TOPICS IN ALGEBRA
BANACH CENTER PUBLICATIONS, VOLUME 26, PART 1
PWN-POLISH SCIENTIFIC PUBLISHERS
WARSAW 1990

STRONGLY INDEFINITE QUADRATIC FORMS AND WILD ALGEBRAS

NIKOLAOS MARMARIDIS

Department of Mathematics, University of Ioannina
Ioannina, Greece

Let A be a one-point extension of some algebra B by an indecomposable B-module M. Assume that the ordinary quiver of A does not contain oriented cycles and that M belongs to a preinjective component of B. In the present paper we prove that under certain conditions, which imply that A is of wild representation type, the Euler form attached to A is strongly indefinite.

Introduction

Let A be a finite-dimensional k-algebra whose ordinary quiver Q_A does not have oriented cycles. One can define on A two quadratic forms, the Tits form t_A via the bound quiver of A, and the Euler form q_A via the Cartan matrix of A (see [2], [3], [10]).

Let A be an algebra of wild representation type of the form

$$\begin{bmatrix} B & M \\ 0 & k \end{bmatrix},$$

where M is an indecomposable B-module belonging to a preinjective component $\mathbf{C}$.

Let $\boldsymbol{K} = \operatorname{Hom}(M, B\text{-mod})$ be the vector space category attached to M and $\operatorname{ind}\boldsymbol{K}$ the full subcategory of $\boldsymbol{K}$ having as objects the indecomposable ones of $\boldsymbol{K}$.

In the present paper the following Theorem 2.8 is proved:

If the preinjective component $\mathbf{C}$ *of the Auslander–Reiten quiver of* B *is of tree type* (1.1), *if the category* $\operatorname{ind}\boldsymbol{K}$ *contains as full subcategory one of the categories*

This paper is in final form and no version of it will be submitted for publication elsewhere.

$\mathbf{L}$ of List A (see Preliminaries), and if the connected components of $\mathbf{L}$ are path-incomparable (2.1), then the Euler form q_A is strongly indefinite (1.2). Further, if gl.dim $A \leqslant 2$, *then the Tits form t_A is strongly indefinite.*

The proof of this theorem is based on the combinatorial Lemma 2.2 and on the evaluation of q_A on the dimension vector of some special A-module (cf. 2.3 and 2.4 of [8]). Moreover, our present method is completely different from the methods used in our previous papers [5], [6], where we attacked similar problems using tilting theory.

Note that there is a long list of papers concerning the above-mentioned quadratic forms and their relation to classification problems of representation theory (see for example [2], [10]).

1. Preliminaries

Throughout this paper all k-algebras A are associative, unitary, basic and finite-dimensional over a field k. The field k is assumed to be algebraically closed. All A-modules are finitely generated left A-modules. The corresponding module category is denoted by A-mod.

As usual, any algebra A will be represented in the form kQ/J, where Q is the *ordinary quiver* of A and J is an *admissible ideal* [4]. We denote by $Q(0)$ (resp. $Q(1)$) the set of vertices (resp. of arrows) of Q.

An algebra A is said to be *directed* if Q does not have oriented cycles. *In the following all algebras are assumed to be directed.*

Let Γ_A be the *Auslander–Reiten quiver* of A [4]. The vertex of Γ_A corresponding to the indecomposable A-module X will be denoted by $[X]$. Let τ (resp. τ^-) be the translation of Γ_A determined by $D\mathrm{Tr}$ (resp. $\mathrm{Tr}D$). A path φ: $[X_0]\to[X_1]\to\ldots\to[X_n]$ of irreducible maps in Γ_A is said to be *sectional* provided $\tau[X_{i+1}] \neq [X_{i-1}]$ for all i, $1 < i < n$ [1].

To any connected component $\mathbf{C}$ of the Auslander–Reiten quiver Γ_A of some algebra A, there is associated its τ-orbit graph $G_A(\mathbf{C})$ [10].

Definition 1.1. A preinjective component $\mathbf{C}$ of Γ_A is said to be of *tree type* if its τ-orbit graph $G_A(\mathbf{C})$ is a tree (with simple edges).

Let $A = kQ/J$ be any connected algebra. Write α for the idempotent of kQ corresponding to the vertex α of $Q(0)$. Let W be a *system of relations* for the ideal J [2]. For any two vertices α, $\beta\in Q(0)$, denote by $r(\alpha, \beta, W)$ the cardinality of $W\cap(\alpha J\beta)$. Let t_A: $\mathbf{Z}^{Q(0)}\to\mathbf{Z}$ be the quadratic form given by the formula

$$t_A(x) = \sum_{\alpha\in Q(0)} x_\alpha^2 - \sum_{\alpha\to\beta\in Q(1)} x_\alpha x_\beta + \sum_{\alpha,\beta\in Q(0)} r(\alpha, \beta, W)x_\alpha x_\beta.$$

This form is called the *Tits form* of A [2].

