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ARITHMETICAL ASPECTS OF SATURATION OF SINGULARITIES

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Introduction

The starting point in this work is the following fact: Assume that $C = \operatorname{Spec} \mathcal{O}$ is an irreducible algebroid curve over an algebraically closed field of characteristic zero, and consider the absolute saturation $\tilde{\mathcal{O}}$ of the local ring $\mathcal{O}$ (see [11], [10] and [3] for three equivalent definitions of saturation). Then the semigroup of values $\tilde{S}$ of $\tilde{\mathcal{O}}$ has the following arithmetical property

(A) If $\gamma, \gamma_1, \ldots, \gamma_l \in \tilde{S}$, $\gamma \geqslant \gamma_i$, and $e = \text{g.c.d.}(\gamma_1, \ldots, \gamma_l)$ $(e \geqslant 0)$ then $\gamma + e \in \tilde{S}$.

In fact, as seen in any of the quoted references, $\tilde{S}$ is the minimum semigroup of $\boldsymbol{Z}_+ = \{n \in \boldsymbol{Z} \mid n \geqslant 0\}$ having the property (A) and containing the characteristic exponents $\beta_0, \beta_1, \ldots, \beta_g$ of the Puiseux expansion corresponding to a general plane projection of C.

In the sequel we will use the definition of saturation introduced in our previous work [3] and in [9], which is given in the following terms: Let A be a commutative ring with unit, F its quotient field, $\bar{A}$ the integral closure of A in F, and D the set of nonzero divisors in A. The ring A is said to be *saturated with respect to a subset* $T \subset D$ if the following property holds for it.

(P_T) If $z \in A$, $z_1, \ldots, z_r, w_1, \ldots, w_s \in D$ are such that either $zz_i^{-1} \in \bar{A}$ or $z_i \in T$ (resp. either $zw_j^{-1} \in \bar{A}$ or $w_j \in T$) and $(z_1 \ldots z_r)(w_1 \ldots w_s)^{-1} \in \bar{A}$ then $z(z_1 \ldots z_r)(w_1 \ldots w_s)^{-1} \in A$.

If $T = \emptyset$, A will simply said to be *saturated*. The saturation of a ring A with respect to a subset $T \subset D$ is the minimum saturated ring with respect to S' between A and $\bar{A}$, and it will be denoted by $\tilde{A}_T$. If $T = \emptyset$, we will only set $\tilde{A}$ for $\tilde{A}_{\emptyset}$.

If $A = \mathcal{O}$ is a one dimensional complete local domain and if $T = \{w\}$, where w is a nonzero element in the maximal ideal of $\mathcal{O}$, then, associated to

$\mathcal{O}$ we have its semigroup of values $S = \{v(z) \mid z \in \mathcal{O}, z \neq 0\} \subset \boldsymbol{Z}_+$, ($v$ is the valuation associated with $\bar{\mathcal{O}}$). If $\mathcal{O}$ is saturated with respect to T and if $m = v(w)$, then it follows from Bezout's theorem that S verifies

(A_m) If $\gamma \in S$, $\gamma_1, \ldots, \gamma_l \in S$ are such that either $\gamma \geqslant \gamma_i$ or $\gamma_i = m$ (for each i) then one has $\gamma + e \in S$, where $e = \text{g.c.d.}(\gamma_1, \ldots, \gamma_l)$, $e \geqslant 0$.

Note that if S verifies (A_m) for a certain element $m \in S$, then S determines m and it is determined by the elements $\beta'_0, \beta'_1, \ldots$, given inductively by

$$\beta'_0 = m, \quad \beta'_{i+1} = \min\{\gamma \in S \mid \text{g.c.d.}(\beta'_0, \beta'_1, \ldots, \beta'_i, \gamma) < \text{g.c.d.}(\beta'_0, \ldots, \beta'_i)\}.$$

In fact, S is the minimum semigroup verifying (A_m) which contains the elements β'_i, and any element of S is of type $\beta'_i + \lambda e'_i$, where $\lambda \in \boldsymbol{Z}_+$ and $e'_i = \text{g.c.d.}(\beta'_0, \ldots, \beta'_i)$. In the case that $m = \min(S - \{0\})$, to say that $\mathcal{O}$ (resp. S) verifies (P_T) (resp. (A_m)) is equivalent to say that $\mathcal{O}$ (resp. S) verifies (P_φ) (resp. (A)). This is the connection of the saturation with our starting point, and in this setting, above definition of saturation looks as an arithmetical one and it seems to be adequate for arithmetical purposes.

