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THE VANISHING INVARIANTS IN INTERSECTION HOMOLOGY OF COMPLEX SPACES

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The vanishing theorem for the intersection homology of a singular Stein space depends in particular on the local vanishing invariant $b(p, q)$ introduced in [FiKp] for perversities p and q. Hence, we give an estimate for this number, and also for the other invariant $a_0(p, q)$, which came up in the local duality theory for intersection homology, in terms of holomorphic data of the pure dimensional complex space X; the general properties of these invariants, like monotony, and their significance in duality theory have been studied in [FiKp]. The case of a complex space that is a set-theoretic complete intersection at every point turns out to be particularly simple. In the general case we have to measure how much a germ X_x differs from a complete intersection: for $x \in X$ set

$$t_x := \min\{r;\ X_x \text{ is homeomorphic to a germ } V(C_0^N; f_1, \ldots, f_{N-n+r})\}.$$

Hence, $t_x = 0$ means that X_x is homeomorphic to a complete intersection. Furthermore, for $\dim^{\mathbf{C}} X = m$, set

$$t_i := \max\{t_x;\ x \in S_{2m-2i}\} \quad \text{and} \quad \operatorname{tab}(X) := \max\{t_x;\ x \in X\}.$$

The main estimate is this: let

$$b := \min_{d \leqslant i}(i - t_i - 2),$$

where d is the complex codimension of the analytic subset of X, in which X is not a homology manifold; then, for arbitrary perversities $p \subset q$ (cf. [KpFi, Sect. 2]), the invariants $a_0(p, q)$ and $b(p, q)$ are at least b. Moreover, if p is any perversity, and $\tilde{X}$ the normalization of X, then

$$m - 1 - \operatorname{tab}(\tilde{X}) \subset p \subset m + 1 + \operatorname{tab}(\tilde{X}).$$

That means in particular: if $\tilde{X}$ is a local complete intersection, then all perversities $p \leqslant m-1$ resp. $p \geqslant m+1$ are dualizing and quasi-isomorphic to o resp. to t.

In the second section we study in detail the intersection homology of cones over a smooth hypersurface in a projective space $\boldsymbol{P}_d$ for fairly general coefficients. The results have been used in [FiKp, Sect. 4].

1. An estimate for the vanishing invariants

Throughout the paper, X denotes a complex space of pure complex dimension m,

$$X = (X = X_{2m} \supset \Sigma \supset \ldots \supset X_{2(m-d)} \supset \ldots \supset X_{2(m-u)} \supset \neq X_{2(m-u-1)} = \emptyset)$$

a complex Whitney stratification satisfying condition A and B, R a PID, L a locally constant sheaf on X with finitely generated stalks and d the maximal natural number i such that $U_{2i} := X \setminus X_{2(m-i)}$ is an R-homology manifold. We use the notation of [Bo, V] and [FiKp], which differs a little bit from that in the basic articles [GoMPh$_1$] and [GoMPh$_2$].

1.1. Definition. Let $S^{\cdot}$ be a complex of sheaves on X, assume that $S^{\cdot}$ is X-cc. For an open subset $W \subset X$ set

$$a_W(S^{\cdot}) := \sup\{a;\ H^j S^{\cdot}|_W = 0, j \leqslant a\},$$

$$b_W(S^{\cdot}) := \sup\{b;\ H^j S^{\cdot}|_{W \cap S_{2(m-i)}} = 0,\ j \geqslant 2i-1-b,\ d \leqslant i \leqslant u\}.$$

For a family of supports ϕ on X set

$$a_\phi(S^{\cdot}) := \max\{a_W(S^{\cdot});\ E(\phi) \subset W \subset X\},$$

$$b_\phi(S^{\cdot}) := \max\{b_W(S^{\cdot});\ E(\phi) \subset W \subset X\},$$

$$a_{\phi o}(S^{\cdot}) := \begin{cases} a_\phi(S^{\cdot}), & \text{if } H^{a+1} S^{\cdot} \text{ is torsionfree for } a := a(S^{\cdot}), \\ a_\phi(S^{\cdot}) - 1 & \text{otherwise.} \end{cases}$$

