its fold curve when $\chi(M)$ is even, and having one cusp point when $\chi(M)$ is odd.* The same result is contained in [23], where one can find interesting examples: in particular, that of an immersion of $\boldsymbol{RP}^2$ into $\boldsymbol{R}^3$ whose projection to $\boldsymbol{R}^2$ has a connected fold curve which contains a single cusp (thus answering a question posed in [20, p. 156]).

*Figure 7b.* In Figure 7b and 7c the number of cusps has the right parity, so we cannot decide on such grounds whether the curves can or cannot be the apparent contour of a sphere embedded in $\boldsymbol{R}^3$. We use the following

THEOREM ([14, Th.1, p. 49]). *An excellent mapping $f$ from a compact surface $S$ to the plane can be factored through an immersion $g$ into $\boldsymbol{R}^3$ if and only if on each connected component $C_i$ of the fold curve of $f$ the number of cusp points is even or odd according to whether $C_i$ admits an orientable neighbourhood or not.*

(As a matter of fact, the conclusion of this theorem continues to hold for maps $f$: $S \to N$ where $N$ is any surface, with regard to factorization by immersions in $N \times \boldsymbol{R}$ ([23]).)

The contour of Figure 7b could belong to a surface $S$ immersed in $\boldsymbol{R}^3$ only if $S$ contained two Möbius bands. Hence it cannot belong to a sphere immersed in $\boldsymbol{R}^3$. However, the curve of Figure 7b is indeed the contour of a sphere embedded in $\boldsymbol{R}^4$. One can easily figure out the corresponding construction (Fig. 8).