In this paper, we will indicate how above definitions provide elementary technics to approach some arithmetical questions. In Section 1 we consider the case in which $\mathcal{O}$ is the local ring of a plane branch of curve over an algebraically closed field of any characteristic. If S is the semigroup of values of the (absolute) saturation $\tilde{\mathcal{O}}$, we introduce the set of characteristic exponents $\beta_0, \ldots, \beta_g$ of $\mathcal{O}$ to be the elements $\beta_0, \beta_1, \ldots, \beta_g$ generating S in the above sense (relative to property (A)), and we show that these elements are equivalent data to the sequence of multiplicities of the successive blowing ups. So these characteristic exponents are seen to be the same that those introduced in [2] and [9], but our actual introduction seems to be more natural and provides more simple developments and manipulations.

To illustrate this, we obtain in Section 2 the neccessary conditions on a set of characteristic exponents given in [4] in order that this set be the set of characteristic exponents of a strange curve. This proof seems to be more direct and more natural than that given in our previous work.

In Section 3, we analyze the case of characteristic zero or multiplicities prime to the characteristic and we show as the characteristic exponents are given by means of Puiseux's series. To illustrate the philosophy in our paper, we include the elementary proof of the invariance and inversion of characteristic exponents obtained in [8], which improves very much that given by Abhyankar in [1].

In Sections 4 and 5 we consider the case of non algebraically closed fields and the case of algebroid curves with several branches, showing that our approach is also adequate for these situations.

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1. Characteristic exponents of plane branches and saturation

In this section we give an approach, via saturation, to the characteristic exponents of an irreducible plane algebroid curve C over an algebraically closed field k of any characteristic. This approach will result equivalent to that given in [2] and [7] in view of the Theorem 1 below.

For an *irreducible algebroid curve over* k (or branch in short) we will mean the scheme $C = \operatorname{Spec} \mathcal{O}$ where $\mathcal{O}$ is an equicharacteristic complete local ring of dimension 1 such that $\mathcal{O}/\mathfrak{m} \approx k$, $\mathfrak{m}$ being the maximal ideal of $\mathcal{O}$. The branch is said to be plane if $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \leqslant 2$. Since $\mathcal{O}$ is equicharacteristic, it follows from Cohen's theorem that $\mathcal{O}$ has coefficient fields, i.e., ring homomorphisms $k \to \mathcal{O}$ such that the induced map $k \to \mathcal{O}/\mathfrak{m}$ is an isomorphism. The integral closure $\bar{\mathcal{O}}$ of $\mathcal{O}$ in its quotient field is a complete discrete valuation ring, and since k is algebraically closed, every coefficient field $k \to \mathcal{O}$ for $\mathcal{O}$ provides a coefficient field $k \to \bar{\mathcal{O}}$ for $\bar{\mathcal{O}}$ by composing with the ring inclusion $\mathcal{O} \hookrightarrow \bar{\mathcal{O}}$. The normalized valuation associated to $\bar{\mathcal{O}}$ will be denoted by v, and semigroups of subrings of $\bar{\mathcal{O}}$ are defined as above. Each choice of a coefficient field and of a set of generators $\{x_1, \ldots, x_r\}$ of $\mathfrak{m}$ provides a ring homomorphism $\Phi\colon k[[X_1, \ldots, X_r]] \to \mathcal{O}$, given by $X_i \mapsto x_i$, which geometrically corresponds to the embedding $C \hookrightarrow \operatorname{Spec} k[[X_1, \ldots, X_r]]$. The kernel of Φ is the ideal defining the equations for this embedding. If the branch is plane and $r = 2$, $\operatorname{Ker} \Phi$ is a principal ideal (f), and each generator f provides an equation for the embedding in $\operatorname{Spec} k[[X_1, X_2]]$.