Let $\{P_\alpha \mid \alpha\in Q(0)\}$ be a complete set of nonisomorphic projective indecomposable A-modules. The *dimension vector* $\dim M$ of an A-module M is defined as

$$(\dim M)_{\alpha\in Q(0)} = \big(\dim_k \mathrm{Hom}_A(P(\alpha), M)\big)_{\alpha\in Q(0)}.$$

Let C_A be the *Cartan matrix* of A, which is defined as the matrix whose αth column equals the transpose of $\dim P(\alpha)$ (see [10]). For z, w in $\mathbf{Z}^{Q(0)}$ consider the bilinear form $\langle z, w\rangle_A = zC_A^{-T}w^T$. The associated quadratic form $q_A(z) = \langle z, z\rangle_A$ is called the *Euler form* of A.

For any two A-modules X and Y, write ${}^0(X, Y)_A$ for the k-dimension of $\mathrm{Hom}_A(X, Y)$ and ${}^i(X, Y)_A$ for the k-dimension of $\mathrm{Ext}^i_A(X, Y)$, $i \geqslant 1$.

On the set S of the dimension vectors of A-modules the Euler form q_A coincides with the function $f(\dim M) = \sum_{i=0}^{\infty}(-1)^i\, {}^i(M, M)_A$ (*) [10]. Since A is a directed algebra, the global dimension of A is finite, which implies that the sum above is finite.

Moreover, if the global dimension of A is less than or equal to 2, then the Tits form t_A coincides with q_A [2].

In the remaining part of the paper we will often use the above (*) homological interpretation.

Let q be any quadratic form with integral coefficients.

Definition 1.2. The form q is said to be *strongly indefinite* if there is a positive integral vector z with $q(z) < 0$.

Let C and B be two finite-dimensional algebras and M a B-C-bimodule. Consider the triangular matrix algebra

$$A = \begin{bmatrix} B & M \\ 0 & C \end{bmatrix}.$$

We will identify, as usual, the category A-mod with the corresponding comma category (see [9]). The A-modules will be represented by triples (X, f, Y) with X a C-module, Y a B-module and f a C-module homomorphism from X to $\mathrm{Hom}_B(M, Y)$.

Let $\underline{D}$ be the full subcategory of B-mod consisting of the modules Y with $\mathrm{Hom}(M, Y_i) \neq 0$ for any indecomposable direct summand Y_i of Y. The representation type of A depends on the representation type of B and on the representation type of the full subcategory $\underline{C}$ of $\mathrm{mod}\, A$ consisting of the triples (X, f, Y) with $Y \in \underline{D}$.

If $C = k$, then the algebra A is called a *one-point extension* of B and if M is an indecomposable B-module, then the representation type of $\underline{C}$ can be determined using vector space categories as follows (see [9], [10]):

Set $\boldsymbol{K} = \mathrm{Hom}(M, B\text{-mod})$ for the vector space category $(\underline{D}, \mathrm{Hom}(M, B\text{-mod}))$. The category $\underline{C}$ is representation equivalent to the subspace category $U(\boldsymbol{K})$. Consequently, in order to determine the representation type of $\underline{C}$ we have to find $\boldsymbol{K}$ and the representation type of $U(\boldsymbol{K})$.

Write ind$K$ for the full subcategory of K with objects the indecomposable objects of K. By Nazarova's theorem ([7], [9]), we know that if any object of ind$K$ is one-dimensional, then $U(K)$ is of wild representation type if and only if ind$K$ contains as full subcategory one of the categories (I) to (VI) of List A below. (The points of List A denote one-dimensional indecomposable objects. Two points x and y are connected by a path of arrows if and only if the vector space of their morphisms is nonzero.) Further, List A contains two more categories. Category (VII) consists of two indecomposable objects, a one-dimensional and a two-dimensional one, and no nonzero morphisms between them. The endomorphism ring of the two-dimensional object is trivial. Category (VIII) consists only of an m-dimensional indecomposable object, $m \geqslant 3$, with trivial endomorphism ring. (In List A, two-dimensional objects are denoted by squares and m-dimensional objects, $m \geqslant 3$, are denoted by triangles.)

It is rather well known that if the subcategory ind$K$ of some vector space category K contains as full subcategory (VII) or (VIII) of List A, then the category $U(K)$ has wild representation type.

