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A CLASS OF PROJECTIVE REPRESENTATIONS OF HYPEROCTAHEDRAL GROUPS AND SCHUR Q-FUNCTIONS

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Introduction

Let H_n denote the hyperoctahedral group on n symbols, i.e. the wreath product of $\mathbf{Z}_2^n := \mathbf{Z}_2 \times \ldots \times \mathbf{Z}_2$ and S_n where $\mathbf{Z}_2$ is the cyclic group of order 2 and S_n is the symmetric group on n symbols acting on $\mathbf{Z}_2^n$ by the formula $sb_p = b_{s(p)}$ for generators $b_1, \ldots, b_n$ of $\mathbf{Z}_2^n$ and any $s \in S_n$.

It is well known (see [5]) that $H^2(H_n, \mathbf{C}^*) \approx \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ for $n \geqslant 4$ where H_n acts trivially on $\mathbf{C}^*$. We are concerned in this article with projective representations of H_n associated with the cocycle of type $\langle 1, 1, -1 \rangle$ (in the notation of [9]). Projective representations mentioned above are related with a class of linear representations of a double cover $\tilde{H}_n$ of H_n.

To define $\tilde{H}_n$ let us denote by G_n the group generated by $a_1, \ldots, a_n$ subject to relations $a_1^2 = a_2^2 = \ldots = a_n^2 = z$, $z^2 = 1$, $a_p a_q = z a_q a_p$ for $p, q = 1, \ldots, n$, $p \neq q$. G_n is a finite group of order 2^{n+1} and is a double cover of $\mathbf{Z}_2^n$; the projection sends a_p into b_p. The symmetric group S_n acts on G_n by $sa_p = a_{s(p)}$. We set $\tilde{H}_n$ to be the semidirect product of G_n and S_n defined by means of the above action. Obviously, we have an epimorphism of groups θ_n: $\tilde{H}_n \to H_n$ such that $\theta_n(a_p) = b_p$, $\theta_n(s) = s$ for $s \in S_n$ and $\operatorname{Ker} \theta_n = \{1, z\}$. A linear representation of $\tilde{H}_n$ (or an $\tilde{H}_n$-module) is called *negative* (see [2]) if z acts as multiplication by -1. The assignment $T \mapsto T\theta_n$ establishes a 1-1 correspondence between equivalence classes of projective representations of H_n associated with the cocycle of type $\langle 1, 1, -1 \rangle$ and equivalence classes of negative representations of $\tilde{H}_n$ (or negative $\tilde{H}_n$-modules).

The aim of this paper is to give a description of the ring of negative $\tilde{H}_n$-supermodules for all n in terms of Schur Q-functions along the lines presented in [7] for projective representations of S_n. The basic novelty in [7]

This paper is in final form and no version of it will be submitted for publication elsewhere.

is an application of the theory of semisimple superalgebras (see [6]) and we use the same approach in the present paper. The relationship between negative $\tilde{H}_n$-supermodules and Q-functions was sketched in [11] by Sergeev and our article contains an elaboration and extension of his ideas.

One can extract information about simple negative $\tilde{H}_n$-modules from that on simple negative $\tilde{H}_n$-supermodules in a way presented in § 5 of [7]. A comparison of this approach with more standard description of simple $\tilde{H}_n$-modules using the semidirect product structure of $\tilde{H}_n$ will be given elsewhere.

Another approach to problems considered in this paper can be found in [3] and [4].

§ 1. Split conjugacy classes of $\tilde{H}_n$

Conjugacy classes of $\tilde{H}_n$ are of the form $\theta_n^{-1}(C)$ where C is a conjugacy class of H_n or $\theta_n^{-1}(C)$ splits into two conjugacy classes of $\tilde{H}_n$ (see [6]). From the point of view of negative representations only the second type of conjugacy classes is interesting (see [7] § 3C). We call them split conjugacy classes. First of all we recall the well-known description of conjugacy classes of H_n which comes from [12] (see also [8]).