DEFINITION 1. The *characteristic exponents of the plane branch* $C = \operatorname{Spec} \mathcal{O}$ are the numbers $\beta_0, \beta_1, \ldots, \beta_g$ determined by the semigroup S of the ring $\tilde{\mathcal{O}}$, i.e.,

$$\beta_0 = \min(S - \{0\}),\ \beta_{i+1} = \min\{\gamma \in S \mid \text{g.c.d.}(\beta_0, \ldots, \beta_i, \gamma) < \text{g.c.d.}(\beta_0, \ldots, \beta_i)\}.$$

Note that the characteristic exponents only depend on the ring structure of $\mathcal{O}$, and so, in particular, they are independent on the coefficient field.

Now, let us examine the behaviour of the characteristic exponents by a blow up of C. For it, take a branch $C = \operatorname{Spec} \mathcal{O}$ and consider the ring $\tilde{\mathcal{O}}$.

LEMMA 1. *$\tilde{\mathcal{O}}$ is an Arf ring* (see [6]), i.e., *$\tilde{\mathcal{O}}$ verifies the following property: If $z, z' \in \tilde{\mathcal{O}}$, $w \in \tilde{\mathcal{O}}$, $w \neq 0$, and zw^{-1}, $z'w^{-1} \in \tilde{\mathcal{O}}$ then $zz'w^{-1} \in \tilde{\mathcal{O}}$.*

Proof. It is trivial from the definition of saturation since either $zz'^{-1} \in \bar{\mathcal{O}}$ or $z'z^{-1} \in \bar{\mathcal{O}}$. □

From the Lemma 1, it follows that if $w \in \tilde{\mathcal{O}}$, $w \neq 0$, then the set $\tilde{\mathcal{O}}(w) = \{z \in \bar{\mathcal{O}} \mid zw \in \tilde{\mathcal{O}}\}$ is a subring of $\bar{\mathcal{O}}$ containing $\tilde{\mathcal{O}}$.

LEMMA 2. *Let k be a field, k' a finite extension of k and $\mathcal{O}'$ a subring of $k'[[t]]$, t an indeterminate, containing $k[[x]]$ for some $x \in k'[[t]]$ such that $0 < \operatorname{ord}_t(x) < \infty$. Then $\mathcal{O}'$ is an 1-dimensional complete local ring.*

Proof. First of all, the ring extension $k[[x]] \subset k'[[t]]$ is integral. In fact, from the finiteness of k'/k it is obvious that $k[[x]] \subset k'[[x]]$ is integral. On the other hand, from Weierstrass preparation theorem if $x = a_m t^m + a_{m+1} t^{m+1} + \ldots$ then

$$(*)\quad X-(a_m T^m + a_{m+1} T^{m+1} + \ldots) = V(X, T)(T^m + b_{m-1}(X) T^{m-1} + \ldots + b_0(X))$$

with V a unit, so t is integral over $k'[[x]]$. Moreover, since $b_j(0) = 0$, it follows that $k'[[t]] = k'[[x]][t]$ since from $(*)$ one has

$$t^\alpha = c_{\alpha,0}(x) + c_{\alpha,1}(x)t + \ldots + c_{\alpha,m-1}(x)t^{m-1}$$

where $\operatorname{ord} c_{\alpha,j} \to \infty$ as $\alpha \to \infty$. Thus, the extension is integral and hence $\mathcal{O}'$ is one dimensional, and moreover, each maximal ideal of $\mathcal{O}'$ is forced to be $\bar{\mathfrak{m}} \cap \mathcal{O}'$ ($\bar{\mathfrak{m}}$ the maximal ideal of $k'[[t]]$), so $\mathcal{O}'$ is local. $\mathcal{O}'$ is Noetherian since $k'[[x]]$ is a $k[[x]]$-module of finite type and hence $\mathcal{O}'$ is finitely generated as module over $k[[x]]$. Finally $\mathcal{O}'$ is complete: in fact, $\mathcal{O}'$ is a $k[[x]]$-module of finite type (since it is torsion free), so it is complete for the q-adic topology (q the maximal ideal of $k[[x]]$), and since $q\mathcal{O}'$ is primary for the maximal ideal of $\mathcal{O}'$ it follows that $\mathcal{O}'$ is complete. □