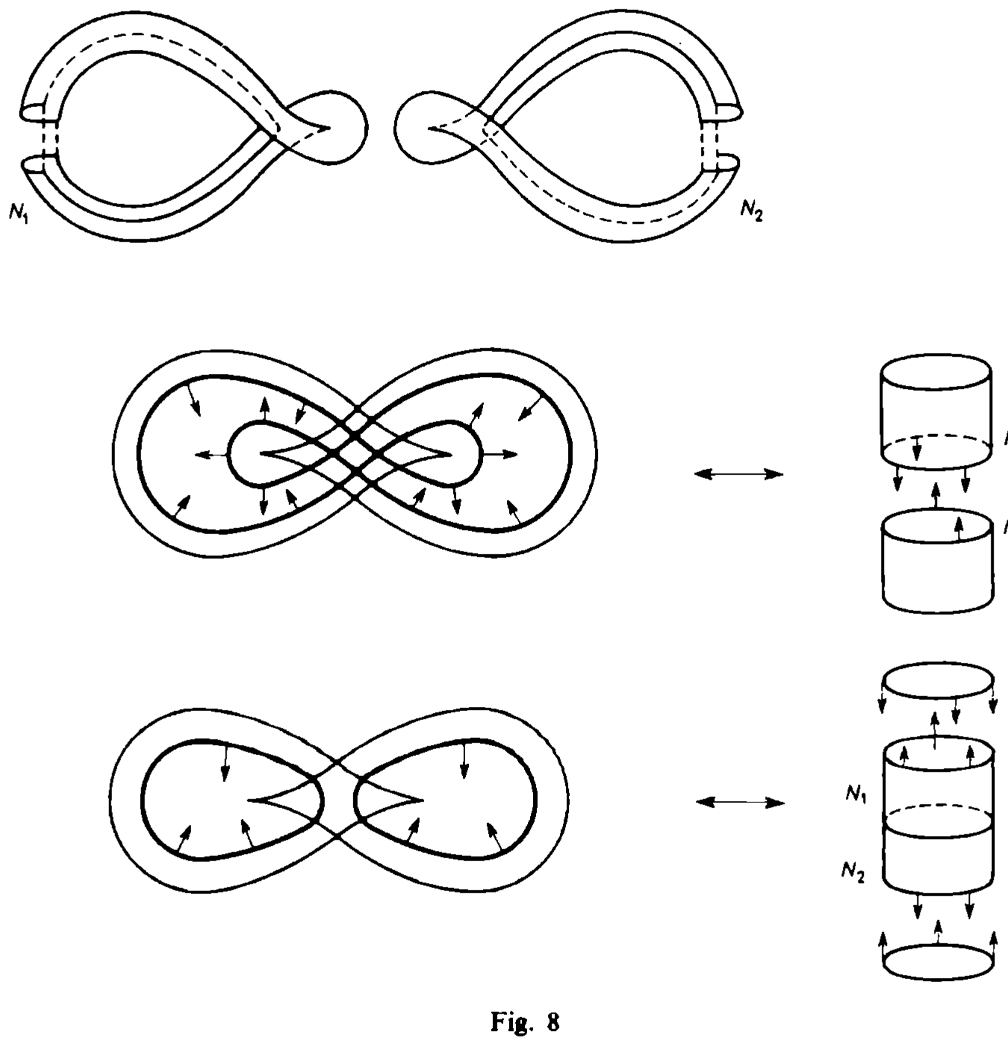



Fig. 8

The curve of Figure 7b cannot be the contour of any other surface. One way to see this is as follows: let $V$ be a surface with boundary $\partial V$, and let $\partial V$ have $k$ connected components $B_1, \ldots, B_k$. If $V$ is immersed in the Euclidean plane by a mapping $f$, the natural orientation of the plane induces an orientation on $V$. We can then orient $\partial V$ as the boundary of $V$: $B_1, \ldots, B_k$ are now oriented circles.

Let $n_i$ be the number of times the unit normal (or the unit tangent) vector to $f(B_i)$ turns around as one goes through $B_i$ in the sense of its orientation; $n = \sum_{i=1}^{k} n_i$ is called the *normal degree* of $\partial V$ immersed by $f$. By [14, Th.3, p. 59], $n = \chi(V)$.

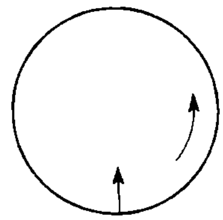



Fig. 9

Now, let's go back to a generic map $f$: $S \to \boldsymbol{R}^2$, where $S$ is a surface without boundary. If $\{\gamma_i\}$ are the irreducible components of the apparent contour $\gamma$, and $\{C_i\}$ are the corresponding components of the fold curve in $S$, it is possible to take tubular neighbourhoods $N_i$ of each $C_i$ in $S$ such that $N_i \cap N_j = \emptyset$ if $i \neq j$, and

1) each $N_i$ is either a cylinder or a Möbius band, and $\partial N_i$ has two or one connected components, respectively;

2) $S \setminus \{\bigcup_i N_i\}$ is an orientable manifold with boundary $B = \bigcup_i \partial N_i$, and $f | S \setminus \{\bigcup_i N_i\}$ is an immersion of this manifold into the plane.

We fix the standard orientation on $\boldsymbol{R}^2$ and compute $n$ = normal degree of $f(B)$. We see that $\chi(S \setminus \{\bigcup_i N_i\}) = \chi(S)$; hence $n = \chi(S)$. In the case of Figure 7b we compute as in Figure 10b.

We see, in this way, that the Euler–Poincaré characteristic of the surface must be 2 and the surface must be a sphere. (If $\partial N_i$ had just one component, as in Figure 10c, the value of the normal degree would have been the same for that component, that is 1. The value $n_i$ which is associated to each $\gamma_i$ does not depend on whether $N_i$ is orientable or not, and this enables us to calculate $\chi(S)$, provided we know the direction towards which $S$ is being "folded" for at least one point of each component of the contour.)

*Figure 7c.* If $N$ is the tubular neighbourhood of the fold curve $C$

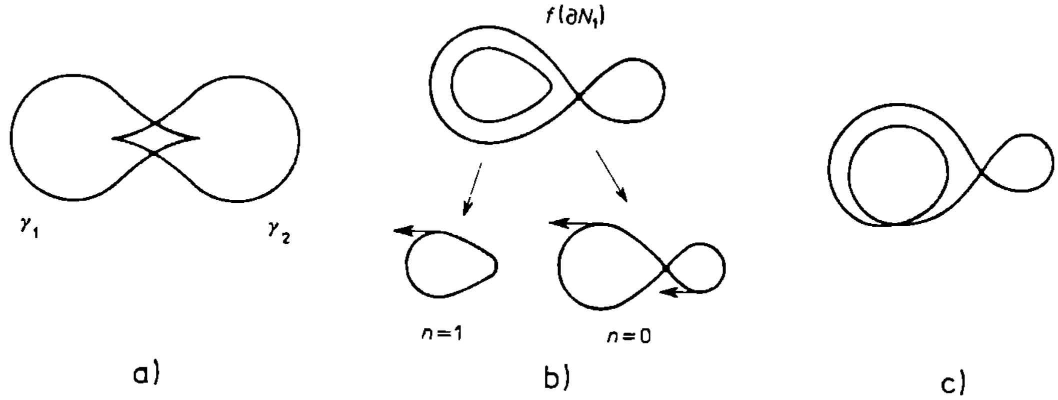



Fig. 10

corresponding to the contour shown in this figure, $\partial N$ must have two components (otherwise $S \setminus N$ would be an orientable surface with boundary of odd characteristic, which would entrain that also $\chi(S)$ id odd, and this is impossible because $\gamma$ has an even number of cusps). The projection of these components into $\boldsymbol{R}^2$ gives rise to the curves shown in Figure 11b. These two curves can be interpreted either as the boundaries of two surfaces (a punctured torus and a disk, Fig. 11c), or as the boundary of one surface (a disk with a hole, Fig. 11d).