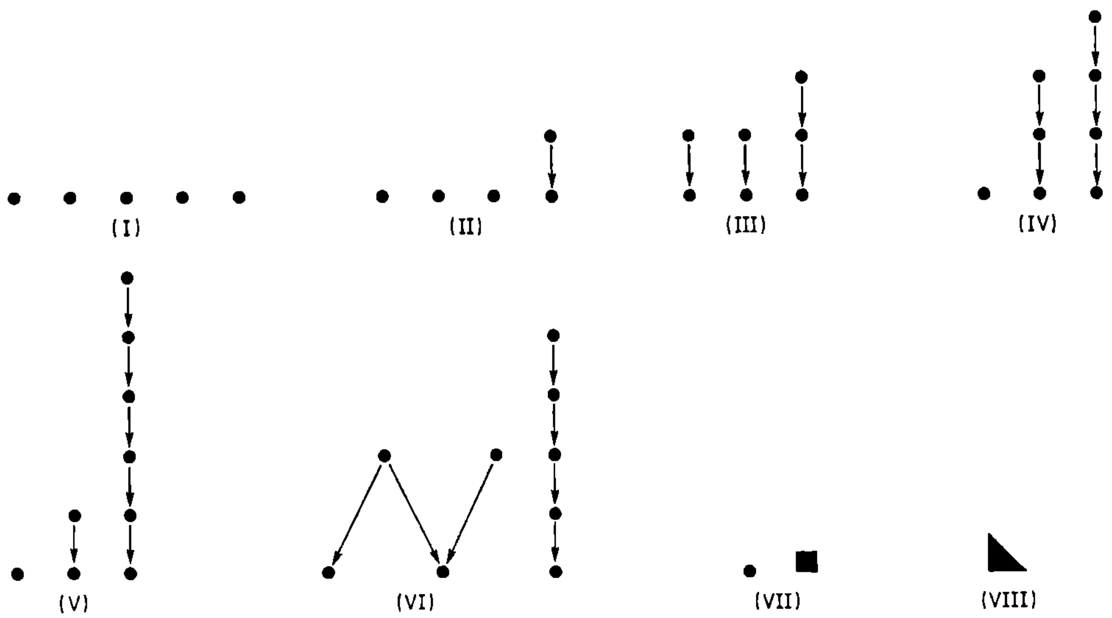

Fig. 1. List A

Let L be any full subcategory of K such that any of its indecomposable objects is one-dimensional. If x and y are objects of ind$L$, then we define: $x \leqslant y$ if and only if $\mathrm{Hom}(x, y) \neq 0$. This relation is a partial ordering on ind$L$.

Any linearly ordered full subcategory H of ind$L$ will be called a *chain* of ind$K$. Two chains H and H' of ind$K$ are said to be *isomorphic* if they have the same number of objects.

We say that a morphism φ: $x \to y$ of ind$K$ *corresponds to an arrow of* C if

φ equals $\mathrm{Hom}_A(M, f)$, where $f\colon X \to Y$ is an irreducible morphism of A-mod. A chain $\boldsymbol{H}$ of $\mathrm{ind}\,\boldsymbol{K}$ will be called *sectional* if the morphisms of $\boldsymbol{H}$ correspond to arrows of $\boldsymbol{C}$, and if these arrows form a sectional path of $\boldsymbol{C}$ [1].

2. Evaluating the forms on particular vectors

From now on we assume that B is an algebra with a preinjective component $\boldsymbol{C}$ and M is an indecomposable B-module such that $[M] \in \boldsymbol{C}$. Let $\boldsymbol{K}$ be the vector space category $\mathrm{Hom}_B(M, B\text{-mod})$. In the following any indecomposable object $\mathrm{Hom}_B(M, X)$ of $\boldsymbol{K}$ will be denoted just by the corresponding small letter x.

DEFINITION 2.1. Two subsets $\boldsymbol{X}$, $\boldsymbol{Y}$ of $\mathrm{ind}\,\boldsymbol{K}$ are said to be *path-incomparable* if for any $x \in \boldsymbol{X}$ and $y \in \boldsymbol{Y}$, there is no path of $\boldsymbol{C}$ connecting $[X]$ and $[Y]$.

