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CYCLES OF ISOTROPIC SUBSPACES AND FORMULAS FOR SYMMETRIC DEGENERACY LOCI

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We give a short proof of the formulas for symmetric and antisymmetric degeneracy loci. We use a presentation of those loci as the image of a proper morphism with a nontrivial generic fibre. A generalization for twisted symmetric and antisymmetric morphisms is also given.

Introduction

Formulas for degeneracy loci are among the most useful tools of intersection theory. Let E and F be vector bundles of ranks e and f on a scheme X and let $\varphi\colon F \to E$ be a bundle map between them. The locus of points in X where the rank of φ does not exceed a given integer r is called the *degeneracy locus of* φ, denoted by $D_r(\varphi)$ (or for short D_r). It was R. Thom who observed in [16] that the cohomology class of the degeneracy locus of a "general" map $\varphi\colon F \to E$ should be a polynomial in the Chern classes of E and F. The corresponding polynomial, generalizing Giambelli's formula for the degree of determinantal varieties, was found by Porteous in [14]:

$$[D_r] = \operatorname{Det}[c_{e-r-p+q}(E-F)], \qquad 1 \leqslant p, q \leqslant f-r.$$

In [7] and [6] the following variant was investigated: $F = E^\vee$ and $\varphi\colon E^\vee \to E$ is symmetric or antisymmetric (in the latter case we assume that r is even). More precisely, if φ is symmetric and general enough, then

$$[D_r] = 2^{e-r}\operatorname{Det}[c_{e-r-2p+q+1}(E)], \qquad 1 \leqslant p, q \leqslant e-r,$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

and for a sufficiently general antisymmetric morphism,

$$[D_r] = \mathrm{Det}\,[c_{e-r-2p+q}(E)], \qquad 1 \leqslant p, q \leqslant e-r-1.$$

The usual method to compute such formulas is first to "simplify" the equations of a degeneracy locus by constructing a commutative diagram of varieties

$$\begin{array}{ccl} Z & \hookrightarrow & \mathbf{G} \ (= \text{a Grassmannian bundle}) \\ \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\pi} \\ D_r & \hookrightarrow & X \end{array}$$

such that Z is the subscheme of zeros of a section of a vector bundle H on G, $\mathrm{codim}_G Z = \mathrm{rank}\, H$, and the morphism η: $Z \to D_r$ is proper and birational. Then using Grothendieck's fundamental formula $[Z] = c_{\mathrm{top}}(H) \cap [G]$ (see [5], Théorème 2), we push forward the class of Z via π to get the desired class of D_r.

The purpose of this paper is to show how certain geometric constructions with a nontrivial generic fibre can be applied to get the formulas for degeneracy loci. We illustrate this method in the situation of the degeneracy loci associated with maps with symmetries. In order to do this we use a geometric construction invented in [15] (see also [8]). This gives us a short proof of the formulas established previously in [7] and [6] by rather complicated combinatorial arguments. Moreover, we give a generalization of those formulas for twisted maps.

The results presented here were announced in [3], p. 52.

This paper is dedicated with gratitude to the mathematician whose works introduced me several years ago into the mysteries of intersection theory.

1. Notations, conventions and preliminaries

We will use the notations and conventions of [2]. If X is an algebraic scheme over a field K, then $A_{\cdot}(X)$ denotes the Chow group of cycles modulo rational equivalence. If X is smooth, then $A_{\cdot}(X)$ is a commutative ring with the multiplicative structure given by intersection theory. If $D \subset X$ is a purely dimensional subscheme then $[D] \in A_{\cdot}(X)$ is the class of the fundamental cycle associated with D, i.e. if $D = D_1 \cup \ldots \cup D_n$ is a minimal decomposition into irreducible components then

$$[D] = \sum_{i=1}^{n} (\mathrm{length}\, O_{D,D_i})[D_i]$$

where O_{D,D_i} is the local ring of D along D_i.