Let $b_I = \prod_{i \in I} b_i \in \mathbf{Z}_2^n$ for $I \subset \{1, \ldots, n\}$; by the support $\operatorname{supp} t$ of the cycle $t = (i_1, \ldots, i_p)$ we mean the set $\{i_1, \ldots, i_p\}$. Every element $b_I s$ of H_n can be uniquely written (up to order) as a product

$$b_I s = (b_{I_1} s_1) \ldots (b_{I_p} s_p) \tag{1}$$

where $s_k \in S_n$, $b_{I_k} \in \mathbf{Z}_2^n$, $s = s_1 \ldots s_p$ is a product of disjoint cycles and $I_k \subset \operatorname{supp} s_k$ for $k = 1, \ldots, p$. If $I \subset \operatorname{supp} s$ and s is a cycle then the type of $b_I s$ is defined as the ordered pair $((-1)^{\#(I)}, \text{length } s)$. The type of an element of the form (1) is a sequence of types of indicated factors, i.e. $\{((-1)^{\#(I_k)}, \text{length } s_k)\}$. Two elements of H_n are in the same conjugacy class if and only if they have the same type. Hence conjugacy classes of H_n are indexed by pairs of partitions (α, β) such that $|\alpha| + |\beta| = n$. The partition α corresponds to all factors of type $(1, *)$ and β corresponds to all factors of type $(-1, *)$. We write $C_{\alpha,\beta}$ for the corresponding conjugacy class. We recall (see [8]) that

$$\#(C_{\alpha,\beta}) = \frac{2^n n!}{z_\alpha 2^{l(\alpha)} z_\beta 2^{l(\beta)}} \tag{2}$$

where $l(\alpha)$ is the length of α, α contains α_i parts equal to i and $z_\alpha = 1^{\alpha_1} \alpha_1! \, 2^{\alpha_2} \alpha_2! \ldots$

We denote by $(H_n)_0$ the subgroup of H_n consisting of all elements $b_I s$ with $\#(I)$ even. It is of index 2 in H_n. The conjugacy class $C_{\alpha,\beta}$ is called *even* if $C_{\alpha,\beta} \subset (H_n)_0$ and *odd* otherwise. Let $D_{\alpha,\beta} = \theta_n^{-1}(C_{\alpha,\beta})$.

LEMMA 1 ([9]). *$D_{\alpha,\beta}$ splits into two conjugacy classes of $\tilde{H}_n$ if and only if:*

1) for $D_{\alpha,\beta}$ even we have $\beta = \emptyset$ and $\alpha \in \mathrm{OP}(n)$,

2) for $D_{\alpha,\beta}$ odd we have $\alpha = \emptyset$ and $\beta \in \mathrm{SP}_-(n)$.

Here $\mathrm{OP}(n)$ *is the set of all partitions of n with odd parts,* $\mathrm{SP}(n)$ *is the set of all partitions of n with all parts distinct (so-called strict partitions) and* $\mathrm{SP}_+(n)$, $\mathrm{SP}_-(n)$ *denote all partitions from* $\mathrm{SP}(n)$ *of even and odd length, respectively.*

For $\alpha \in \mathrm{OP}(n)$ we define D_α^+ to be the conjugacy class of $\tilde{H}_n$ contained in $D_{\alpha,\emptyset}$ and containing an element of S_n of cycle type α; then the other conjugacy class contained in $D_{\alpha,\emptyset}$ is equal to zD_α^+ and we denote it by D_α^-. From (2) it follows that

$$\#(D_\alpha^+) = \#(D_\alpha^-) = 2^{n-l(\alpha)} z_\alpha^{-1} n! \tag{3}$$

Let $(\tilde{H}_n)_0 = \theta_n^{-1}((H_n)_0)$ and $(\tilde{H}_n)_1 = \theta_n^{-1}(H_n \backslash (H_n)_0)$. We have $\tilde{H}_n = (\tilde{H}_n)_0 \cup \cup (\tilde{H}_n)_1$ and this decomposition allows us to define a $\mathbf{Z}_2$-grading on the group algebra of $\tilde{H}_n$ or a structure of a superalgebra. Consequently, we can consider $\mathbf{Z}_2$-graded modules over $\tilde{H}_n$ or $\tilde{H}_n$-supermodules. There are two kinds of simple $\tilde{H}_n$-supermodules: of type M and of type Q (see [6]). There exists a close relationship between the number of simple negative $\tilde{H}_n$-supermodules and number of split conjugacy classes, see [6], Proposition 4.14. Hence we get

COROLLARY 1. 1) *The number of non-isomorphic simple negative $\tilde{H}_n$-supermodules of type M is equal to* $\#(\mathrm{SP}_+(n))$.

2) *The number of non-isomorphic simple negative H_n-supermodules of type Q is equal to* $\#(\mathrm{SP}_+(n))$.

3) *The number of non-isomorphic simple negative H_n-supermodules is equal to* $\#(\mathrm{SP}(n))$.

Proof follows from Lemma 1, Proposition 4.14 of [6] and the classical fact that $\#(\mathrm{OP}(n)) = \#(\mathrm{SP}(n))$; see [1].