LEMMA 2.2. *Let $\boldsymbol{C}$ be a preinjective component of tree type. If* $\mathrm{ind}\,\boldsymbol{K}$ *contains as full subcategory $\boldsymbol{L}$ one of the categories* (II)–(VI) *of List A and if the connected components of $\boldsymbol{L}$ are path-incomparable, then*:

(α) *Either there is some full subcategory $\boldsymbol{L}'$ of* $\mathrm{ind}\,\boldsymbol{K}$ *such that $\boldsymbol{L}'$ is one of the categories* (II)–(VI) *of List A and any morphism between two objects of $\boldsymbol{L}'$ corresponds to an arrow of $\boldsymbol{C}$, or* $\mathrm{ind}\,\boldsymbol{K}$ *contains as full subcategory $\boldsymbol{L}'$ one of the categories* (VII) *or* (VIII). *In both cases the connected components of $\boldsymbol{L}'$ are path-incomparable.*

(β) *If all morphisms of some chain $\boldsymbol{H}$ of $\boldsymbol{L}$ correspond to arrows of $\boldsymbol{C}$, then either $\boldsymbol{H}$ is sectional or* $\mathrm{ind}\,\boldsymbol{K}$ *contains as full subcategory $\boldsymbol{L}'$ one of the categories* (I) *or* (II) *of List A. Again the connected components of $\boldsymbol{L}'$ are path-incomparable.*

Proof. (α) Let $\boldsymbol{H}$ be any of the chains of $\boldsymbol{L}$ and $\varphi\colon x \to y$ some morphism between two neighbouring vertices of $\boldsymbol{H}$ which does not correspond to an arrow of $\boldsymbol{C}$. Since $\boldsymbol{C}$ is a preinjective component, there is some chain of irreducible morphisms $f_i\colon Z_i \to Z_{i+1}$, $0 \leqslant i \leqslant n-1$, with $X = Z_0$, $Y = Z_n$ and such that the morphism

$$\psi = \mathrm{Hom}(M, f_0 \circ \ldots \circ f_{n-1})\colon\ x = z_0 \to z_1 \ldots \to z_{n-1} \to z_n = y$$

is a nonzero morphism of $\mathrm{ind}\,\boldsymbol{K}$. Any z_i is path-incomparable to any connected component $\boldsymbol{J} \neq \boldsymbol{H}$ of $\boldsymbol{L}$, because x and y have this property.

If there is some i, $1 \leqslant i \leqslant n-1$, such that $\dim_k z_i \geqslant 3$, then (VIII) of List A is contained as full subcategory in $\mathrm{ind}\,\boldsymbol{K}$.

If there is some i, $1 \leqslant i \leqslant n-1$, such that $\dim_k z_i = 2$, then z_i together with any other object of $\boldsymbol{L} \backslash \boldsymbol{H}$ form a full subcategory $\boldsymbol{L}'$ of $\mathrm{ind}\,\boldsymbol{K}$ which coincides with (VII) of List A.

If for any i, $1 \leqslant i \leqslant n-1$, $\dim_k z_i = 1$, then we replace φ by the path ψ and we obtain from $\boldsymbol{H}$ a new chain $\boldsymbol{H}'$. This chain $\boldsymbol{H}'$ is path-incomparable to any connected component $J \neq H$ of L. If there are some morphisms of H' which do not correspond to arrows of C, we repeat the above procedure until we achieve finally some full subcategory L' of L which either (i) coincides with (VII) or (VIII) of List A, or (ii) has the property that there is a chain H'' obtained from H such that any morphism of H'' corresponds to an arrow of C. Obviously the number of objects of H'' is greater than or equal to the number h of objects of H. We choose a linear subchain H' with maximal object x the maximal object of H together with $h-1$ consecutive objects of H'' starting with the unique neighbour of x. Now, H' is isomorphic to H and its morphisms correspond to arrows of C. This procedure can be applied to any chain of L. This closes the proof for the cases where L consists only of chains (cases (II)–(V)).

We are now going to study the remaining case (VI) of List A. Now, L consists of a chain H and another component N of the form shown in Fig. 2.

(N)

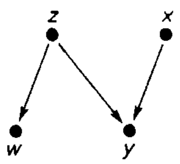

Fig. 2

By the preceding discussion we may assume, without loss of generality, that the morphisms of H correspond to arrows of C. If the morphisms $x \to y$ or $z \to y$ of (N) do not correspond to arrows of C, then we can find as before nonzero morphisms ψ: $x_1 \to y$ and ω: $z_1 \to y$ corresponding to arrows of C.

Notice that x_1 and z_1 are path-incomparable to the chain H. Moreover, there is no path between $[X_1]$ and $[Z_1]$, since $G_B(C)$ is a tree. Hence, there is no nonzero morphism between x_1 and z_1.

If the k-dimension of x_1 or z_1 is $\geqslant 2$, then we conclude as before that ind$K$ contains as full subcategory (VII) or (VIII) of List A.

If $\dim_k x_1 = \dim_k z_1 = 1$, there are two cases:

(i) There is no nonzero morphism from x_1 or z_1 to w. Now, the objects w, x_1, z_1 and any two neighbouring objects a, b of H form a full subcategory of ind$K$ which coincides with (II) of List A.