If E is a vector bundle over X then $c_i(E)$, the Chern classes of E, and $s_i(E)$, the Segre classes of E, as well as polynomials in them are treated as operators acting on $A_.(X)$. Let $I = (i_1, \ldots, i_k) \in \mathbf{Z}^k$ be an arbitrary sequence of integers. We will use the following *Schur polynomials*:

$$s_I(E) := \det [s_{i_p - p + q}(E)]_{1 \leqslant p,q \leqslant k}.$$

Recall that if $f\colon X \to Y$ is a proper morphism then it induces an additive map $f_*\colon A_.(X) \to A_.(Y)$ such that $f_*[V] = \deg(f|_V)[f(V)]$ if $\dim f(V) = \dim V$, and 0 otherwise. In particular, if f establishes a birational isomorphism of V and $f(V)$ then $[f(V)] = f_*[V]$. If X and Y are nonsingular then a morphism $f\colon X \to Y$ induces a ring homomorphism $f^*\colon A_.(Y) \to A_.(X)$. If X, Y are possibly singular and $f\colon X \to Y$ is flat or a regular imbedding, then there exists an (additive) Gysin morphism $f^*\colon A_.(Y) \to A_.(X)$. In particular, such a morphism exists if f is a section of a vector bundle (see [2], Corollary 6.5).

In general it is difficult to compute $f^*[W]$ for $W \subset Y$. But if $W \subset Y$ is irreducible, Cohen–Macaulay and $\operatorname{codim}_X f^{-1} W = \operatorname{codim}_Y W$, then $f^*[W] = [f^{-1} W]$, where $f^{-1} W$ stands for the schematic preimage of W in X (see [10], Lemma 9).

The following formula was proved in [7]. Let $E \to X$ be a vector bundle over X and let $\pi\colon G_r(E) \to X$ be the Grassmannian bundle of r-subbundles of E endowed with the tautological sequence $0 \to R \to E_G \to Q \to 0$ of vector bundles on $G = G_r(E)$.

LEMMA 1.1. *For* $\alpha \in A_.(X)$,

$$\pi_*\big(s_I(R) \cdot s_J(Q) \cap \pi^* \alpha\big) = s_{j_1 - r, \ldots, j_q - r, i_1, \ldots, i_r}(E) \cap \alpha,$$

where $I \in \mathbf{Z}^r$, $J \in \mathbf{Z}^q$ *and* $q = \operatorname{rank}(E) - r$.

By a *partition* we mean a sequence $I = (i_1, \ldots, i_k)$ of integers where $i_1 \geqslant \ldots \geqslant i_k \geqslant 0$. With every partition I one associates its *Ferrers' diagram* $D_I = \{(a, b) \in \mathbf{Z} \times \mathbf{Z},\ 1 \leqslant b \leqslant k,\ 1 \leqslant a \leqslant i_b\}$. For given two partitions $I = (i_1, i_2, \ldots)$, $J = (j_1, j_2, \ldots)$ we will write $I \subset J$ if $D_I \subset D_J$ (or equivalently, if $i_h \leqslant j_h$ for every h). Moreover, for a given partition I, denote by $I^\sim$ the *conjugate* partition of I defined by the condition: $(a, b) \in D_{I^\sim} \Leftrightarrow (b, a) \in D_I$.

COROLLARY 1.2. *Let I be a partition such that $I \subset (q, \ldots, q)$ (r times). Let $\bar{I} \subset (r, \ldots, r)$ (q times) be the partition for which $D_{\bar{I}}$ is the complement of $D_{I^\sim}$ in $(q, \ldots, q)$ (r times). Then in the above notations $\pi_*(s_I(R) \cdot s_J(Q) \cap \pi^* \alpha) = \pm \alpha$ if $J = \bar{I}$, and $= 0$ otherwise. In particular, if $X = \operatorname{Spec} K$ is a point, we see that $\pm s_{\bar{I}}(Q)$ is the Poincaré-dual cycle of $s_I(R)$ in the Chow ring of the Grassmannian $G_r(K^{r+q})$.*

COROLLARY 1.3. *If $J = (j_1, \ldots, j_q)$ is a partition, then $\pi_*\big(s_J(Q)\big) \neq 0$ only if $J \supset (r, \ldots, r)$ (q times).*

2. Use of constructions with a nontrivial generic fiber

Assume that we have a commutative diagram of schemes over a field K

$$\begin{array}{ccccc} W^0 & \hookrightarrow & W & \hookrightarrow & \mathbf{G} \\ \downarrow & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\pi} \\ D^0 & \overset{j}{\hookrightarrow} & D & \overset{i}{\hookrightarrow} & X \end{array}$$

with the following properties:

1° D is a closed, irreducible subscheme of a smooth scheme X, D^0 is an open subscheme of D. Let $W^0 = \pi^{-1}(D^0)$.