§ 2. Character ring of negative $\tilde{H}_n$-supermodules

The group superalgebra of $(\tilde{H}_n, (\tilde{H}_n)_0)$ over $\mathbf{C}$ is a semisimple superalgebra. We consider the category of all $\tilde{H}_n$-supermodules and their $\tilde{H}_n$-homomorphisms (morphisms of degree 1 are allowed) and its full subcategory of negative $\tilde{H}_n$-supermodules (see [6] for details). We write T_n^- for the Grothendieck group of the category of negative $\tilde{H}_n$-supermodules, $n \geqslant 1$. The group T_n^- is a free abelian group with a basis consisting of classes of simple negative $\tilde{H}_n$-supermodules. By Corollary 1 the rank of T_n^- is equal to the number $\#(\mathrm{SP}(n))$ of strict partitions of n. There exists a scalar product [,] on T_n^- defined by $[P, N] = \dim_{\mathbf{C}} \mathrm{HOM}_{\tilde{H}_n}(P, N)$ for negative $\tilde{H}_n$-supermodules P, N (see [6], § 4). We quote from [6] the characterization of simple negative $\tilde{H}_n$-supermodules.

LEMMA 2. *If P, N are simple negative $\tilde{H}_n$-supermodules, then*

$$[P, N] = \begin{cases} 1 & \text{if } P \approx N \text{ is of type } M, \\ 2 & \text{if } P \approx N \text{ is of type } Q, \\ 0 & \text{if } P \not\approx N. \end{cases}$$

We consider the direct sum $T^- = \bigoplus_{n \geq 0} T_n^-$ assuming that $T_0^- = \mathbf{Z}$. We are going to define a ring structure on T^-.

To this end notice that $G_m \hat{\times} G_n \approx G_{m+n}$ where $\hat{\times}$ is the operation of twisted product introduced in [2] (see also [7], § 4). Here G_m is generated by $a_1, \ldots, a_m$ and G_n by $a_{m+1}, \ldots, a_{m+n}$ subject to the defining relations. Informally, the operation $\hat{\times}$ reflects the fact that the a's of G_m and the a's of G_n anticommute with each other (where the role of -1 is played by z). The isomorphism $G_m \hat{\times} G_n \approx G_{m+n}$ induces an injection $\tau(m, n)$: $\tilde{H}_m \hat{\times} \tilde{H}_n \to \tilde{H}_{m+n}$ such that $\operatorname{Im} \tau(m, n) = \theta_{m+n}^{-1}(H_m \hat{\times} H_n)$ where we identify $H_m \times H_n$ with a subgroup of H_{m+n} in the obvious way.

Let P be a negative $\tilde{H}_m$-supermodule and N a negative $\tilde{H}_n$-supermodule. Then $\tilde{H}_m \hat{\times} \tilde{H}_n$ acts on $P \otimes N$ by the formula

$$(a, b)(x \otimes v) = (-1)^{\deg(b)\deg(x)} ax \otimes bv$$

for homogeneous $b \in \tilde{H}_n$, $x \in P$. An easy calculation shows that this formula defines the structure of a negative $\tilde{H}_m \hat{\times} \tilde{H}_n$-supermodule on $P \otimes N$.

If $[P]$ and $[N]$ denote classes of P and N in T_m^-, T_n^-, respectively, then we define their product by the formula

$$[P][N] = [\tilde{H}_{m+n} \otimes_{\tilde{H}_m \hat{\times} \tilde{H}_n} (P \otimes N)]$$

where the tensor product is in the sense of supermodules (i.e. graded) and the structure of the right $\tilde{H}_m \hat{\times} \tilde{H}_n$-supermodule on H_{m+n} comes from the embedding $\tau(m, n)$. Obviously $[P][N] \in T_{m+n}^-$. This defines a multiplication on T^- which is commutative and associative as is easily seen.

As in the usual (non-graded) case one can replace classes of supermodules by their characters (see [6], Theorem 4.12). This means that we can consider T_n^- as a free abelian group with a basis consisting of characters of simple negative $\tilde{H}_n$-supermodules. Therefore we call elements of T_n^- negative (virtual) characters of $\tilde{H}_n$. If $\varphi \in T_n^-$ then we write $\varphi_\alpha = \varphi(x)$ for $x \in D_\alpha^+$, $\alpha \in \mathrm{OP}(n)$; consequently $\varphi(y) = -\varphi_\alpha$ for $y \in D_\alpha^-$. The character φ vanishes on all remaining conjugacy classes of $\tilde{H}_n$. If also $\psi \in T_n^-$, then

$$(4) \qquad [\varphi, \psi] = \frac{1}{\#(\tilde{H}_n)} \sum_{x \in \tilde{H}_n} \varphi(x^{-1})\psi(x) = \frac{1}{2^{n+1} n!} \sum_{\alpha \in \mathrm{OP}(n)} 2 \# (D_\alpha^+) \varphi_\alpha \psi_\alpha$$

$$= \frac{1}{2^n n!} \sum_{\alpha \in \mathrm{OP}(n)} 2^{n-l(\alpha)} z_\alpha^{-1} n! \, \varphi_\alpha \psi_\alpha = \sum_{\alpha \in \mathrm{OP}(n)} 2^{-l(\alpha)} z_\alpha^{-1} \varphi_\alpha \psi_\alpha$$

by (3) and the fact that $x \in D_\alpha^\pm$ implies $x^{-1} \in D_\alpha^\pm$.