(ii) There is some nonzero morphism from x_1 or z_1 to w, say from x_1 to w. As before we can find some nonzero morphism λ: $x_1 \to t$ of ind$K$ corresponding to an arrow of C. Notice that t is path-incomparable to H. If $\dim_k t \geqslant 2$, then we conclude using the same argumentation as before. If $\dim_k t = 1$, then consider the full subcategory N' of ind$K$ consisting of the objects x_1, z_1, y, t. Again it is easy to see that the only morphisms of N' are the k-multiples of λ, ψ and ω. Hence, N' is of the form (N) and its nonzero morphisms correspond to arrows of C. The full subcategory L' of ind$K$ consisting of N' and H coincides with (VI) of List A and has path-incomparable components.

(β) Assume that some chain H is nonsectional, i.e. H contains two morphisms $x \to y \to z$, with $[X] = \tau[Z]$. Consider the Auslander–Reiten sequence $0 \to X \to J \to Z \to 0$. The middle term J is the direct sum of Y and at least another indecomposable module U, with $\dim_k u = 1$, because the function $\dim_k \mathrm{Hom}_B(M, -)$ is additive and $\dim_k x = \dim_k y = \dim_k z = 1$. Moreover, for any other indecomposable direct summand V of J not equal to Y or U we must have $\mathrm{Hom}_B(M, V) = 0$. Of course, the corresponding object u of $\mathrm{ind}\,\boldsymbol{K}$ is path-incomparable to the remaining connected components of $\boldsymbol{L}$ and there is no nonzero morphism between u and y. Using u, y and objects of the other connected components of $\boldsymbol{L}$ we see that in each case $\mathrm{ind}\,\boldsymbol{K}$ contains as full subcategory one of the categories (I) or (II) of List A.

LEMMA 2.3. *Let $\boldsymbol{C}$ be a preinjective component and let $[X]$, $[Y]$ be two vertices of $\boldsymbol{C}$. If $\mathrm{Ext}^i(X, Y) \neq 0$, for some $i \geqslant 1$, then there is a path in $\boldsymbol{C}$ starting at $[Y]$ and ending at $\tau[X]$.*

Proof. We will prove our lemma by induction on i.

For $i = 1$, we have $\mathrm{Ext}_B^1(X, Y) \cong D\overline{\mathrm{Hom}}_B(Y, \tau X) \neq 0$. Since $\boldsymbol{C}$ is a preinjective component, there is a path from $[Y]$ to $\tau[X]$.

Assume that our lemma is true for any $n < i$. We are going to prove it for $n = i \geqslant 2$. Consider the short exact sequence

$$(*) \qquad 0 \to Y \xrightarrow{i} I(Y) \to L \to 0,$$

with $Y \xrightarrow{i} I(Y)$ an injective hull of Y. Now, $\mathrm{Ext}_B^i(X, Y) \cong \mathrm{Ext}_B^{i-1}(X, L) \neq 0$. Hence, there is an indecomposable direct summand L' of L such that $\mathrm{Ext}_B^{i-1}(X, L') \neq 0$. Since $[Y] \in \boldsymbol{C}$ it follows that $[L'] \in \boldsymbol{C}$. By the inductive assumption, there is a path in $\boldsymbol{C}$ starting at $[L']$ and ending at $\tau[X]$. Moreover, from the above exact sequence $(*)$ we obtain a path in $\boldsymbol{C}$ from $[Y]$ to $[L']$. Hence, there is a path starting at $[Y]$ and ending at $\tau[X]$.

LEMMA 2.4. *If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are path-incomparable subsets of $\mathrm{ind}\,\boldsymbol{K}$, then for any $x \in \boldsymbol{X}$ and $y \in \boldsymbol{Y}$ we have $\mathrm{Ext}_B^i(X, Y) = 0$, for any $i \geqslant 1$.*

Proof. If $\mathrm{Ext}_B^i(X, Y) \neq 0$, then there is a path from $[Y]$ to $\tau[X]$, by Lemma 2.3. Hence, there is a path from $[Y]$ to $[X]$. This is impossible, because $\boldsymbol{X}$ and $\boldsymbol{Y}$ are path-incomparable.

LEMMA 2.5. *Let $\boldsymbol{C}$ be a preinjective component of tree type and let*

$$\varphi = f_0 \circ f_1 \circ \ldots \circ f_{n-1}\colon [X_0] \to [X_1] \to \ldots \to [X_n], \qquad n \geqslant 1,$$

be a path in $\boldsymbol{C}$, where f_i is the arrow from $[X_i]$ to $[X_{i+1}]$, $0 \leqslant i \leqslant n-1$. If φ is sectional, then there is no path in $\boldsymbol{C}$ with domain $[X_0]$ and range $\tau[X_n]$.

Proof. We will prove our lemma by induction on n.