2° π is proper and $\pi(W) = D$.

3° There is an open covering $\{U_\alpha\}$ of X such that for every α, $\pi^{-1}(U_\alpha) \simeq U_\alpha \times G$, where G is a smooth variety and $A_{.}(G)$ satisfies the Poincaré duality.

4° There exists a closed subscheme $F \subset G$ such that for every α, $\pi^{-1}(D^0 \cap U_\alpha) = (D^0 \cap U_\alpha) \times F$ in $U_\alpha \times G$.

5° There exists a cycle $\mathbf{F}^d$ in $\mathbf{G}$ such that for every α, the restriction of $[\mathbf{F}^d]$ to $A_{.}(\pi^{-1} U_\alpha) = A_{.}(U_\alpha \times G)$ is $[U_\alpha \times F^d]$ where $[F^d]$ is the Poincaré-dual of $[F]$ in $A_{.}(G)$.

PROPOSITION 2.1. *Under the above assumptions, the following equality holds in* $A_{.}(X)$:

$$[D] = \pi_*([W] \cdot [\mathbf{F}^d]).$$

Proof. Let $y = [W] \cdot [\mathbf{F}^d] \in A_{.}(\mathbf{G})$, $x = [D]$ in $A_{.}(D)$. We have to check that $i_*(x) = \pi_*(y)$. Let $\mathbf{G}^0 = \pi^{-1}(X-(D-D^0))$ and $\pi^0 = \pi|_{\mathbf{G}^0}$: $\mathbf{G}^0 \to X-(D-D^0)$. Let k: $\mathbf{G}^0 \hookrightarrow \mathbf{G}$ be the inclusion. Moreover, denote by i^0 the inclusion $D^0 \hookrightarrow X-(D-D^0)$. Then it suffices to prove that the following equality holds in $A(X-(D-D^0))$:

$$(*) \qquad i^0_* j^*(x) = \pi^0_* k^*(y).$$

Indeed, this implies that the image of $i_*(x) - \pi_*(y)$ in $A(X-(D-D^0))$ is zero, i.e. is represented by a cycle on $D-D^0$. Since $\dim(D-D^0) < \dim D$ and both $i_*(x)$ and $\pi_*(y)$ are in $A_{\dim D}(X)$, $i_*(x) = \pi_*(y)$ in $A(X)$. The equality $(*)$ is equivalent to

$$(**) \qquad [D^0] = \pi^0_*([W^0] \cdot k^*[\mathbf{F}^d]).$$

To prove $(**)$ take $U = U_\alpha$ such that $D^0 \cap U \neq \emptyset$. Then using a dimension argument as above, together with 4°, 5°, we can reduce our problem to the situation where $\mathbf{G} = X \times G$, $W = D \times F$ and π: $X \times G \to X$, η: $D \times F \to F$ are the projections. Then the assertion follows from the following

CLAIM. *Let G, D, F, F^d be as above and let π_D: $D \times G \to D$ be the projection. Then the following equality holds in* $A_{.}(D)$:

$$(\pi_D)_*([D \times F] \cdot [D \times F^d]) = [D].$$

To prove the claim consider the map $p\colon D \to \mathrm{pt} = \operatorname{Spec} K$. It follows from the cartesian square

$$\begin{array}{ccc} D\times G & \xrightarrow{\pi_D} & D \\ {\scriptstyle p\times 1}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{pt}\times G & \xrightarrow{\tau} & \mathrm{pt} \end{array}$$