We are going to describe multiplication in T^- in terms of negative characters.

For $\varphi \in T_m^-$, $\psi \in T_n^-$ we have a negative character $\varphi \hat{\times} \psi$ of $\tilde{H}_m \hat{\times} \tilde{H}_n$ defined by

$$(\varphi \hat{\times} \psi)\overline{(x, v)} = \varphi(x)\psi(v)$$

where $\overline{(x, v)}$ is the residue class of (x, v) in $\tilde{H}_m \hat{\times} \tilde{H}_n$. If χ^P, χ^N are characters corresponding to $\tilde{H}_m$-supermodule P and $\tilde{H}_n$-supermodule N, respectively, then obviously $\chi^P \hat{\times} \chi^N = \chi^{P\otimes N}$.

The correspondence between supermodules and characters gives the following formula for multiplication in T^-:

$$\varphi \cdot \psi = \mathrm{Ind}_{\tilde{H}_m \hat{\times} \tilde{H}_n}^{\tilde{H}_{m+n}} \varphi \hat{\times} \psi$$

where $\varphi \in T_m^-$, $\psi \in T_n^-$ and Ind means the operation of taking the induced character. To compute this product explicitly we need the following

LEMMA 3. *Let $\gamma \in \mathrm{OP}(m+n)$ and let us identify $\tilde{H}_m \hat{\times} \tilde{H}_n$ with its image under the injection $\tau(m, n)$. Then*

$$D_\gamma^+ \cap (\tilde{H}_m \hat{\times} \tilde{H}_n) = \bigcup_{\alpha\cup\beta=\gamma} D_\alpha^+ \hat{\times} D_\beta^+$$

$$D_\gamma^- \cap (\tilde{H}_m \hat{\times} \tilde{H}_n) = \bigcup_{\alpha\cup\beta=\gamma} D_\alpha^- \hat{\times} D_\beta^+ = \bigcup_{\alpha\cup\beta=\gamma} D_\alpha^+ \hat{\times} D_\beta^-$$

is a partition of the left hand sides into the conjugacy classes of $\tilde{H}_m \hat{\times} \tilde{H}_n$ where the summation runs through all $\alpha \in \mathrm{OP}(m)$, $\beta \in \mathrm{OP}(n)$ and $\alpha \cup \beta$ is the partition of $m+n$ consisting of all parts of α and β. Moreover,

$$D_\alpha^+ \hat{\times} D_\beta^+ = D_\alpha^- \hat{\times} D_\beta^-, \quad z(D_\alpha^+ \hat{\times} D_\beta^+) = D_\alpha^- \hat{\times} D_\beta^+ = D_\alpha^+ \hat{\times} D_\beta^-.$$

Proof is similar to that of Lemma 4.2 in [7].

LEMMA 4. *Let $\varphi \in T_m^-$, $\psi \in T_n^-$ and $\gamma \in \mathrm{OP}(m+n)$. Then*

$$(\varphi \cdot \psi)_\gamma = \sum_{\alpha\cup\beta=\gamma} \frac{z_\gamma}{z_\alpha z_\beta} \varphi_\alpha \psi_\beta.$$

Proof. From the definition of the induced character and Lemma 3 we get

$$\begin{aligned}
(\varphi \cdot \psi)_\gamma &= \frac{\#(\tilde{H}_{m+n})}{\#(\tilde{H}_m \hat{\times} \tilde{H}_n)\,\#(D_\gamma^+)} \sum_{w\in D_\gamma^+} (\varphi \cdot \psi)(w) \\
&= \frac{2^{m+n+1}(m+n)!}{2^{m+n+1}\, m!\, n!\, 2^{m+n-l(\gamma)} z_\gamma^{-1} (m+n)!} \sum_{\alpha\cup\beta=\gamma} \varphi_\alpha \psi_\beta \,\#(D_\alpha^+)\,\#(D_\beta^+) \\
&= \frac{z_\gamma}{2^{m+n-l(\gamma)} m!\, n!} \sum_{\alpha\cup\beta=\gamma} 2^{m-l(\alpha)} z_\alpha^{-1}\, m!\, 2^{n-l(\beta)} z_\beta^{-1}\, n!\, \varphi_\alpha \psi_\beta \\
&= \sum_{\alpha\cup\beta=\gamma} \frac{z_\gamma}{z_\alpha z_\beta} \varphi_\alpha \psi_\beta
\end{aligned}$$

since $l(\gamma) = l(\alpha) + l(\beta)$.