COROLLARY 1. *With notations as in Lemma 1, for each w the ring $\tilde{\mathcal{O}}(w)$ is again the ring of an irreducible algebroid curve. Moreover, the ring $\tilde{\mathcal{O}}(w)$ is again saturated and it only depends on the value $m = v(w)$, so it can be represented by $\tilde{\mathcal{O}}(m)$.*

Proof. The first part follows from Lemma 2. For the second part, it is clear that $\tilde{\mathcal{O}}(w)$ is saturated (in fact saturated with respect to $\{w\}$), and if $w' \in \tilde{\mathcal{O}}$ verifies $v(w) = v(w')$, then

$$z \in \tilde{\mathcal{O}}(w) \Leftrightarrow zw^{-1} \in \tilde{\mathcal{O}} \Leftrightarrow (zw'^{-1})(w'w^{-1}) \in \tilde{\mathcal{O}} \Leftrightarrow zw'^{-1} \in \tilde{\mathcal{O}} \Leftrightarrow z \in \tilde{\mathcal{O}}(w').$$

So $\tilde{\mathcal{O}}(w) = \tilde{\mathcal{O}}(w')$. □

Now, take a plane branch $C = \operatorname{Spec} \mathcal{O}$, a coefficient field $k \hookrightarrow \mathcal{O}$ for it (identify k to the corresponding subfield of $\mathcal{O}$), and a set of two generators x, y of $\mathfrak{m}$. Assume $n = v(x) \leqslant v(y) = m$. Consider the Euclid algorithm to find the greatest common divisor r_s of m, and n,

$$\begin{aligned} m &= hn + r_1, \\ n &= h_1 r_1 + r_2, \\ &\cdots\cdots\cdots\cdots \qquad ; n > r_1 > \ldots > r_s. \\ r_{s-2} &= h_{s-1} r_{s-1} + r_s, \\ r_{s-1} &= k r_s, \end{aligned}$$

and set $m' = m \cdot r_s^{-1}$, $n' = n \cdot r_s^{-1}$. Since g.c.d$(m', n') = 1$, take $\sigma, \tau \in \mathbf{Z}$ such that $\tau m' - \sigma n' = 1$, and set $x' = y^\tau \cdot x^{-\sigma}$, $y' = y^{n'} x^{-m'}$, and $\alpha = \operatorname{Res}(y^{n'} \cdot x^{-m'})$. Then our main proposition states as below.

PROPOSITION 1. *One has* $\tilde{\mathcal{O}}(m) = k[[x', y'-\alpha]]_{x'}^{\sim}$.

Proof. The complete local ring $\tilde{\mathcal{O}}(m)$ contains k and both x', y', since $yx' = yy^\tau x^{-\sigma} \in \tilde{\mathcal{O}}$, $yy' = yy^{n'} \cdot x^{-m'} \in \tilde{\mathcal{O}}$, and it is obviously saturated with respect to x', so one has $k[[x', y'-\alpha]]_{x'}^{\sim} \subset \tilde{\mathcal{O}}(m)$.

To see the converse consider the set $A = k[[x]] + yk[[x', y'-\alpha]]_{x'}^{\sim}$. Since if $l > \left[\frac{m}{n}\right]$ one has $x^l = y \cdot \frac{x^l}{y} \in yk[[x', y'-\alpha]]_{x'}^{\sim}$, as it is clear that $y = x'^{m'} y'^{-\sigma}$, $x = x'^{n'} y'^{-\tau}$ implies $x^l y^{-1} \in k[[x', y'-\alpha]]_{x'}^{\sim}$. It follows that A is a ring containing $k[[x]]$ and such that any element in A can be written in the form $p(x) + yz$ where $z \in k[[x', y'-\alpha]]_{x'}^{\sim}$ and $p(x)$ a polynomial of degree less than $\left[\frac{m}{n}\right]$. So since A contains k, x and y, if we prove that A is saturated then one would have that $\mathcal{O} \subseteq A$ and, hence, by above remark $\tilde{\mathcal{O}}(m) \subset A(m) = k[[x', y'-\alpha]]_{x'}^{\sim}$ as required.