If $p \subset q$ are two perversities and

$$\begin{tikzcd} P_p^{\cdot} L \arrow[rr, dashed] & & P_q^{\cdot} L \arrow[dl] \\ & Q_{pq}^{\cdot} L \arrow[ul, "{[1]}"] & \end{tikzcd}$$

is the associated distinguished triangle (for the definition of the relation "$\subset$" cf. [FiKp, 2.3]), then we set in particular

$$a(p, q) := a_\phi^L(p, q) := a(Q_{pq}^{\cdot} L),$$

$$b(p, q) := b_\phi^L(p, q) := b(Q_{pq}^{\cdot} L),$$

$$a_0(p, q) := a_{\phi o}^L(p, q) := \min\{a(Q_{pq}^{\cdot}(\operatorname{Tors} L)),\ a_0(Q_{pq}^{\cdot}(L/\operatorname{Tors} L))\}.$$

The properties of these invariants have been discussed in [FiKp]. We have to use Lemma 0.1 in [Kp$_2$] in this form:

$$(1.1.1)\qquad H^j D_X R[-2m]|_{S_{2m-2i}} \cong \mathscr{H}^R_{2m-j}|_{S_{2m-2i}} = 0 \quad \text{for } j \geqslant i+t_i+1.$$

As a consequence we obtain this result:

1.2. THEOREM. *Set* $b := \min_{d \leqslant i \leqslant u} \{i-t_i-2\}$. *If* $p \subset q$, *then*

$$a_0(p, q) \geqslant b \quad \text{and} \quad b(p, q) \geqslant b.$$

Proof. Let us first prove $a_0(p) = a_0(o, p) \geqslant b$ for $L = R$. By [FiKp, 3.6] we may assume $p = t$. If $i \geqslant d$, then $H^j D_X R[-2m]|_{S_{2m-2i}} = 0$ for $j \geqslant 2i-b-1$ by (1.1.1). Now consider the distinguished triangle

$$\begin{tikzcd} P_0^{\cdot} \arrow[rr, "\phi \circ \mu_{0t}"] & & D_X R[-2m] \arrow[dl] \\ & Q^{\cdot} \arrow[ul, "{[1]}"] & \end{tikzcd}$$

where $\phi: P_t^{\cdot} \to D_X R[-2m]$ denotes the canonical morphism, cf. [Bo, V.9.4]. We obtain $H^j Q^{\cdot}|_{S_{2m-2i}} = 0$ for $j \geqslant 2i-b-1$ and every i; thus, by [FiKp, 3.8] and biduality, the sheaves $H^j DQ^{\cdot}[-2m]$ vanish for $j \leqslant b+1$ and $H^{b+2} DQ^{\cdot}[-2m]$ is torsionfree. Since $a_0(t) \geqslant 0$, we may assume $b \geqslant 0$. Then the dual triangle

$$\begin{tikzcd} P_t^{\cdot} \cong DP_0^{\cdot}[-2m] \arrow[dr, "{[1]}"'] & & {}_X R \arrow[ll] \\ & DQ^{\cdot}[-2m] \arrow[ur] & \end{tikzcd}$$

yields: $H^j P_t^{\cdot} = 0$ for $1 \leqslant j \leqslant b$, and $H^{b+1} P_t^{\cdot}$ is torsionfree. Since, by [KpFi, 6.5 (iii)], $a^T(p) \geqslant a_0^R(p)$ for a torsion module T, we obtain $a_0^L(p) \geqslant a_0^R(p) \geqslant b$. By 2.6, 3.6, and 3.7 of [FiKp], this implies that

$$a_0(p, q) \geqslant a_0(o, q) \geqslant a_0(o, t) = a_0(t) \geqslant b$$

and

$$b(p, q) \geqslant b(o, t) = a_0(o, t) = a_0(t) \geqslant b. \qquad \square$$

In the special case that $b = d-2$ the situation is particularly simple for sufficiently large perversities: we say that a normal space X is of *type* E *with respect to* L if

$$H^j P_t^{\cdot}(L/\mathrm{Tors}\, L)|_{S_{2(m-d)}} = \begin{cases} 0, & \text{if } j \geqslant 1, j \neq d, \\ \text{nonzero torsion}, & \text{if } j = d. \end{cases}$$

Set $c := d$, if X is of type E, and $c := d-1$ otherwise.