For $n = 1$, let $\psi = g_0 \circ g_1 \circ \ldots \circ g_s\colon [X_0] \to [Y] \to \ldots \to \tau[X_1]$ be some path from $[X_0]$ to $\tau[X_1]$, with $g_0\colon [X_0] \to [Y]$. The neighbour $[Y]$ of $[X_0]$ does not coincide with $[X_1]$, since there is no path from $[X_1]$ to $\tau[X_1]$. Considering the vertices $[X_0]$, $[X_1]$, $[Y]$, the arrows f_0, g_0 and the path $g_1 \circ \ldots \circ g_s$ we observe that the graph $G_B(C)$ associated to C is not a tree. This is a contradiction.

Assume that our lemma is true for any $m < n$. We are going to prove it for $m = n$. Let $\psi = g_0 \circ g_1 \circ \ldots \circ g_s$: $[X_0] \to [Y] \to \ldots \to \tau[X_n]$ be some path from $[X_0]$ to $\tau[X_n]$, with g_0: $[X_0] \to [Y]$. The neighbour $[Y]$ of $[X_0]$ does not coincide with $[X_1]$, since there is no path from $[X_1]$ to $\tau[X_n]$ by our induction hypothesis.

Considering the vertices $[X_0]$, $[X_1]$, $[Y]$, the paths φ, $g_1 \circ \ldots \circ g_s$ and the arrow g_0 we observe that the graph $G_B(C)$ is not a tree. This is again a contradiction.

LEMMA 2.6. *Let C be a preinjective component and let*

$$\varphi = f_0 \circ f_1 \circ \ldots \circ f_{n-1}: [X_0] \to [X_1] \to \ldots \to [X_n], \quad n \geqslant 1,$$

be a path in C, where f_i is the arrow from $[X_i]$ to $[X_{i+1}]$, $0 \leqslant i \leqslant n-1$.

α) $\mathrm{Hom}_B(X_n, X_0) = 0$ *and* $\mathrm{Ext}^i_B(X_0, X_n) = 0$ *for any* $i \geqslant 1$.

β) *If φ is sectional and C is of tree type, then* $\mathrm{Ext}^i_B(X_n, X_0) = 0$ *for any* $i \geqslant 1$.

Proof. α) It is obvious that $\mathrm{Hom}_B(X_n, X_0) = 0$, since C is a preinjective component. If $\mathrm{Ext}^i_B(X_0, X_n) \neq 0$ for some $i \geqslant 1$, then, by 2.3, there is a path in C starting at $[X_n]$ and ending at $\tau[X_0]$. This implies the existence of an oriented cycle in C containing $[X_0]$. This is impossible.

β) If $\mathrm{Ext}^i_B(X_n, X_0) \neq 0$, for some $i \geqslant 1$, then, by Lemma 2.3, there is a path in C with domain $[X_0]$ and range $\tau[X_n]$. This is impossible because of 2.5.

Let A be an algebra such that its ordinary quiver Q_A does not have oriented cycles. Assume that A can take the form

$$\begin{bmatrix} B & M \\ 0 & k \end{bmatrix},$$

where Γ_B has a connected preinjective component C and M is an indecomposable B-module with $[M]$ in C.

Let S be an additional simple injective A-module which does not belong to B. If π: $P(S) \to S$ is the projective cover of S, then the kernel of π equals M and we have the short exact sequence

$$0 \to M \to P(S) \to S \to 0.$$

For any A-module X we consider the induced exact sequence:

$$\begin{aligned} 0 &\to \mathrm{Hom}_A(S, X) \to \mathrm{Hom}_A(P(S), X) \to \mathrm{Hom}_A(M, X) \to \\ &\to \mathrm{Ext}^1_A(S, X) \to \mathrm{Ext}^1_A(P(S), X) \to \mathrm{Ext}^1_A(M, X) \to \ldots \\ &\to \mathrm{Ext}^i_A(S, X) \to \mathrm{Ext}^i_A(P(S), X) \to \mathrm{Ext}^i_A(M, X) \to \ldots \end{aligned}$$

Because of this sequence we obtain:

$$(*) \qquad {}^0(S, X)_A - {}^1(S, X)_A = {}^0(P(S), X)_A - {}^0(M, X)_A,$$

$$(**) \qquad {}^i(M, X)_A = {}^{i+1}(S, X)_A \quad \text{for any } i \geqslant 1.$$

For any A-module Y we denote by Y^n the direct sum of n copies of Y.