To see that A is saturated take $z \in A$, $z_1, \ldots, z_r$, $w_1, \ldots, w_s \in D$ such that zz_i^{-1}, zw_j^{-1}, $(z_1 \ldots z_r)(w_1 \ldots w_s)^{-1} \in \tilde{A} = \tilde{\mathcal{O}}$. We claim that $w = z(z_1 \ldots z_r)(w_1 \ldots w_s)^{-1} \in A$. We shall distinguish two cases: 1) If $v(z) < m$. 2) If $v(z) \geqslant m$. In the first case set $z = x^\gamma \varepsilon(x)(1+yz')$, $z_i = x^{\gamma_i} \varepsilon_i(x)(1+yz_i')$, $w_j = x^{\gamma_j} \varepsilon_j'(x)(1+yw_j')$ where $\gamma_i, \delta_j \leqslant \gamma < \left[\frac{m}{n}\right]$ and $\varepsilon, \varepsilon_i, \varepsilon_j'$ units in $k[[x]]$. Then

$$w = x^{\gamma'}(1+yz)(1+yz_1') \ldots (1+yz_r')(1+yw_1') \ldots (1+yw_s')\varepsilon'(x),$$

where $\gamma' = \gamma + \sum \gamma_i - \sum \delta_j$, $(1+yz_j') = (1+yw_j')^{-1} \in k + yk[[x', y']]_{x'}^{\sim}$ and $\varepsilon'(x)$ is a unit in $k[[x]]$. So w can be written in the form $w = x^{\gamma'} \varepsilon'(x)(1+yz')$ with $z' \in k[[x', y'-\alpha]]_{x'}^{\sim}$, so $w \in A$. In the second case write $z = yz'$ and either $z_i = yz_i'$ or $z_i = x^{\gamma_i} \varepsilon_i(x)(1+yz_i')$ (resp. either $w_j = yw_j'$ or $w_j' = x^{\delta_j} \varepsilon_j(x)(1+yw_j')$) where z', z_i', $w_j' \in k[[x', y'-\alpha]]_{x'}$ and ε_i, ε_i' are units in $k[[x]]$. As above

$$w = y \cdot z' \cdot \frac{z_1^* \ldots z_r^*}{w_1^* \ldots w_s^*} \varepsilon(x)(1+yz^*),$$

where $\varepsilon(x)$ is a unit in $k[[x]]$, $z^* \in k[[x', y']]_{x'}^{\sim}$ and either $z_i^* = z_i'$ or $z_i^* = x^{\gamma_i}$ (resp. $w_j^* = w_j'$ or $w_j^* = x^{\delta_j}$). Since $x = x'^{n'} y'^{-\tau}$, y' is a unit in $k[[x', y'-\alpha]]_{x'}^{\sim}$ it follows that $z'(z_1^* \ldots z_r^*)(w_1^* \ldots w_s^*)^{-1} \in k[[x', y'-\alpha]]_{x'}^{\sim}$, and hence $w \in A$. This completes the proof of the proposition.