1.3. COROLLARY. *Let $o+c \subset p$, let X be normal and L be locally free. Assume that $d = \min_{d \leqslant i}(i-t_i)$ and $d \geqslant 2$, then $a_0(p) = d-2 \leqslant b(p) \leqslant d-1$, and $b(p) = d-1$ iff X is not of type* E *and $p(2d) = d-1$.*

Proof. While Theorem 1.2 implies that

$$a_0(p) \geqslant d-2 \quad \text{and} \quad b(p) \geqslant d-2,$$

[KpFi, 3.5] and [KpFi, 3.7] yield

$$b(p) \geqslant b(t) = a_0(t) = a_0(p) \leqslant a(p) \leqslant c-1.$$

Since $a(p)+b(p)+3 \leqslant k_p = 2d$ by [FiKp, 3.3], we obtain $d-2 \leqslant b(p) \leqslant d-1$. If X is not of type E, then $c = d-1$ and thus $a_0(p) = d-2$; if X is of type E, then $a_0(p) = d-2$ follows from the very definition. For the rest we consider a point $x \in S_{2(m-d)}$; then Poincaré duality for the link L of x yields

$$H^j P^{\cdot}_{t,x} = \begin{cases} R^l \oplus T, & \text{if } j = d, \\ R^l, & \text{if } j = d-1, \\ 0, & \text{if } j \neq 0, d-1, d. \end{cases}$$

Hence, $b(p) = d-2$ if $p(2d) \geqslant d$. That holds in particular if X is of type E. Finally, for $p(2d) = d-1$, we obtain by [FiKp, 3.12] that $b(p) \geqslant 2d-2-p(2d) = d-1$ and thus $b(p) = d-1$. □

The invariant $b(-)$ is of "*real* nature", i.e., it compares the vanishing of the local intersection homology on a stratum S with the real codimension of S: the significant line, above which all local intersection homology groups vanish, is a line parallel to the graph of the perversity t. In the complex case the naturally arising (local) vanishing conditions depend on the *complex* codimension of S: the characteristic line is a line parallel to that defined by the middle perversity m. Here is an example, which generalizes [GoMPh, 5.6.2]:

1.4. PROPOSITION. *If p is an arbitrary perversity, then*

$$m-\operatorname{tab}(\tilde{X})-1 \subset p \subset m+\operatorname{tab}(\tilde{X})+1.$$

In particular, a perversity q is dualizing on X if $q \subset m-\operatorname{tab}(\tilde{X})-1$ or if $m+\operatorname{tab}(\tilde{X})+1 \subset q$.

Proof. Since "$\subset$" is an antisymmetric relation, it is easy to see that the first statement is equivalent to this:

$$t \cong m+\operatorname{tab}(\tilde{X})+1 \quad \text{and} \quad o \cong m-\operatorname{tab}(\tilde{X})-1.$$

We may assume that $X = \tilde{X}$ and thus

$$H^j P^{\cdot}_t \cong H^j D_X R^{\cdot}[-2m] \overset{L}{\otimes} L = {}_X\mathscr{H}^L_{2m-j},$$

since $H_X^{\cdot} P_p^{\cdot} \cong \pi_* H_{\tilde{X}}^{\cdot} P_p^{\cdot}$ as in the proof of [KpFi, 2.1]. Then, for $p = t$ and $q = m + \mathrm{tab}(X) + 1$, condition (iii) of [FiKp, 1.3] is satisfied by (1.1.1); consequently $t \cong m + \mathrm{tab}(X) + 1$, and in particular $lt \leqslant m + \mathrm{tab}(X) + 1$.