and [2], Proposition 1.7, that

$$(\pi_D)_*([D\times F]\cdot[D\times F^d]) = (\pi_D)_*((p\times 1)^*[\mathrm{pt}\times F]\cdot(p\times 1)^*[\mathrm{pt}\times F^d])$$
$$= (\pi_D)_*(p\times 1)^*([\mathrm{pt}\times F]\cdot[\mathrm{pt}\times F^d]) = p^*\tau_*(\mathrm{pt}\times\mathrm{pt}) = p^*(\mathrm{pt}) = [D]$$

as needed. ∎

3. A new proof of the formulas for symmetric degeneracy loci

Let $\varphi\colon E\to E^\vee$ be a symmetric (resp. antisymmetric) morphism of vector bundles over a purely dimensional scheme X over a field K. Let p be a natural number such that $2p \leqslant e = \operatorname{rank} E$. Let $\pi\colon \mathbf{G} = G_{e-p}(E)\to X$ be the Grassmannian bundle of $(e-p)$-subbundles of E. Let $R \hookrightarrow E_{\mathbf{G}}$ be the tautological bundle on $\mathbf{G}$. We define $W\subset \mathbf{G}$ as the subscheme of zeros of the composite morphism

$$R \hookrightarrow E_{\mathbf{G}} \xrightarrow{\pi^*\varphi} E_{\mathbf{G}}^\vee \twoheadrightarrow R^\vee .$$

Let $D = D_{2p}$. Let $D^0 = \{x\in X \mid \operatorname{rank}\varphi_x = 2p\}$ and define $W^0 = \pi^{-1}(D^0)$. We get a diagram

$$\begin{array}{ccccc} W^0 & \hookrightarrow W & \hookrightarrow & \mathbf{G} & = G_{e-p}(E) \\ \downarrow & \downarrow{\scriptstyle \eta=\pi|_W} & & \downarrow{\scriptstyle \pi} & \\ D^0 & \hookrightarrow D & \hookrightarrow & X & \end{array}$$

in which $\pi(W) = D$, because if φ_x has an isotropic subspace of dimension $e-p$ then $\operatorname{rank}\varphi_x \leqslant 2p$. The construction from Chap. 14.1 in [2] gives the localized top Chern class $\mathbf{W}\in A_m(W)$, where $m = \dim\mathbf{G} - \binom{e-p+1}{2}$ (resp. $m = \dim\mathbf{G} - \binom{e-p}{2}$). Let U be a vector space of dimension e. The fibre of π is $G_{e-p}(U)$. By construction the generic fiber of $\pi|_W$ is

$G^{\mathrm{iso}}_{e-p}(U) = \{L\subset U \mid L$ is an $(e-p)$-dimensional subspace which is isotropic with respect to a symmetric (resp. antisymmetric) form of rank $2p$ on $U\}$.

To define the determinantal class $\mathbf{D}\in A_{m-\dim(F^d)}(D)$ (F^d is the dual cycle to F in $G_{e-p}(U)$) we need the expression of the class $[F^d]$ in $A_.(G_{e-p}(U))$ in terms of the Schur polynomials applied to the tautological vector bundles on $G_{e-p}(U)$.

LEMMA 3.1. *The dimension of $G^{\mathrm{iso}}_{e-p}(U)$ is $(p^2-p)/2$ (resp. $(p^2+p)/2$).*

This can be calculated by applying the above construction to X being the affine space of $e\times e$ symmetric (respectively antisymmetric) matrices endowed with the canonical (tautological) morphism φ. We have $\operatorname{codim}_G(W) = \binom{e-p+1}{2}$ (resp. $\operatorname{codim}_G(W) = \binom{e-p}{2}$) (see [15], [8]). Then, using the formulas for $\dim D$ (see [9], [11]) we compute the dimension of the generic fiber which is $G^{\mathrm{iso}}_{e-p}(U)$.

Let ϱ_k denote the partition $(k, k-1, \ldots, 2, 1)$.

PROPOSITION 3.2. *Let $0\to R\to U_G\to Q\to 0$ be the tautological sequence on $G = G_{e-p}(U)$. Then*

(i) $2^p[F^d] = s_{\varrho_{p-1}}(R^\vee)$ *(resp.* $[F^d] = s_{\varrho_p}(R^\vee)$*),*

(ii) $2^p[F^d] = \varepsilon(p-1)s_{\varrho_{p-1}}(Q^\vee)$ *(resp.* $[F^d] = \varepsilon(p)s_{\varrho_p}(Q^\vee)$*), where* $\varepsilon(p) = (-1)^{1+2+\ldots+p}$.