COROLLARY 2. *Let* $\mathcal{O} = k[[x, y]]$ *as above with* $n = v(x) \leqslant v(y) = m$. *Then one has*

$$\tilde{\mathcal{O}}_y = k + x\tilde{\mathcal{O}}(m).$$

Proof. The set at right hand side member is a ring containing $k[[x]]$. Moreover, it also contains y since $y = xy^{-1}z$, where $z = y^2x^{-1} \in \hat{\mathcal{O}}$, and it is saturated with respect to $\{y\}$, so by Lemma 2 one has $\tilde{\mathcal{O}}_y \subset k + x\tilde{\mathcal{O}}(m)$. On the other hand, $\tilde{\mathcal{O}}(m) = k[[x', y'-\alpha]]_{x'}$ and $x \cdot x' = x \cdot y^{\tau} \cdot x^{-\sigma} \in \tilde{\mathcal{O}}_y$ and $x \cdot y' = xy^{n'}x^{-m'} \in \tilde{\mathcal{O}}_y$ so $\tilde{\mathcal{O}}(m) \subset \tilde{\mathcal{O}}_y(n)$ and hence $k + \tilde{\mathcal{O}}(m) \subset \tilde{\mathcal{O}}_y$. □

COROLLARY 3. *Let $\mathcal{O}$ be the local ring of a plane branch, x a transversal parameter for it, and $\mathcal{O}'$ the quadratic transform of $\mathcal{O}$. Then $\tilde{\mathcal{O}}'_x = \tilde{\mathcal{O}}(n)$ where $n = v(x)$ is the multiplicity of $\mathcal{O}$.*

Proof. Take another transversal parameter y, such that $\{x, y\}$ be a system of generators of $\mathfrak{m}$. Then, since $v(y) = v(x)$, one has $m = n$ and so $m' = n' = 1$, so take $\sigma = -1$, $\tau = 0$. One has $x' = x$ and $y' = y \cdot x^{-1}$, hence $\tilde{\mathcal{O}}(n) = k[[x, yx^{-1} - \alpha]]_x^{\sim} = \tilde{\mathcal{O}}'_x$. □

THEOREM 1. *The characteristic exponents of $\mathcal{O}$ given in Definition* 1 *are the characteristic exponents introduced in* [2], *Chapter* 3.

Proof. Above results allow us to write down the characteristic exponents of $\mathcal{O}'$ in terms that those of $\mathcal{O}$. In fact, if the multiplicity of $\mathcal{O}'$ is that of $\mathcal{O}$ then $\hat{\mathcal{O}}(n) = \tilde{\mathcal{O}}'$ and so one has

$$\beta_0 = \beta'_0 \quad \text{and} \quad \beta'_v = \beta_v - \beta_0, \quad v \geqslant 1.$$

If the multiplicity n_1 of $\mathcal{O}'$ is less than those of $\mathcal{O}$ one has (with notations as in Corollary 3) that $\tilde{\mathcal{O}}(n+n_1) = \tilde{\mathcal{O}}'_x(n_1) = \tilde{\mathcal{O}}'(n)$ so $\tilde{\mathcal{O}}(\beta_1) = \tilde{\mathcal{O}}'(n)$. (It is a simple matter to check that $\beta_1 = n + n_1$). We shall distinguish two cases according as $n_1 \mid n$ or $n_1 \nmid n$. In the first case is clear that $\beta'_0 = n_1 = \beta_1 - \beta_0$ and $\beta'_v = (\beta_{v+1} - \beta_1) + \beta_0$, $v \geqslant 1$. □

Note that above formulas coincide with those given in [2], 3.2.11, so the characteristic exponenets given here are the same than those given there.

Remark 1. The main Proposition 1 is also true if we do not assume that x is transversal. Thus if $\{x, y\}$ is a system of generators for $\mathfrak{m}$ and if x', y' are defined in the same way, then one can prove $k[[x, y]]_x^{\sim}(m) = k[[x'y']]_{x'}^{\sim}$. The proof is just that given before, taking into account the easy fact that the ring A is saturated with respect to x.

Using this formula one can deduce the Theorem 1 in [3]. The proof given in [3] has a gap because for the various considered elements x_i, y_i it is assumed that $v(x_i) < v(y_i)$ which can not be true. To fill this gap one would need relative saturation. Note that

$$k[[x_i, y_i]]_{x_i}^{\sim}(\beta_{i+1} - \beta_i) = k[[x_{i+1}, y_{i+1}]]_{x_{i+1}}^{\sim}$$

so the Theorem 1 in [3] follows directly from this fact.