LEMMA 2.7. *If X is a B-module such that for any indecomposable direct summand X_i of X we have $[X_i] \in C$ and $\mathrm{Hom}_B(M, X_i) \neq 0$, then the value of the Euler form q_A at $\dim_k X + \dim_k S^n$ equals*

$$\sum_{i=0}^{\infty} (-1)^i \, {}^i(X, X)_B + n^2 - {}^0(M^n, X)_B.$$

Proof. We have

$$q_A(\dim_k X + \dim_k S^n) = \sum_{i=0}^{\infty} (-1)^i \, {}^i(X, X)_A$$
$$+ \sum_{i=0}^{\infty} (-1)^i \, {}^i(S^n, S^n)_A + \sum_{i=0}^{\infty} (-1)^i \, {}^i(X, S^n)_A + \sum_{i=0}^{\infty} (-1)^i \, {}^i(S^n, X)_A.$$

Since S is an injective A-module, we have ${}^i(S^n, S^n)_A = 0$ and ${}^i(X, S^n)_A = 0$, for $i \geqslant 1$. Moreover, ${}^0(X, S^n)_A = 0$, because no factor of X is isomorphic to S, and ${}^0(S^n, S^n)_A = n^2$, since S is a simple A-module. Hence,

$$q_A(\dim_k X + \dim_k S^n) = \sum_{i=0}^{\infty} (-1)^i \, {}^i(X, X)_A + n^2 + \sum_{i=0}^{\infty} (-1)^i \, {}^i(S^n, X)_A.$$

We replace ${}^0(S^n, X)_A - {}^1(S^n, X)_A$ by ${}^0(P(S)^n, X)_A - {}^0(M^n, X)_A$, and ${}^i(S^n, X)_A$ for $i \geqslant 2$ by ${}^{i-1}(M^n, X)_A$ because of $(*)$ and $(**)$. We have ${}^0(P(S)^n, X)_A = 0$, since no factor of X is isomorphic to S. Now,

$$q_A(\dim_k X + \dim_k S^n)$$
$$= \sum_{i=0}^{\infty} (-1)^i \, {}^i(X, X)_A + n^2 - {}^0(M^n, X)_A + \sum_{i=2}^{\infty} (-1)^i \, {}^{i-1}(M^n, X)_A.$$

Finally, since B-mod is a full subcategory of A-mod closed under extensions, ${}^i(Z, Z')_A = {}^i(Z, Z')_B$ for all B-modules Z, Z' and $i \geqslant 0$. So,

$$q_A(\dim_k X + \dim_k S^n)$$
$$= \sum_{i=0}^{\infty} (-1)^i \, {}^i(X, X)_B + n^2 - {}^0(M^n, X)_B + \sum_{i=2}^{\infty} (-1)^i \, {}^{i-1}(M^n, X)_B.$$

Since $\mathrm{Hom}_B(M, X_i) \neq 0$ for any indecomposable direct summand X_i of X and since $\boldsymbol{C}$ is a preinjective component we see by 2.3 that ${}^i(M^n, X)_B = 0$ for $i \geqslant 1$. Hence,

$$q_A(\dim_k X + \dim_k S^n) = \sum_{i=0}^{\infty} (-1)^i \, {}^i(X, X)_B + n^2 - {}^0(M^n, X)_B.$$

THEOREM 2.8. *Let $\boldsymbol{C}$ be a preinjective component of tree type. If* $\operatorname{ind}\boldsymbol{K}$ *contains some full subcategory $\boldsymbol{L}$ which coincides with one of the vector space categories of List A and if the connected components of $\boldsymbol{L}$ are path-incomparable, then the Euler form q_A is strongly indefinite.*

Proof. Because of Lemma 2.2, we may assume that $\operatorname{ind}\boldsymbol{K}$ contains as full subcategory $\boldsymbol{L}$ one of the categories of List A with the additional property that the morphisms of $\boldsymbol{L}$ correspond to arrows of $\boldsymbol{C}$ and that any chain $\boldsymbol{H}$ of $\boldsymbol{L}$ is sectional.

Let $\{x_1, \ldots, x_s\}$ be the set of objects of $\boldsymbol{L}$ and $\{X_1, \ldots, X_s\}$ the corresponding set of preinjective indecomposable modules. Let X be the direct sum $(X_1)^{n_1} \oplus \ldots \oplus (X_s)^{n_s}$. Since $\operatorname{Ext}^i_B(U, U)$ is 0 for any preinjective indecomposable B-module U and for $i \geqslant 1$, we see by Lemmata 2.4 and 2.6 that ${}^i(X, X)_B = 0$ for $i \geqslant 1$. Hence, the formula for $q_A(\dim_k X + \dim_k S^n)$ obtained by 2.7 is ${}^0(X, X)_B + n^2 - {}^0(M^n, X)_B$.