§ 3. The characteristic map

We are going to describe the ring structure of T^- in terms of certain functions introduced by Schur in [10] and now called *Schur Q-functions*. For updated account of Q-functions and notation, see [7], § 4B.

Let $\Gamma = \mathbf{Z}[q_1, q_2, \ldots]$ where the symmetric functions q_i in indeterminates $x_1, x_2, \ldots$ are defined by the formula

$$Q(t) := \sum_{k \geq 0} q_k t^k = \prod_{i \geq 1} (1 + x_i t)(1 - x_i t)^{-1}. \tag{5}$$

One knows that $\Gamma_{\mathbf{Q}} = \Gamma \otimes_{\mathbf{Z}} \mathbf{Q} = \mathbf{Q}[p_1, p_3, \ldots]$ where $p_k = \sum_{i \geq 1} x_i^k$ is the power series function ([7], § 4B).

The ring Γ is a graded ring and $\deg q_k = \deg p_k = k$. We write Γ_n for the subgroup of all homogeneous elements of degree n. We denote $p_\mu = p_{\mu_1} p_{\mu_2} \ldots$ for $\mu \in \mathrm{OP}(n)$. Notice that $\{p_\mu\}$, $\mu \in \mathrm{OP}(n)$, is a basis of $\Gamma_n \otimes \mathbf{Q}$ over $\mathbf{Q}$.

The ring $\Gamma_{\mathbf{C}} := \Gamma \otimes_{\mathbf{Z}} \mathbf{C}$ (and hence Γ) is equipped with a scalar product $[\ ,\ ]$ such that

$$[p_\alpha, p_\beta] = 2^{-l(\alpha)} z_\alpha \delta_{\alpha\beta}. \tag{6}$$

We define the *characteristic map* ch: $T^- \to \Gamma_{\mathbf{C}}$ by the formula

$$\mathrm{ch}(\varphi) = \sum_{\alpha \in \mathrm{OP}(n)} z_\alpha^{-1} \varphi_\alpha p_\alpha \quad \text{for } \varphi \in T_n^-. \tag{7}$$

PROPOSITION 1. *The characteristic map is an isometric homomorphism of rings.*

Proof. Let $\varphi, \psi \in T_n^-$; we have

$$\begin{aligned}[\mathrm{ch}(\varphi), \mathrm{ch}(\psi)] &= \sum_{\alpha,\beta \in \mathrm{OP}(n)} z_\alpha^{-1} z_\beta^{-1} [p_\alpha, p_\beta] \varphi_\alpha \psi_\beta \\ &= \sum_{\alpha \in \mathrm{OP}(n)} z_\alpha^{-2} 2^{-l(\alpha)} z_\alpha \varphi_\alpha \psi_\alpha = [\varphi, \psi]\end{aligned}$$

by (4) and (6).

By Lemma 4 we have $\mathrm{ch}(\varphi \cdot \psi) = \mathrm{ch}(\varphi)\mathrm{ch}(\psi)$ for $\varphi \in T_m^-$, $\psi \in T_n^-$.

Now we are going to define explicitly a *negative $\tilde{H}_n$-supermodule* which corresponds in a sense to the basic spin representation of the symmetric group S_n (see [7], § 2C).

Let L_n be the Clifford algebra over $\mathbf{C}$ of a quadratic form of rank n. There exists an algebra basis $y_1, \ldots, y_n$ of L_n such that $y_p^2 = -1$ and $y_p y_q = -y_q y_p$ for $p \neq q$. Hence the monomials $\{y_I = \prod_{i \in I} y_i\}$, $I \subset \{1, \ldots, n\}$, form a linear basis of L_n over $\mathbf{C}$ (we abuse the notation here since y_I is determined by I up to sign). The structure of $\tilde{H}_n$-module on L_n is given by

$$a_p y_I = y_p y_I, \quad a_p \text{ generators of } G_n,$$

$$s y_I = y_{s(I)}, \quad s \in S_n.$$

If we define a grading on L_n by setting $(L_n)_0$ to be spanned by y_I with $\#(I)$ even and $(L_n)_1$ by y_I with $\#(I)$ odd, then L_n becomes an $\tilde{H}_n$-supermodule. We have $zy_I = a_p^2 y_I = y_p^2 y_I = -y_I$ so that L_n is a *negative* $\tilde{H}_n$-*supermodule.*

LEMMA 5. *If* ξ^n *is the character of* L_n, *then* $\xi^n_\alpha = 2^{l(\alpha)}$ *for* $\alpha \in \mathrm{OP}(n)$.