Proof. Assume first that $e = 2p$, i.e., that the corresponding form is nondegenerate. Then by Grothendieck's formula $[F]$ is the top Chern class of the bundle $S_2R^\vee$ (resp. $\Lambda^2R^\vee$) (see [5], Théorème 2). Using the formulas from [12] (see also [13], I.10) we get

$$[F] = 2^p s_{\varrho_p}(R^\vee) \qquad (\text{resp. } [F] = s_{\varrho_{p-1}}(R^\vee)).$$

To prove (i) in this case, we recall that the Poincaré-dual cycle of $s_I(R^\vee)$ in $A_.(G_p(U^\vee)) = A_.(G_p(U))$ is $s_J(R^\vee)$ where D_J is the complement of D_I in the square $p\times p$ (see [2], Proposition 14.6.3). On the other hand, the assertion (ii) follows in this case from another description of the Poincaré-dual cycle with the help of the $s_J(Q^\vee)$'s, which is given in Corollary 1.2.

In general let U be a vector space endowed with a symmetric (resp. antisymmetric) form of rank $2p$. Then we can find subspaces U' and A of dimensions $2p$ and $e-2p$ such that $U = U'\oplus A$ and the form restricted to U' is nondegenerate. Then we have a (closed) imbedding $i\colon G' = G_p(U')\subsetneq G = G_{e-p}(U)$ defined by $L\mapsto L\oplus A$ for $L\in G_p(U')$. One sees easily (with the help of Lemma 3.1) that $i_*[G^{\mathrm{iso}}_p(U')] = [G^{\mathrm{iso}}_{e-p}(U)]$. Observe that the restriction of the tautological sequence $0\to R\to U_G\to Q\to 0$ on G to G' is $0\to R'\oplus A_{G'}\to U'\oplus A_{G'}\to Q'\to 0$ where R', Q' denote the tautological bundles of rank p on G'. Therefore if $\mathscr{P}(R^\vee)$ (resp. $\mathscr{P}(Q^\vee)$) is a polynomial in the Chern classes of $R^\vee$ (resp. of $Q^\vee$), then $i^*(\mathscr{P}(R^\vee)) = \mathscr{P}(R'^\vee)$ (resp. $i^*(\mathscr{P}(Q^\vee)) = \mathscr{P}(Q'^\vee)$). Write $[F^d] = 2^{-p}s_{\varrho_{p-1}}(R^\vee)$ (resp. $[F^d] = s_{\varrho_p}(R^\vee)$) in case (i), and $[F^d] = 2^{-p}\varepsilon(p-1)s_{\varrho_{p-1}}(Q^\vee)$ (resp. $[F^d] = \varepsilon(p)s_{\varrho_p}(Q^\vee)$) in case (ii). We conclude that

$$\begin{aligned}
[\mathrm{pt}] &= i_*[\mathrm{pt}] = i_*([G^{\mathrm{iso}}_p(U')]\cdot i^*[F^d]) \quad (\text{by the preliminary step})\\
&= i_*[G^{\mathrm{iso}}_p(U')]\cdot[F^d] \quad (\text{by the projection formula for } i)\\
&= [G^{\mathrm{iso}}_{e-p}(U)]\cdot[F^d],
\end{aligned}$$

which is the desired equality. ■

We define the *determinantal class* $\mathbf{D} \in A_{m-\dim(F^d)}(D)$ by $2^p \mathbf{D} = \eta_*(s_{\varrho_{p-1}}(R^\vee) \cap \mathbf{W})$, resp. $\mathbf{D} = \eta_*(s_{\varrho_p}(R^\vee) \cap \mathbf{W})$. It follows easily from the formula for the top Chern class of the second symmetric power (see [12], [13], I.10) that this definition makes sense. It is easy to see that the formation $(X, E, \varphi) \rightsquigarrow \mathbf{D}(\varphi\colon E \to E^\vee)$ commutes with Gysin maps and proper push forward in the sense of [2], Theorem 14.3 (d). Now we are ready to give a short proof of the degeneracy loci formulas for maps with symmetries.