Now, using the fact that for any two path-incomparable connected components $\boldsymbol{X}$ and $\boldsymbol{Y}$ of $\operatorname{ind}\boldsymbol{K}$ and for any $x \in \boldsymbol{X}$ and any $y \in \boldsymbol{Y}$ we have $\operatorname{Hom}_B(X, Y) = 0$ and $\operatorname{Hom}_B(Y, X) = 0$, we are able to determine the value of q_A at $\dim_k X + \dim_k S^n$ for cases (I)–(VIII):

(I) $\displaystyle\sum_{s=1}^{5} n_s^2 + n^2 - n(n_1 + \ldots + n_5).$

(II) $\displaystyle\sum_{s=1}^{5} n_s^2 + n^2 + n_4 n_5 - n(n_1 + \ldots + n_5).$

(III) $\displaystyle\sum_{s=1}^{7} n_s^2 + n^2 + n_1 n_2 + n_3 n_4 + n_5(n_6 + n_7) + n_6 n_7 - n(n_1 + \ldots + n_7).$

(IV) $\displaystyle\sum_{s=1}^{8} n_s^2 + n^2 + n_2(n_3 + n_4) + n_3 n_4 + n_5(n_6 + \ldots + n_8)$

$+ n_6(n_7 + n_8) + n_7 n_8 - n(n_1 + \ldots + n_8).$

(V) $\displaystyle\sum_{s=1}^{9} n_s^2 + n^2 + n_2 n_3 + n_4(n_5 + \ldots + n_9) + n_5(n_6 + \ldots + n_9)$

$+ n_6(n_7 + \ldots + n_9) + n_7(n_8 + n_9) + n_8 n_9 - n(n_1 + \ldots + n_9).$

(VI) $\sum_{s=1}^{9} n_s^2 + n^2 + n_1 n_2 + n_2 n_3 + n_3 n_4 + n_5(n_6 + \ldots + n_9)$

$+ n_6(n_7 + \ldots + n_9) + n_7(n_8 + n_9) + n_8 n_9 - n(n_1 + \ldots + n_9).$

(VII) $\sum_{s=1}^{2} n_s^2 + n^2 - n(n_1 + 2n_2).$

(VIII) $n_1^2 + n^2 - mnn_1.$

In each case, the form q_A evaluated at the vector $v = (n, n_1, \ldots, n_s)$ given below takes negative value.

(I) (2, 1, 1, 1, 1, 1), (II) (4, 2, 2, 2, 1, 1), (III) (6, 2, 2, 2, 2, 2, 1, 1),
(IV) (8, 4, 2, 2, 2, 2, 2, 1, 1), (V) (30, 15, 10, 10, 4, 4, 4, 4, 4, 6),
(VI) (12, 4, 3, 2, 5, 2, 2, 2, 2, 2), (VII) (2, 1, 2), (VIII) (1, 1).

Clearly, the assumptions of 2.8 imply that A is of wild representation type. This justifies the title of our paper.

Under the same assumptions as in the above theorem we obtain

COROLLARY 2.9. *If* gl.dim $A \leqslant 2$, *then the Tits form* t_A *is strongly indefinite.*

Note. There is in preparation a joint paper with J. A. de la Peña, where Corollary 2.9 is shown to be valid in a much more general situation.

References

[1] R. Bautista and S. Brenner, *Replication numbers for non-Dynkin sectional subgraphs in finite Auslander–Reiten quivers and some properties of Weyl roots*, Proc. London Math. Soc. (2) 27 (1983), 212–220.

[2] K. Bongartz, *Algebras and quadratic forms*, J. London Math. Soc. (2) 28 (1983), 461–469.

[3] P. Gabriel, *Représentations indécomposables*, in: Sém. Bourbaki (1973/74), Lecture Notes in Math. 431, Springer, Berlin 1975, 1–71.

[4] —, *Auslander–Reiten sequences and representation-finite algebras*, in: Proc. ICRA II (Ottawa 1979), Lecture Notes in Math. 831, Springer, Berlin 1980, 1–71.

[5] N. Marmaridis, *One point extensions of linear quivers and quadratic forms*, J. Pure Appl. Algebra, to appear.

[6] —, *One point extensions of trees and quadratic forms*, Fund. Math. 134 (1990), 15–35.

[7] L. A. Nazarova, *The representations of partially ordered sets of infinite representation type*, Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), 963–991 (in Russian).

[8] J. A. de la Peña, *On the representation type of one point extensions of tame concealed algebras*, Manuscripta Math., to appear.

[9] C. M. Ringel, *On algorithms for solving vector space problems II. Tame algebras*, in: Proc. Workshop ICRA II (Ottawa 1979), Lecture Notes in Math. 831, Springer, Berlin 1980, 137–287.

[10] —, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math. 1099, Springer, Berlin 1984.