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be an odd partition of length m and let $s = s_1 \ldots s_m$ be an element from S_n expressed as a product of disjoint cycles, s_k a cycle of length α_k; moreover let $I_k = \operatorname{supp} s_k$. An easy calculation shows that

$$sy_I = \begin{cases} y_I & \text{if } I = \bigcup I_p,\ p \in \{1, \ldots, m\}, \\ \pm y_K,\ K \neq I, & \text{otherwise.} \end{cases}$$

Hence the trace of the action of s on L_n is 2^m as required.

LEMMA 6. *We have* $\mathrm{ch}(\xi^n) = q_n$, $[\xi^n, \xi^n] = 2$ *and* L_n *is a simple* $\tilde{H}_n$-*supermodule of type Q.*

Proof. By definition (7), Lemma 5 and Lemma 4.5 of [7] we have

$$\mathrm{ch}(\xi^n) = \sum_{\alpha \in \mathrm{OP}(n)} 2^{l(\alpha)} z_\alpha^{-1} p_\alpha = q_n.$$

By (4) and Lemma 4.5 of [7] we have

$$[\xi^n, \xi^n] = \sum_{\alpha \in \mathrm{OP}(n)} 2^{l(\alpha)} z_\alpha^{-1} = q_n(1, 0, 0, \ldots).$$

Since by (5) $q_n(1, 0, 0, \ldots) = 2$ we are done.

Let us recall (see [6] or [7], § 2A) that L_n is a simple $\tilde{H}_n$-supermodule if it is isomorphic to $M_n \oplus M'_n$ as an $\tilde{H}_n$-module and M_n, M'_n are associate simple $\tilde{H}_n$-modules, i.e. M_n, M'_n are isomorphic as abelian groups and $b.y$ in M'_n corresponds to $(-1)^{\deg(b)} b.y$ in M_n, $b \in \tilde{H}_n$, $y \in M'_n$.

Let $c_n = i(\sqrt{n})^{-1}(y_1 + \ldots + y_n) \in L_n$, $i^2 = -1$; notice that $c_n^2 = 1$ and $sc_n = c_n$ for $s \in S_n$. We define

$$M_n = \{w + wc_n;\ w \in (L_n)_0\}, \qquad M'_n = \{w - wc_n;\ w \in (L_n)_0\}.$$

A straightforward computation shows that M_n, M'_n are $\tilde{H}_n$-submodules of L_n. Moreover, $\sigma\colon M_n \to M'_n$ defined by $\sigma(w + wc_n) = w - wc_n$, $w \in (L_n)_0$, is an isomorphism of abelian groups and

$$\sigma\big(b(w + wc_n)\big) = \sigma(bw + bwc_n) = (-1)^{\deg(b)} b\sigma(w + wc_n).$$

This gives the required decomposition of L_n as an $\tilde{H}_n$-module. Obviously, M_n and M'_n are simple $\tilde{H}_n$-modules by $[\xi^n, \xi^n] = 2$.

LEMMA 7. *If* $L(t) = \sum_{n \geq 0} \xi^n t^n$, *then* $L(t)L(-t) = 1$.

The same arguments as in the proof of Lemma 4.16 of [7] apply here.

The Schur Q-functions are polynomials in the q_k and are defined as pfaffians of certain skew-symmetric matrices thanks to the property $Q(t)Q(-t) = 1$ of the generating function of the q_k (see [7], § 4B). By Lemma 7 the same procedure applies when the q_k are replaced by the ξ^k and the operations are performed in the character ring T^- instead of the ring Γ. In other terms for each partition ν from $\mathrm{SP}(n)$ there exist elements $Q_\nu \in \Gamma$ and $\xi^\nu \in T^-$ such that $\deg Q_\nu = |\nu|$ and $Q_n = q_n$. We have explicit recurrence relations for ξ^ν (as well as for Q_ν):

$$\xi^{\nu_1\nu_2} = \xi^{\nu_1}\xi^{\nu_2} + 2\sum_{i=1}^{\nu_2}(-1)^i \xi^{\nu_1+i}\xi^{\nu_2-i}, \qquad \nu_1, \nu_2 > 0 \tag{8}$$

$$\xi^\nu = \sum_{j=2}^{k}(-1)^j \xi^{\nu_1\nu_j}\xi^{\nu_2\ldots\hat{\nu}_j\ldots\nu_k} \quad \text{for } k = l(\nu) \text{ even, } k > 2, \tag{9}$$

$$\xi^\nu = \sum_{j=1}^{k}(-1)^{j-1}\xi^{\nu_j}\xi^{\nu_1\ldots\hat{\nu}_j\ldots\nu_k} \quad \text{for } k = l(\nu) \text{ odd, } k > 2. \tag{10}$$