Figure 11c corresponds to an orientable $S$ (torus), while the punctured disk of Figure 11d lies in a Klein bottle (in fact, the curve of Figure 11a) illustrates the usual way in which the Klein bottle is drawn). The answer to the question of Example 1 is negative in all cases.

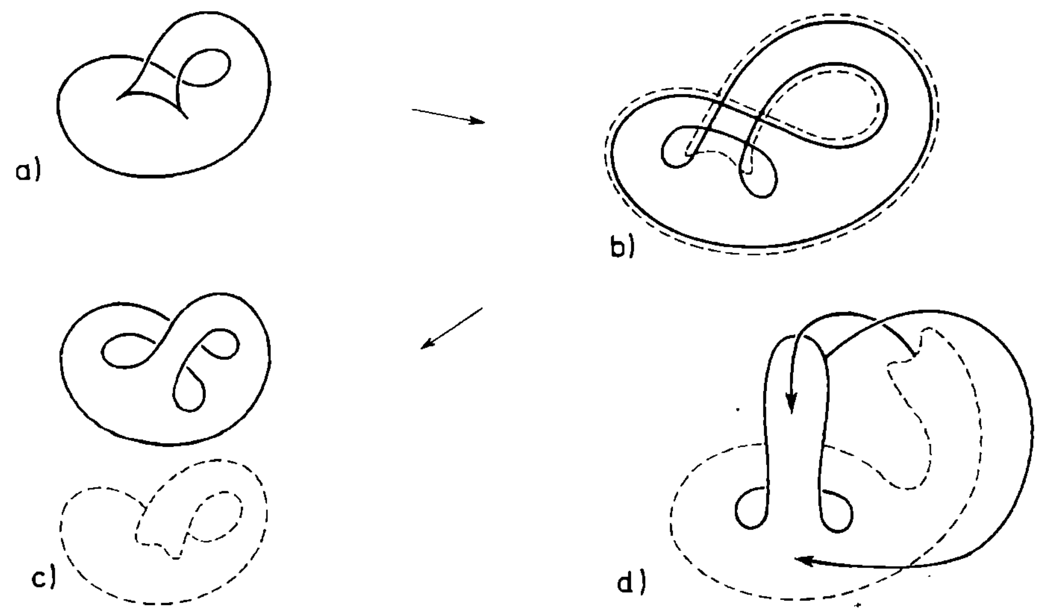



Fig. 11

If $S$ is orientable, we can interprete in the following way the relation $n = \chi(S)$. We choose an orientation on $S$ and consider the two surfaces $S^+$ and $S^-$ made up by all points at which $f$ preserves or (respectively) reverses orientation. $C$ can be oriented as the boundary of $S^+$. Let $\Phi$: $C \to \boldsymbol{RP}^1$ be the mapping which associates to a point $x \in C$ the tangent line to the apparent contour in $f(x)$, translated to the origin. We give to $\boldsymbol{RP}^1$ the orientation induced, through its double covering $\pi$: $S^1 \to \boldsymbol{RP}^1$, by the counterclockwise orientation of $S^1 \hookrightarrow \boldsymbol{R}^2$. One sees that $\sum_i \deg \Phi | C_i$ equals the value of the normal degree of the boundary $B = \partial(S \setminus \{\bigcup_i N_i\})$. Since $n = \chi(S)$, we have $\sum_i \deg \Phi | C_i = \chi(S)$. (This result has been generalized in [19] to the case of a generic mapping, with values in the Euclidean plane, of any orientable compact manifold of even dimension.) So, we have described an alternative procedure [cfr. 20] verifing that the Euler–Poincaré characteristic of any orientable surface with the contour shown in Figure 7c must be 0.