THEOREM 3.3. (a) *The image of* $\mathbf{D}$ *in* $A_{\dim X - c}(X)$ *is equal to* $2^{e-2p} s_{\varrho_{e-2p}}(E^\vee) \cap [X]$ (*resp.* $s_{\varrho_{e-2p-1}}(E^\vee) \cap [X]$ *in the antisymmetric case*). *Here* $c = \binom{e-2p+1}{2}$ (*resp.* $c = \binom{e-2p}{2}$).

(b) *Each irreducible component of* D *has codimension at most* c *in* X. *If* $\operatorname{codim}_X D = c$, *then* $\mathbf{D}$ *is a positive cycle whose support is* D.

(c) *If* $\operatorname{codim}_X D = c$ *and* X *is Cohen–Macaulay, then* D *is Cohen–Macaulay and* $[D] = \mathbf{D}$.

Proof. (a) We give two proofs. By definition we have $\mathbf{W} = c_{\text{top}}(S_2 R^\vee) \cap [\mathbf{G}]$ (resp. $\mathbf{W} = c_{\text{top}}(\Lambda^2 R^\vee) \cap [\mathbf{G}]$). Therefore, using the formulas for the top Chern classes of the corresponding tensor operations (see [12], [13], I.10), we obtain

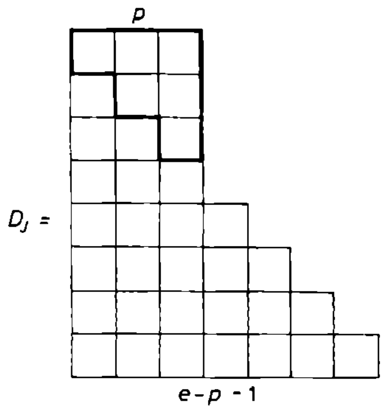
$$\mathbf{D} = 2^{e-2p} \pi_*(s_{\varrho_{p-1}}(R^\vee) \cdot s_{\varrho_{e-p}}(R^\vee) \cap [\mathbf{G}]),$$

$$\text{resp. } \mathbf{D} = \pi_*(s_{\varrho_p}(R^\vee) \cdot s_{\varrho_{e-p-1}}(R^\vee) \cap [\mathbf{G}]).$$

Consider, for example, the latter case. By the Littlewood–Richardson rule (see [13], I.9) we know that the product $s_{\varrho_p}(R^\vee) \cdot s_{\varrho_{e-p-1}}(R^\vee)$ is a sum $\sum_I \alpha_I s_I(R^\vee)$ where $|I| = |\varrho_{p-1}| + |\varrho_{e-p-1}|$(¹) and $\alpha_I \neq 0$ only if $I \supset \varrho_{e-p-1}$. There exists only one partition J among such I's for which $J \supset (p, \ldots, p)$ $(e-p$ times), namely the partition

$$J = (e-p-1, e-p-2, \ldots, p+1, p, \ldots, p).$$

Pictorially:



(¹) For a partition J we denote by $|J|$ the sum of its parts.

The Littlewood–Richardson rule implies that $s_J(R^\vee)$ appears in the above product with multiplicity 1. Finally, by Corollary 1.3 and Lemma 1.1 applied to the Grassmannian $G_p(E^\vee) \simeq \mathbf{G}$ (with the tautological quotient $E^\vee \twoheadrightarrow R^\vee$), the image of $\mathbf{D}$ in $A_{\dim X - c}$ is $s_{\varrho_{e-2p-1}}(E^\vee) \cap [X]$.