LEMMA 8. 1) $\mathrm{ch}(\xi^\nu) = Q_\nu$ *for* $\nu \in \mathrm{SP}(n)$,

2) $[\xi^\nu, \xi^\mu] = 2^{l(\nu)}$, $\nu, \mu \in \mathrm{SP}(n)$.

Proof. Part 1) follows from the construction of Q_ν, ξ^ν and Proposition 1. Part 2) is an immediate consequence of Proposition 1 and the formula $[Q_\nu, Q_\mu] = 2^{l(\nu)}\delta_{\nu\mu}$, see [7], Theorem 4.8.

PROPOSITION 2. *For every* $\nu \in \mathrm{SP}(n)$ *the element* $\zeta^\nu := 2^{-[l(\nu)/2]}\xi^\nu$ *belongs to* $T^- \cap \mathbf{Q}[\zeta^1, \zeta^2, \ldots]$ *where* [ ] *denotes the integral part symbol.*

Proof. For $l(\nu) = 1$ we have $\zeta^\nu = \xi^\nu$ and the assertion is obvious. For $l(\nu) = 2$ we use the formula

$$\xi^m \xi^n = 2\eta^{m,n} \quad \text{for some } \eta^{m,n} \in T^-$$

which follows from Proposition 5.1 of [7] since L_n is a simple negative $\tilde{H}_n$-supermodule of type Q by Lemma 6. Now by (8) it follows that $\xi^{\nu_1\nu_2}$ is divisible by 2 in T^- and belongs to $\mathbf{Q}[\zeta^1, \zeta^2, \ldots]$. For $l(\nu) > 2$ the assertion of the lemma follows from formulas (9) and (10) by induction on $l(\nu)$.

COROLLARY 2. *For strict partitions* ν, μ *we have*

$$[\zeta^\nu, \zeta^\nu] = \begin{cases} 1 & \text{for } l(\nu) \text{ even,} \\ 2 & \text{for } l(\nu) \text{ odd,} \end{cases}$$

$$[\zeta^\nu, \zeta^\mu] = 0 \qquad \text{for } \nu \neq \mu.$$

Proof follows from Proposition 2 and Lemma 8.

COROLLARY 3. *For any* $\nu \in \mathrm{SP}(n)$ *we have*

$$Q_\nu = \sum_{\alpha \in \mathrm{OP}(n)} 2^{[l(\nu)/2]} z_\alpha^{-1} \zeta_\alpha^\nu p_\alpha.$$

Proof follows from Lemma 8, Proposition 2 and (7).

COROLLARY 4. *For any* $\alpha \in \mathrm{OP}(n)$ *we have*

$$p_\alpha = \sum_{\nu \in \mathrm{SP}(n)} 2^{-([(l(\nu)+1)/2]+l(\alpha))} \zeta^\nu_\alpha Q_\nu.$$

Proof. By Corollary 4 and (6) we have $[p_\alpha, Q_\nu] = 2^{[l(\nu)/2]-l(\alpha)} \zeta^\nu_\alpha$; hence the result by $[Q_\nu, Q_\mu] = 2^{l(\nu)} \delta_{\nu\mu}$, see Theorem 4.8 of [7].

COROLLARY 5. *The set* $\{\zeta^\nu\}$, $\nu \in \mathrm{SP}(n)$, *forms the basis of* T^- *consisting of characters corresponding to simple negative* $\tilde{H}_n$*-supermodules.*

Proof. By Corollary 1 the number of ζ^ν, $l(\nu)$ even, is the same as the number of simple negative $\tilde{H}_n$-supermodules of type M, and the number of ζ^ν, $l(\nu)$ odd, is the same as the number of simple negative $\tilde{H}_n$-supermodules of type Q. Hence, by Lemma 2 and Corollary 2, $\{\zeta^\nu\}$ is, up to sign, the basis consisting of classes of simple negative $\tilde{H}_n$-supermodules. We have to show that ζ^ν and not $-\zeta^\nu$ is the character corresponding to a simple negative $\tilde{H}_n$-supermodule. To this end we need to know that $\zeta^\nu(1) = \zeta^\nu_{(1^n)} > 0$; this, however, follows from Corollary 3 and Proposition 4.13 of [7].