Now [FiKp, 1.7] with $p = t$ yields

$$o = t^* \cong (lt)^* \geqslant (m + \mathrm{tab}(X) + 1)^* = m - \mathrm{tab}(X) - 1.$$

For the second part it suffices to consider the case that $m + \mathrm{tab}(X) + 1 \subset q$, as we may go over to the complementary perversities, by [FiKp, 1.6 v)]. In that situation $lt \subset q$, and [FiKp, 1.8] applies with $p = t$. □

2. Intersection homology of iterated projective cones

We want to illustrate the results of the first section by considering a special class of singular varieties, which is also of interest in connection with the theorems of Lefschetz type in intersection homology. Let Z denote a smooth hypersurface $Z = V(\boldsymbol{P}_d; f)$ in some projective space $\boldsymbol{P}_d$. Up to homeomorphy, Z is known to depend only on the degree g of f, since the complex manifolds of this type form a smooth holomorphic family over a Zariski-open subset of some complex number space (of course the polynomial f has to be irreducible). The proof of that fact can be extended to the corresponding family of iterated cones of a fixed dimension over Z. Since a typical such manifold Z is the hypersurface $Z := V(\boldsymbol{P}_d; \sum_{j=0}^{d} z_j^g)$, in complex dimension m such an iterated projective cone is homeomorphic to the projective algebraic variety

$$_m X_d^g := V(\boldsymbol{P}_{m+1}; \sum_{j=0}^{d} z_j^g).$$

In the case of isolated singularities ($d = m$) and rational coefficients the intersection homology has been calculated in [Bo, I.5.4].

We use these notations: For $r \in \boldsymbol{N}$ set $\boldsymbol{Z}_r := \boldsymbol{Z}/r\boldsymbol{Z}$, in particular $\boldsymbol{Z}_0 \cong \boldsymbol{Z}$; the endomorphism of $\boldsymbol{Z}_r$ determined by the multiplication with a natural number g is indicated by "$\cdot g$"; the abstract abelian group $H^j(\boldsymbol{P}_m, \boldsymbol{Z}_r)$ is written as H^j.

2.1. THEOREM. *Let the m-dimensional complex space X be an iterated projective cone over a smooth hypersurface of degree g in* $\boldsymbol{P}_d$ *with* $d, g \geqslant 2$, *and set* $b := \dfrac{g-1}{g}((g-1)^d - (-1)^d)$.

Then, for $d \leqslant m$ *or* $p = m$, *X has the following hypercohomology* $H^j(X, P_p^{\cdot} \boldsymbol{Z}_r)$:

j \ p	o	μ^j_{om}	m	μ^j_{mt}	t	μ^j_{ot}
$\leqslant d-2$		$\cong$	H^j	$\cong$	H^j	$\cong$
$d-1$		direct $\hookrightarrow$	$H^j \oplus Z_r^b$	$\cong$	$H^j \oplus Z_r^b$	direct $\hookrightarrow$
$d \leqslant j \leqslant 2m-d,\ d+j \equiv 0\ (2)$	H^j	$\twoheadrightarrow$	gH^j	$\hookrightarrow$		$\cdot g$
$d \leqslant j \leqslant 2m-d,\ d+j \equiv 1\ (2)$		direct $\hookrightarrow$	$H^j \oplus Z_r^b$	$\twoheadrightarrow$	H^j	
$2m-d+1$	$H^j \oplus Z_r^b$	$\cong$	$H^j \oplus Z_r^b$	$\twoheadrightarrow$		$\twoheadrightarrow$
$\geqslant 2m-d+2$	H^j	$\cong$	H^j	$\cong$		$\cong$

For $d = m+1$ we have $\boldsymbol{H}^j(X, P_p) \cong \boldsymbol{H}^j(x, P_m)$ for every perversity p.

Proof. We may assume that $X = V(\boldsymbol{P}_{m+1};\ \sum_{i=0}^{d} z_i^g)$. We obviously have $d \leqslant m+1$. Since $d \geqslant 2$, the hypersurface X is a normal variety; its singular locus $\Sigma = V(\boldsymbol{P}_{m+1}; z_0, \ldots, z_d) \cong \boldsymbol{P}_{m-d}$ is of (complex) codimension d. In particular, X is a manifold iff $m = d+1$. We shall prove Theorem 2.1 in three steps. First of all we investigate the homomorphisms μ^j_{ot}. In a second step we use them to calculate the local intersection homology of X. Eventually we describe the global intersection homology data that concern the perversity m. Denote with ψ: $X \hookrightarrow \boldsymbol{P}_{m+1}$ the canonical inclusion, moreover, set $P_p^{\cdot} := P_p^{\cdot} \boldsymbol{Z}_r$.