The second proof is even more elementary. It uses only Lemma 1.1 and does not use the Littlewood–Richardson rule. We have

$$\begin{aligned}\mathbf{D} &= \pi_*\left[\varepsilon(p) s_{\varrho_p}(Q^\vee)\cdot s_{\varrho_{e-p-1}}(R^\vee) \cap [\mathbf{G}]\right] \quad \text{(by Proposition 3.2 (ii))}\\ &= \varepsilon(p) s_{e-2p-1, e-2p-2, \dots, 1, 0, -1, \dots, -p+1, -p, p, p-1, \dots, 2, 1}(E^\vee) \cap [X]\\ &\qquad \text{(by Lemma 1.1 applied to the Grassmannian } G_p(E^\vee) \simeq \mathbf{G}\\ &\quad \text{endowed with the tautological sequence } 0 \to Q^\vee \to E^\vee_{\mathbf{G}} \to R^\vee \to 0)\\ &= s_{e-2p-1, e-2p-2, \dots, 2, 1}(E^\vee) \cap [X].\end{aligned}$$

Indeed, if we reorder the last segment of length $2p$ through the rule

$$s_{(\dots, i, i', \dots)}(E^\vee) = -s_{(\dots, i'-1, i+1, \dots)}(E^\vee)$$

we see that

$$\varepsilon(p) s_{(\dots, -1, -2, \dots, -p, p, p-1, \dots, 2, 1)}(E^\vee) = s_{(\dots, 0, \dots, 0)}(E^\vee).$$

To prove (b) and (c) we should pass to the "generic case". For a given symmetric (resp. antisymmetric) morphism of vector bundles on X we define $\bar{X} = S_2 E^\vee$ (resp. $\bar{X} = \Lambda^2 E^\vee$) treated as a scheme. Observe that φ induces a section s: $X \to \bar{X}$. There exists a canonical (tautological) bundle map $\bar{\varphi}$: $\bar{E} \to \bar{E}^\vee$ on $\bar{X}$, where $\bar{E} = E_{\bar{X}}$, such that $s^*(\bar{\varphi}) = \varphi$. If X is Cohen–Macaulay, then it was proved in [9] and [11] that $D(\bar{\varphi})$ is also Cohen–Macaulay. Therefore, if D is of codimension c in X, then from Lemma 9 in [10] we infer that D is Cohen–Macaulay. Now since the formation $(X, E, \varphi) \rightsquigarrow \mathbf{D}(\varphi$: $E \to E^\vee)$ commutes with Gysin maps and proper push forward, it suffices to prove the remaining assertions locally on $\bar{X}$. Therefore we can assume that X is the affine space of $e \times e$ symmetric (resp. antisymmetric) matrices over the field K. It follows from [15], [8] that the degeneracy locus under consideration is defined by the vanishing of the $(2p+1)$th order minors (resp. $(2p+2)$th order pfaffians) and it is an irreducible variety of codimension c. Moreover, in this case the equality $[D] = \mathbf{D}$ is a consequence of Proposition 2.1. ∎

Let us notice that the class of D_r with odd $r = 2p-1$, say, can be easily derived from the above formula. Namely, writing $\mathbf{1}$ for the trivial line bundle on X, consider the symmetric morphism $\varphi' = \varphi \oplus \mathbf{1}$: $E \oplus \mathbf{1} \to E^\vee \oplus \mathbf{1}$ of vector bundles on X. Then the ideals defined by $(2p-1)$th order minors of φ and the $2p$th order minors of φ' are equal. In particular, the codimension of $D_{2p}(\varphi')$ is $\binom{e-2p+1}{2}$. Using the linearity formula (see [13], I.5.9) and writing $q = e-2p+1$ we get

$$[D_{2p-1}(\varphi)] = [D_{2p}(\varphi')] = 2^q s_{\varrho_q}(E^\vee \oplus \mathbf{1}) \cap [X] = 2^q s_{\varrho_q}(E^\vee) \cap [X].$$

So summing up we have proved the following formulas:

THEOREM 3.4. (a) *Let* φ: $E^\vee \to E$ *be symmetric and let* $O_D = \mathrm{Coker}(\Lambda^{e-q+1}\varphi\colon \Lambda^{e-q+1}E^\vee \otimes \Lambda^{e-q+1}E^\vee \to O_X)$. *Then under the above assumptions*

$$[D] = 2^q s_{\varrho_q}(E) \cap [X].$$

(b) *Let* φ: $E^\vee \to E$ *be antisymmetric and let* D *be the subscheme defined by vanishing of the* $(2p+2)$*th order pfaffians of* φ. *Let* $q = e-2p$. *Then*

$$[D] = s_{\varrho_{q-1}}(E) \cap [X].$$

(A passage from the above formulas to the formulas stated in the Introduction follows from [13], I.(3.4), (3.5).)