## 4. Surfaces embedded in $\boldsymbol{RP}^n$

Now we shall consider, from a global point of view, central projections of surfaces embedded in $\boldsymbol{RP}^n$. The global results that we have surveyed in Section 3 regarded exclusively mappings with values in an Euclidean plane (or some other *orientable* manifold). Here we go over to a new situation and we need a few more definitions.

Let $S$ be a smooth compact connected surface without boundary. Given an embedding $\varphi$: $S \to \boldsymbol{RP}^n$, we realize central projection by fixing an $(n-3)$-hyperplane $P$ transversal to $\varphi(S)$ and then taking the set $\Sigma$ of all $(n-2)$-hyperplanes containing $P$. We define a map $\pi$: $\boldsymbol{RP}^n \setminus P \to \Sigma$ by associating to any $x \in \boldsymbol{RP}^n \setminus P$ the $(n-2)$-hyperplane containing $x$ and $P$. We call $P$ the *center* of this projection. It can be proved, using methods of [22], that, given *any* embedding $\varphi$: $S \to \boldsymbol{RP}^n$, if we choose generically a center of projection $P$ we obtain a mapping $\pi \circ \varphi$: $S \to \Sigma$ which is stable. Since $\Sigma$ is a 2-dimensional projective space, this means that the critical values of $\pi \circ \varphi$ – that is, the $(n-2)$-hyperplanes of $\Sigma$ which are not transversal to $\varphi(S)$ – form a curve with cusps and double points in $\Sigma$. Let $\Delta$ be this curve: we call it the *apparent contour* of the projection.

An approach which is dual to that of central projections is provided by the study of *linear nets* of hyperplanes intersecting $\varphi(S)$. If we take the family $\Sigma^*$ of all $(n-1)$-dimensional hyperplanes containing $P$, we define a projective space of dimension 2 which is dual to $\Sigma$.

Figure 12 illustrates the situation (in the case $n = 3$). If we fix a plane $H$ not containing the center $P$, the elements of $\Sigma$ are in one-to-one correspon-

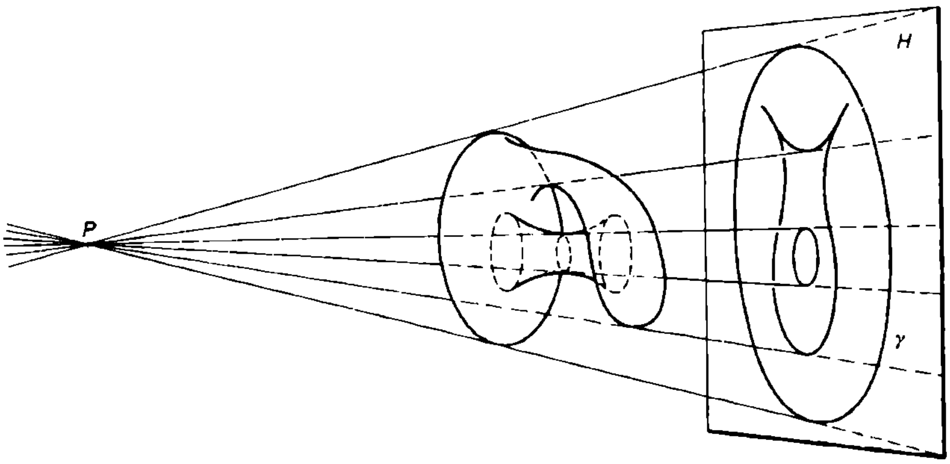



Fig. 12

dence with the points of $H$. A mapping from $S$ to $H$ is thus defined, which associates to any element $x \in S$ the intersection between $H$ and the element of $\Sigma$ that contains $\varphi(x)$. The apparent contour of this function is represented in Figure 12 by the curve $\gamma$, a curve with cusps and double points isomorphic to $\Delta$. The elements of $\Sigma^*$ correspond to the lines of the plane $H$: the last are the points of the dual projective plane $H^*$.