(a) The calculation of the groups $\boldsymbol{H}^j(X, P_o^{\cdot}) \cong H^j(X, \boldsymbol{Z}_r)$ and $\boldsymbol{H}^j(X, P_t^{\cdot}) \cong H_{2m-j}(X, \boldsymbol{Z}_r)$ follows by means of universal coefficient formulas from the corresponding result for $r = 0$ in [Kp$_1$, Beispiel 3.1]. By Theorem 1.2, all invariants $a_0(p, q)$ and $b(p, q)$ are at least $d-2$, since X is a complete intersection. An application of the Main Lemma now yields the desired properties of μ^j_{ot} except that μ^{d-1}_{ot} is a direct inclusion (we shall prove that in part (c)) and that μ^j_{ot} is multiplication by g in the middle dimensions. There exists a commutative diagram

$$(2.1.1)\qquad \begin{array}{ccccc} \boldsymbol{H}^j(X, P_0^{\cdot}) & \cong & H^j(X) & \xleftarrow{\ \psi^j\ } & H^j(P_{m+1}) \\ \downarrow{\scriptstyle \mu^j_{ot}} & & \downarrow{\scriptstyle P_{2m-j}}\ {\scriptstyle \ldots\frown[X]} & & \downarrow{\scriptstyle \gamma}\ {\scriptstyle \ldots\frown[X]} \\ \boldsymbol{H}^j(X, P_t^{\cdot}) & \cong & H_{2m-j}(X) & \xrightarrow{\ \psi_{2m-j}\ } & H_{2m-j}(P_{m+1}) \end{array}$$

The homomorphism γ is multiplication by g, since $[X]$ is homologous to $g[\boldsymbol{P}_m]$ in $\boldsymbol{P}_{m+1}$. The missing description of μ^j_{ot} follows in particular if we can prove:

the homomorphisms ψ_j and ψ^j are
bijective for $j \leqslant 2m-d$,

multiplication by g for $j \geqslant 2m-d+2$;

ψ_{2m-d+1} is surjective and ψ^{2m-d+1} is injective.

For the properties of ψ_j we shall consider the exact homology sequence of the pair $(\boldsymbol{P}_{m+1}, X)$. Since $\boldsymbol{P}_{m+1} \setminus X$ is the total space of a complex vector bundle of rank $m-d+1$ over the d-dimensional Stein manifold $W := \boldsymbol{P}_d \setminus {}_{d-1}X^g_d$, the vanishing theorem for singular cohomology of Stein manifolds [AnFr] yields

$$H_i(\boldsymbol{P}_{m+1}, X) \cong H^{2m+2-i}(\boldsymbol{P}_{m+1} \setminus X) \cong H^{2m+2-i}(W) = 0 \text{ for } 2m+2-i \geqslant d+1.$$

An analogous argument applies to ψ^j. In order to show that ψ_j is multiplication by g for $j \geqslant 2m-d+2$, note that ψ^j and μ^j_{0t} are isomorphisms for $j \leqslant d-2$ in diagram (2.1.1).

(b) Near Σ the variety X is of the form $\boldsymbol{C}^{m-d} \times V(\boldsymbol{C}^{d+1}; \sum_{i=0}^{d} z_i^g)$; hence, $X = (X, \Sigma)$ is a topological stratification.