COROLLARY 6. *If* $\nu \in \mathrm{SP}(n)$, $l(\nu) = m$, *then the degree of the character* ζ^ν *is equal to*

$$2^{n-[m/2]} \frac{(\nu_1 + \dots + \nu_m)!}{\nu_1! \dots \nu_m!} \prod_{i<j} \frac{\nu_i - \nu_j}{\nu_i + \nu_j}.$$

Proof follows from Corollary 3 and Proposition 4.13 of [7] since the degree of ζ^ν is equal to $\zeta^\nu_{(1^n)}$.

COROLLARY 7. $T^-_{\mathbf{Q}} = \mathbf{Q}[\zeta^1, \zeta^2, \dots]$.

Proof. The inclusion $\subset$ follows from Proposition 2 and Corollary 5, the opposite inclusion is obvious.

COROLLARY 8. *The homomorphism* ch: $T^- \to \Gamma_{\mathbf{C}}$ *is an injection and* $\mathrm{ch}(T^-)$ *is a subring of* $\Gamma_{\mathbf{Q}}$ *spanned linearly over* $\mathbf{Z}$ *by* $\{2^{-[l(\nu)/2]} Q_\nu\}$ *where* ν *runs through all strict partitions.*

Proof. ch is an injection by Corollary 5, Proposition 2, Lemma 8 and the fact that $\{Q_\nu\}$, $\nu \in \mathrm{SP}(n)$, is a linear basis of Γ_n over $\mathbf{Z}$ for each n (see Proposition 4.4 of [7]). The remaining assertions follow from the formula $\mathrm{ch}(\zeta^\nu) = 2^{-[l(\nu)/2]} Q_\nu$ which holds true by Lemma 8 and Proposition 2.

COROLLARY 9. *The map* ch *induces an isomorphism of rings* $T^-_{\mathbf{Q}} \approx \Gamma_{\mathbf{Q}}$.

Proof. We have $\mathrm{rank}\, T^-_n = \#(\mathrm{SP}(n)) = \mathrm{rank}\, \Gamma_n$ by Corollary 1 and Proposition 4.3 of [7]. Hence the result follows from Corollary 8.

COROLLARY 10. *The set of all monomials* $\{\zeta^{\nu_1}\zeta^{\nu_2}\ldots\}$ *indexed by strict partitions* ν *is a* $\mathbf{Q}$*-linear basis of* $T_{\mathbf{Q}}^{-}$.

Proof. Since $\mathrm{ch}(\zeta^n) = q_n$ for all n and $\{q_{\nu_1} q_{\nu_2}\ldots\}$, ν running through strict partitions, is a $\mathbf{Z}$-linear basis of Γ (see Proposition 4.3 of [7]) the result follows from Corollary 9.

References

[1] G. E. Andrews, *The Theory of Partitions*, Encyclopedia Math. Appl. vol. 2, Addison–Wesley, Reading 1976.

[2] P. N. Hoffman and J. F. Humphreys, *Hopf algebras and projective representations of* $G \wr S_n$ *and* $G \wr A_n$, Canad. J. Math. 38 (1986), 1380–1458.

[3] —, —, *Projective representations of generalized symmetric groups using* PHS-*algebras*, preprint, 1987.

[4] J. F. Humphreys, *Projective representations of the generalized symmetric groups*, this volume, 299–302.

[5] S. Ihara and T. Yokonuma, *On the second cohomology groups (Schur multipliers) of finite reflection groups*, J. Fac. Sci. Univ. Tokyo, Sect. IA Math. 11 (1965), 155–171.

[6] T. Józefiak, *Semisimple superalgebras*, in: Some Current Trends in Algebra, Proceedings of the Varna Conference 1986, Lecture Notes in Math. 1352, Springer, Berlin 1988, 96–113.

[7] —, *Characters of projective representations of symmetric groups*, Expositiones Math. 7 (1989), 193–247.

[8] A. Kerber, *Representations of Permutation Groups, I*, Lecture Notes in Math. 240, Springer, Berlin 1971.

[9] E. W. Read, *The α-regular classes of the generalized symmetric group*, Glasgow Math. J. 17 (1976), 144–150.

[10] I. Schur, *Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen*, J. Reine Angew. Math. 139 (1911), 155–250.

[11] A. N. Sergeev, *Tensor algebra of the identity representation as a module over Lie superalgebras* $\mathrm{Gl}(n, m)$ *and* $Q(n)$, Mat. Sb. 123 (1984), 422–430 (in Russian).

[12] W. Specht, *Eine Verallgemeinerung der symmetrischen Gruppe*, Schriften math. Sem. Univ. Berlin, Band 1 (1932), 1–32.