Let $T$ be an $(n-1)$-hyperplane passing through a point $y \in \varphi(S)$. We say that $T$ has contact of type $A_k$ with $\varphi(S)$ at $y = \varphi(z)$ if, given an affine chart $(U, \psi)$ with $y \in U$, $\psi(y) = 0 \in \boldsymbol{R}^n$, and a linear functional $\xi$: $\boldsymbol{R}^n \to \boldsymbol{R}$ with $\ker \xi = T$, then the function $\xi \circ (\psi \circ \varphi)$: $S \to \boldsymbol{R}$ has a singularity of type $A_k$ in $z$. When $\varphi$ embeds $S$ as an analytic submanifold of $\boldsymbol{R}^3$, it has been shown [6] that it exists a closed *subanalytic* set $\Gamma \subset \boldsymbol{R}^3$, of codimension $\geqslant 1$, with the property that any $a \in \boldsymbol{R}^3 \setminus \Gamma$ has at most a finite number of tangent planes to $\varphi(S)$, passing through it, whose contact with $\varphi(S)$ is more degenerate than $A_1$. We call $\Sigma^*$ a *regular net for* $\{S, \varphi\}$ if it admits a stratification $\Sigma^* = \Sigma_0^* \cup \Sigma_1^* \cup \Sigma_2^*$ in which each stratum $\Sigma_i^*$ is a submanifold of dimension $i$, with a finite number of connected components, and

$\Sigma_2^*$ = all elements of $\Sigma^*$ which are transversal to $\varphi(S)$;

$\Sigma_1^*$ = all elements of $\Sigma^*$ that have contact of type $A_1$ with $\varphi(S)$ in one point, and are transversal to $\varphi(S)$ at all other points;

$\Sigma_0^*$ =,all elements of $\Sigma^*$ having contact of type $A_2$ with $\varphi(S)$ in one point, or contact of type $A_1$ in two different points (not lying on the same element of $\Sigma$), and are transversal to $\mathscr{S}(S)$ in every other point.

We call the set $\Delta^* = \Sigma_1^* \cup \Sigma_0^*$ the *discriminant* of the net. The condition that $\Sigma^*$ is a regular net of hyperplanes for $\{S, \varphi\}$ is seen to imply that $\Delta^*$ is a curve with cusps and double points, the singularities of which are the elements of $\Sigma_0^*$. (Looking at Figure 12, we visualize $\Delta^*$ as the curve of tangents of the contour $\gamma$.) Keeping in mind the results of [21], it is possible to assert that, if $\varphi$ is taken from an open dense set of embeddings of $S$ in $\boldsymbol{RP}^n$, then for a generic choice of the center $P$, the resulting net $\Sigma^*$ is a regular net of hyperplanes for $\{S, \varphi\}$. So, generically, both the apparent contour $\Delta$ and its dual curve $\Delta^*$ will be curves with cusps and double points lying in their respective projective planes. In this case, we have a precise relation between the following three entities:

(a) the homology class $[\Delta^*]$ defined by $\Delta^*$ in $H_1(\Sigma^*; \boldsymbol{Z}_2)$;
(b) the genus $g$ of the surface $S$;
(c) the homology class $[S]$ induced by $\varphi(S)$ in $H_2(\boldsymbol{RP}^n; \boldsymbol{Z}_2)$.

This relation is expressed by the following

THEOREM ([26, Th.2]). *If S is orientable,* $[\Delta^*] = 0$. *If S is not orientable, when the genus of S is odd*

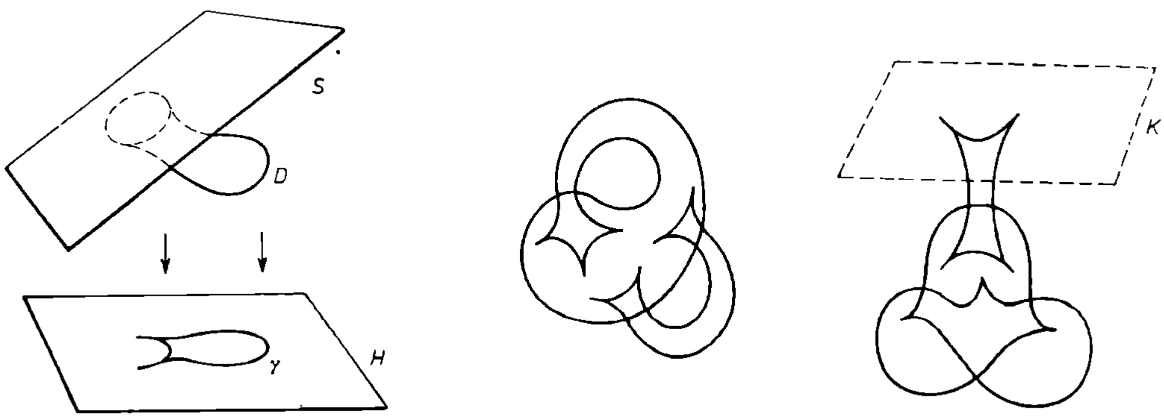
$$[S] = 0 \Leftrightarrow [\Delta^*] \neq 0$$

*and when the genus of S is even*

$$[S] = 0 \Leftrightarrow [\Delta^*] = 0.$$

For orientable surfaces, let us see this in some trivial cases: a two-sheeted hyperboloid projected in the direction of its axis gives rise to a double covering of $\Sigma$, with an empty contour: hence $[\Delta^*] = 0$. A one-sheeted hyperboloid that is mapped to $\Sigma$ in the same way, has a circle as its apparent contour. In such a circumstance, $\Delta$ has no cusps and this entrains that $\Delta^*$ has no inflection points: by an argument of the end of Section 2, $[\Delta^*] = 0$.