We start with the computation of $H^{\cdot} P^{\cdot}_t$ in the special case that $r = 0$. Then, by the Künneth formula, it suffices to consider the case $m = d$ and to compute the local homology ${}_X\mathscr{H}^{\boldsymbol{Z}}_{j,v}$ for the vertex $v := [0, \ldots, 0, 1]$ of the projective cone $X = {}_dX^g_d$. Since then X is nothing but the Thom space of the complex line bundle

$$E := X \setminus \{v\} \to Y_d := {}_{d-1}X^g_d, \; [z_0, \ldots, z_d] \to [z_0, \ldots, z_{d-1}],$$

we obtain ${}_X\mathscr{H}^{\boldsymbol{Z}}_{j,v} \cong \tilde{H}^c_{j-1}(E_0)$, where $E_0 := E \setminus V(X; z_d)$ is the complement of the zero section. The Gysin sequence associated to the bundle E over Y_d is of the following form:

$$H_{j+1}(Y_d) \xrightarrow[\ldots \cap c_1(E)]{\gamma_{j+1}} H_{j-1}(Y_d) \to H^c_j(E_0) \to H_j(Y_d) \xrightarrow[\ldots \cap c_1(E)]{\gamma_j} H_{j-2}(Y_d).$$

Since $E \to Y_d$ is the pullback of the bundle $F := \boldsymbol{P}_{d+1} \setminus \{v\} \to \boldsymbol{P}_d$, we obtain the following commutative diagram

$$\begin{tikzcd}
H_j(Y_d) \arrow[r, "\psi_j"] \arrow[d, "\gamma_j"', "\cdots \cap c_1(E)"] & H_j(P_d) \arrow[d, "\cdots \cap c_1(F)"] \\
H_{j-2}(Y_d) \arrow[r, "\psi_{j-2}"] & H_{j-2}(P_d)
\end{tikzcd}$$

The Chern class $c_1(F)$ generates $H^2(\boldsymbol{P}_d)$, consequently the right vertical arrow is an isomorphism for $2 \leqslant j \leqslant 2d$. The properties of ψ_j in (a) yield (set $m = d-1$): the homomorphism γ_j is

bijective for $2 \leqslant j \leqslant d-2$,

surjective with kernel $\cong \boldsymbol{Z}^b$ for $j = d-1$,

injective with cokernel $\cong T$ for $j = d$, where

$$T := \begin{cases} \boldsymbol{Z}_g, & \text{if } d \equiv 0(2), \\ 0, & \text{if } d \equiv 1(2). \end{cases}$$

Thus, for the link L of X at v we obtain

$$\tilde{H}_j(L) \cong \tilde{H}_j(E_0) \cong \begin{cases} 0, & \text{if } j \leqslant d-2, \\ Z^b \oplus T, & \text{if } j = d-1. \end{cases}$$

Finally, by Poincaré duality for the compact manifold L of real dimension $2d-1$, we obtain for $j \geqslant d$

$$H_j(L) \cong H^{2d-1-j}(L) \cong \operatorname{Hom}(H_{2d-1-j}(L), Z) \cong \begin{cases} Z^b & \text{for } j = d, \\ 0, & \text{if } d+1 \leqslant j \leqslant 2d-2. \end{cases}$$

We now can calculate $H^j P^{\cdot}_{t,x} \cong H^j(L) \cong H_{2d-1-j}(L)$ for arbitrary r and $x \in \Sigma$: the universal coefficient formula for local homology yields

$$\tag{2.1.2} H^j P^{\cdot}_t Z_{r,x} = \begin{cases} Z_r^b \oplus \operatorname{Tor}(T, Z_r), & \text{if } j = d-1, \\ Z_r^b \oplus T \otimes Z_r, & \text{if } j = d, \\ 0, & \text{if } j \neq 0,\ d-1,\ d. \end{cases}$$

Note that for d even

$$\operatorname{Tor}(T, Z_r) \cong Z_{(r,g)} \cong T \otimes Z_r \quad \text{for } r \neq 0.$$

There is only one nontrivial stratum; hence, $P^{\cdot}_p := P^{\cdot}_p Z_r \cong \tau_{\leqslant p(2d)} P^{\cdot}_t Z_r$. By Proposition 1.4, there exist at most three different classes of quasi-isomorphic perversities, which may be represented by o, m and t. For $b \neq 0$ these classes are indeed all different, while for $b = 0$ (i.e., $g = 2$ and $d \equiv 0(2)$) the situation is as follows: if $r = 0$, then $o \cong m \not\cong t$; if $r \neq 0$, $r \equiv 0(2)$, then $o \not\cong m \not\cong t$; if $r \equiv 1(2)$, then $o \cong m \cong t$. Combining this with Proposition 1.4 and [FiKp, 1.6 (iv)] we easily obtain Corollary 2.2 below.