4. A generalization for twisted morphisms with symmetries

In this section we will work with more general morphisms φ: $E^\vee \to E \otimes L$ of vector bundles on X, where L stands for a line bundle. Let us call φ *symmetric* (resp. *antisymmetric*) iff $\varphi^\vee \otimes \mathrm{id}_L = \varphi$ (resp. $\varphi^\vee \otimes \mathrm{id}_L = -\varphi$). Our goal is to describe the class of $D_r(\varphi)$ as a polynomial in the Chern classes of E and $\lambda = c_1(L)$. We need the following equality for the Schur polynomials in the variables $x_1, \ldots, x_e$ (see [12], [13], I.10):

$$(*) \qquad s_I(x_1+1, \ldots, x_e+1) = \sum_{J \subset I} d_{I,J}\, s_J(x_1, \ldots, x_e),$$

where

$$d_{I,J} = \mathrm{Det}\left[\binom{i_p+e-p}{j_q+e-q}\right], \qquad 1 \leqslant p, q \leqslant e.$$

THEOREM 4.1. *Under the assumptions of Theorem* 3.3, *the following formulas hold*:

(a) *If* φ *is symmetric, then*

$$[D_{e-q}(\varphi)] = 2^{-\binom{q}{2}} \sum_{J \subset \varrho_q} 2^{|J|} d_{\varrho_q, J}\, s_J(E) \cdot \lambda^{\binom{q+1}{2}-|J|} \cap [X].$$

(b) *If* φ *is antisymmetric and* $e-q$ *is even, then*

$$[D_{e-q}(\varphi)] = 2^{-\binom{q}{2}} \sum_{J \subset \varrho_{q-1}} 2^{|J|} d_{\varrho_{q-1}, J}\, s_J(E) \cdot \lambda^{\binom{q}{2}-|J|} \cap [X].$$

In order to prove these formulas we recall that the universal formula in question is the same as in the topological situation, i.e. when we replace X by a complex manifold, E by a C^∞-vector bundle on X and $A_\cdot(X)$ by the cohomology groups $H^\cdot(X, \mathbf{Z})$. Now, by the "squaring principle" (see [6], Proposition 9) we can take formally the square root M of the line bundle L, i.e. $L = M^{\otimes 2}$, and reduce the problem of calculating $[D_r(\varphi\colon E^\vee \to E\otimes L)]$ to the same problem about $[D_r(\varphi\otimes \mathrm{id}_M\colon (E\otimes M)^\vee \to E\otimes M)]$. The final formulas follow easily from Theorem 3.4 and (*).

Remark 4.2. The formulas stated in Theorem 4.1 generalize Giambelli's formulas for the degree of projective symmetric and antisymmetric determinantal varieties given in [4].

EXAMPLE 4.3. If φ is symmetric, $q = 2$, then

$$[D_{e-2}(\varphi)] = 4(c_1 c_2 - c_3) + 4c_2\cdot\lambda + 2(e-1)c_1^2\cdot\lambda + (e^2-1)c_1\cdot\lambda^2 + \binom{e+1}{3}\cdot\lambda^3$$

where $c_i = c_i(E)$. This formula has an application to counting the nodes of the discriminant surface. More precisely, let $\Delta \subset \mathbf{P}^3$ be a hypersurface of degree e given by the vanishing of the determinant of the symmetric matrix $[a_{ij}(z_0:z_1:z_2:z_3)]$, $1 \leqslant i, j \leqslant e$, which corresponds to a morphism φ: $O_{\mathbf{P}^3}^{\oplus e} \to O_{\mathbf{P}^3}(1)^{\oplus e}$. For sufficiently general $[a_{ij}]$ the number of nodes on Δ is $[D_{e-2}(\varphi)] \in A_2(\mathbf{P}^3) = \mathbf{Z}$. By the above formula this number is $\binom{e+1}{3}$ (see [1] for details).

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