We will introduce now some illustrations of the theorem in the non-orientable case.




a) b) c)

Fig. 13

EXAMPLE 2. In Figure 13a we have sketched a simple example of an embedding in $\boldsymbol{RP}^3$ of a nonorientable surface $S$, such that $[S] \neq 0$. The surface is a projective plane, and we first embed it as an hyperplane of $\boldsymbol{RP}^3$, then we remove a disk along the dotted line and replace it by a cell $D$ in the way that is shown in the picture. The resulting embedded $\boldsymbol{RP}^2$ is projected to a transversal plane $H$ (we have taken $P$ at $\infty$ in order to visualize the projection). The apparent contour of the projection of center $P$ is isomorphic to the curve $\gamma \subset H$. In our picture, the curve $\Delta$ is self-dual: it has an even number of flexes, it is obviously shrinkable to a point in the projective plane in which it lies, and so $[\Delta^*] = 0$. This is what we expected, since the genus of $S$ is 1, that is, an odd number.

We remark that, by stretching and turning the attached cell $D$, it is easy to obtain, after a projection from the same center $P$ as before, a contour of the surface $S$ that is precisely the one of Figure 7c. The same contour could be produced in quite a similar way (removing a disk, then attaching a cell, finally stretching and turning this cell) from a two-sheeted hyperboloid. This fact tells us that in $\boldsymbol{RP}^3$ the profiles of a torus, of a Klein bottle, of a projective plane, of a sphere, etc. may be indistinguishable. At first sight this seems to be a major difficulty, since we have just seen that it happens *even for irreducible contours*. How can we find out the Euler–Poincaré characteristic of $S$ from its contour? In [26] it is shown that in order to reconstruct the Euler–Poincaré characteristic of a surface $S$ embedded in $\boldsymbol{RP}^n$, starting from the apparent contour $\Delta$, a necessary and sufficient condition is given by the knowledge of two data: *the number of points at which one element of $\Sigma$ intersects $\varphi(S)$, and an orientation (suitably defined) of the normal bundle to the contour in $\Sigma$.*

In Figure 13b we have plotted the apparent contour of a nonorientable surface of genus 5. To see where it comes from, start with the projective plane of Figure 7a and attach two handles to it. Everything can be done in $\boldsymbol{R}^n$, $n \geqslant 4$. The resulting surface is thus homologous to zero in $\boldsymbol{RP}^n$: $[S] = 0$. Now, the dual curve to $\Delta$ is not homologous to zero: it has an odd number of inflection points, which correspond to the cusps of $\Delta$, hence, by the same argument as before, $[\Delta^*] \neq 0$.

The surface of Figure 13c represents a Klein bottle, obtained by joining, through a handle, Boy's surface [17] with the hyperplane $K$. The genus is now even: $[\Delta^*] \neq 0$ (the contour is represented in the picture by the dark line and its dual curve displays an odd number of flexes) and $[S] \neq 0$ ($S \setminus K$ is homologous to the part of $K$ which has been removed in order to attach the handle; hence $S$ is homologous to a 2-dimensional hyperplane).

At this point we may go back to a different question, and consider what can be said about the number $c$ of cusp points presented by a generic apparent contour $\Delta \subset \Sigma$. The conclusion we reach generalizes in some way to the present setting what was known about mappings with values in the

Euclidean plane. The requirements on the embedding $\varphi$ that were needed in order to state and prove the preceding theorem are not necessary any more. Let $\varphi$: $S \to \boldsymbol{R}P^n$ be *any* embedding of a compact connected surface without boundary in $n$-dimensional projective space: *if* $S$ *is orientable, the number* $c$ *of the cusps in the apparent contour is even. If* $S$ *is not orientable and has odd genus, then*

$$c \text{ is odd } \Leftrightarrow [S] = 0;$$

*while, if* $S$ *has even genus,*

$$c \text{ is even } \Leftrightarrow [S] = 0$$

([26; Th.3]).

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