We now consider the complexes (again with coefficients in Z_r)

$$Q^{\cdot}_{om} \cong \tau^{\geqslant 1} \tau_{\leqslant d-1} P^{\cdot}_t \quad \text{and} \quad Q^{\cdot}_{mt} \cong \tau^{\geqslant d} P^{\cdot}_t.$$

From (2.1.2) we derive for $x \in \Sigma$

$$H^j Q^{\cdot}_{om,x} \cong \begin{cases} Z_r^b \oplus \operatorname{Tor}(T, Z_r), & \text{if } j = d-1, \\ 0 & \text{otherwise}, \end{cases}$$

$$H^j Q^{\cdot}_{mt,x} \cong \begin{cases} Z_r^b \oplus T \otimes Z_r, & \text{if } j = d, \\ 0 & \text{otherwise}. \end{cases}$$

Consequently, $a(m, t)$ and $b(o, m)$ are at least $d-1$, while $a(o, m)$ and $b(m, t)$ are at least $d-2$. Equality holds in all cases if $o \not\cong m \not\cong t$, thus in particular if $b \neq 0$. The complexes $Q^{\cdot}_{om}$ and $Q^{\cdot}_{mt}$ are particularly simple, since their cohomology sheaves are different from zero in at most one degree; moreover, these cohomology sheaves are locally constant and thus constant since Σ is simply connected.

(c) For the computation of $H^{\cdot}(X, P^{\cdot}_m)$ we first calculate the modules $H^{\cdot}(X, Q^{\cdot}_{om})$ and $H^{\cdot}(X, Q^{\cdot}_{mt})$. Since these complexes have nonvanishing cohomology in at most one dimension, a simple spectral sequence argument yields

$$\begin{aligned} H^j(X, Q^{\cdot}_{om}) &\cong H^{j-d+1}(\Sigma, H^{d-1}Q^{\cdot}_{om}) \\ &\cong \begin{cases} Z_r^b \oplus \operatorname{Tor}(T, Z_r), & \text{if } d-1 \leqslant j \leqslant 2m-d-1,\ d+j \equiv 1(2), \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

In the same way we obtain

$$H^j(X, Q^{\cdot}_{mt}) = \begin{cases} Z_r^b \oplus T \otimes Z_r, & \text{if } d \leqslant j \leqslant 2m-d,\ d+j \equiv 0(2), \\ 0, & \text{otherwise.} \end{cases}$$

Since μ_{mt}^{d-1} and μ_{om}^{2m-d+1} are bijective, the results obtained so far cover the statements of Theorem 2.1 except for $d \leqslant j \leqslant 2m-d$ (and the fact that the inclusion μ_{om}^{d-1} is direct, which we shall show as the final point of the proof); hence, we now restrict to the case $d \leqslant j \leqslant 2m-d$:

(i) For $d+j \equiv 0(2)$ there is an exact commutative diagram

$$\begin{tikzcd} & H^{j-1}(X, Q^{\cdot}_{mt}) = 0 \arrow[d] & \\ H^j(X, P^{\cdot}_0) \arrow[r] \arrow[dr, dashed, "\mu^j_{0t}"'] & H^j(X, P^{\cdot}_m) \arrow[r] \arrow[d] & H^j(X, Q^{\cdot}_{0m}) = 0 \\ & H^j(X, P^{\cdot}_t) & \end{tikzcd}$$

in particular, $H^j(X, P^{\cdot}_m)$ is isomorphic to $\operatorname{Im}\mu^j_{ot}$.

(ii) For $d+j \equiv 1(2)$ there exists an exact sequence

$$\begin{array}{ccccccccc} H^{j-1}(X, Q^{\cdot}_{0m}) & \longrightarrow & H^j(X, P^{\cdot}_0) & \longrightarrow & H^j(X, P^{\cdot}_m) & \longrightarrow & H^j(X, Q^{\cdot}_{0m}) & \longrightarrow & \operatorname{Ker}\mu^{j+1}_{0m} \longrightarrow 0 \\ \| \wr & & \| \wr & & & & \| \wr & & \| \wr \ (\text{by i}) \\ 0 & & H^j & & & & Z_r^b \oplus \operatorname{Tor}(T, Z_r) & & \operatorname{Tor}(T, Z_r) \end{array}$$

If d is odd or $r = 0$, then T is zero and the sequence reduces to a short split exact sequence of free Z_r-modules. If d is even and $r \neq 0$, then $H^j(X, P^{\cdot}_m)$ is isomorphic to the kernel of a surjective homomorphism $Z_r^b \oplus \operatorname{Tor}(T, Z_r) \to \operatorname{Tor}(T, Z_r)$. Consequently, $H^j(X, P^{\cdot}_m)$ contains exactly r^b elements for $r \neq 0$ and it suffices to show that $H^j(X, P^{\cdot}_m)$ includes a free module Z_r^b. To that end consider the distinguished triangle

(2.1.3)
$$\begin{tikzcd} P^{\cdot}_m Z \overset{L}{\otimes} Z_r \arrow[rr, "\varrho"] & & P^{\cdot}_m Z_r \arrow[dl] \\ & M^{\cdot} \arrow[ul, "{[1]}"] & \end{tikzcd}$$

of [KpFi, 5.15]. For the induced exact sequence in hypercohomology

$$\begin{array}{ccccc} H^{j-1}(X, M^{\cdot}) & \longrightarrow & H^j(X, P^{\cdot}_m Z \overset{L}{\otimes} Z_r) & \longrightarrow & H^j(X, P^{\cdot}_m Z_r) \\ & & \Vert\wr \; [\text{KpFi}, 5.13.1] & & \\ & & H^j(X, P^{\cdot}_m Z) \otimes Z_r & & \end{array}$$

it remains to show that $H^{j-1}(X, M^{\cdot})$ vanishes, if j is odd and d even. Since $H^{\cdot}(P^{\cdot}_m Z)$ is torsionfree, the universal coefficient formula [KpFi, (5.13.1)] yields $H^{\cdot}(P^{\cdot}_m Z \overset{L}{\otimes} Z_r) \cong H^{\cdot}(P^{\cdot}_m Z) \otimes Z_r$.

As a consequence of (2.1.3) and [KpFi, 5.15] since $a(t) \geqslant d-2$, we obtain for $x \in \Sigma$

$$H^j M^{\cdot}_x = \begin{cases} \mathrm{Tor}(T, Z_r), & \text{if } j = d-1, \\ 0, & \text{otherwise.} \end{cases}$$

As usual the global hypercohomology results as

$$H^{j-1}(X, M^{\cdot}) \cong H^{j-1-(d-1)}(\Sigma, \mathrm{Tor}(T, Z_r)),$$

and this module vanishes, since $j-d$ is odd.

We finally have to show that μ^j_{om} is a direct inclusion for $d-1 \leqslant j \leqslant 2m-d$ and $d+j$ odd. For j odd this is again obvious, thus we may assume that j is even and d is odd. In the case $r = 0$ the statement follows immediately from the fact that $H^j(X, Q^{\cdot}_{om} Z)$ is a free abelian group. Since the perversity m is dualizing, the morphism ϱ in (2.1.3) is a quasi-isomorphism, and, by the universal coefficient formula [KpFi, (5.13.1)], the splitting carries over from Z to Z_r. □

2.2. COROLLARY. *There exist precisely three different classes of perversities on X, which can be represented by o, m and t, except in the following cases:*

$g = 2$, d is even and r is odd (then $o \cong m \cong t$),

$r = 0$ (then $o \cong m \not\cong t$).

The perversity m is dualizing for $R = Z$ iff d is odd; all other perversities are dualizing in any case.

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