2. Characteristic exponents of strange branches

According to [4], a plane branch over k, $\operatorname{Spec} \mathcal{O} \to \operatorname{Spec} k$, will be said to be *strange* (of type I) if there exists $z \in \mathfrak{m} - \mathfrak{m}^2$ such that $Dz = 0$ for any $D \in \operatorname{Der}_k(\bar{\mathcal{O}}, \bar{\mathcal{O}})$. If t is a uniformizing for $\bar{\mathcal{O}}$, we have $\bar{\mathcal{O}} = k[[t]]$, so above condition is equivalent to the single on $Dz = 0$ for the particular $D = \dfrac{d}{dt}$.

If $C = \operatorname{Spec} \mathcal{O}$ is strange, and $p = \operatorname{charac}(k) > 0$, then z can be taken in such a way that $v(z) \neq kn_0$ for all $k \in z_+$, $k \not\equiv 0 \pmod p$, n_0 being the multiplicity of $\mathcal{O}$. In [4] is proved that if z is as above then $v(z) = m_0$ where the integer m_0 only depends on the k-algebra $\mathcal{O}$. We will not use this fact in the sequel, so we refer to the quoted paper for its proof. Take a particular z and call m_0 to the corresponding $v(z)$, $m_0 \neq kn_0$ for all $k \not\equiv 0 \pmod p$.

We shall distinguish two cases: In $m_0 > n_0$, take a basis $\{x, y\}$ of $\mathfrak{m}$ such that $y = z$ (so one has $v(x) = n_0$, $v(y) = m_0$). If $m_0 = n_0$, take a basis $\{x, y\}$ of $\mathfrak{m}$ such that $x = z$ and $v(y) = \beta_1$.

Now set $n = v(x)$, $m = v(y)$ and consider the Euclid's algorithm for n, m as in Section 1. Set, in particular, $m' = m \cdot r_s^{-1}$, $n' = n \cdot r_s^{-1}$, r_s being the g.c.d. of n, m.

In case that $m' \not\equiv 0 \pmod p$ take an element $u \in \mathcal{O}$ such that $u^{m'} = y$. In the same way, if $n' \not\equiv 0 \pmod p$, take an element $w \in \mathcal{O}$ such that $w^{n'} = x$. In the first case set $\mathcal{O}^* = k[[x, u]]$ and, in the second one, set $\mathcal{O}^{**} = k[[y, w]]$. Note that $\mathcal{O}^*$ and $\mathcal{O}^{**}$ are both defined if $m' \not\equiv 0$, $n' \not\equiv 0 \pmod p$. We have the following result:

PROPOSITION 2. *Denote by $\mathcal{O}^{(j)}$ the j-th successive quadratic blow up of $\mathcal{O}$. Then, one has*

$$\tilde{\mathcal{O}}^*_x = \mathcal{O}^{(h+h_1+\dots+k-1)\sim} =: \tilde{\mathcal{O}}^{**}_y,$$

in case that $\mathcal{O}^*$ (resp. $\mathcal{O}^{**}$) be defined.

Proof. Choosing σ, τ appropriately, it is seen in [2], 3.4.11 that

$$\mathcal{O}^{(h+h_1+\dots+k-1)} = k[[x', x' \cdot y']]$$

(with notations as in Section 1).

Assume, first, that $m' \not\equiv 0 \pmod p$, and consider $\mathcal{O}^*$. One has

$$x' = u \cdot u^{n'\sigma} \cdot x^{-\sigma} \in \tilde{\mathcal{O}}^*_x,$$

$$x' y' = x' y^{n'} x^{-m'} = x'(u^{n'} \cdot x^{-1})^{m'} \in \tilde{\mathcal{O}}^*_x,$$

so $\mathcal{O}^{(h+h_1+\dots+k-1)\sim} \subset \tilde{\mathcal{O}}^*_x$. Conversely, one has

$$x = x'^{n'} \cdot (x' y')^{-\tau} \cdot x'^{\tau} \in \mathcal{O}^{(h+h_1+\dots+k-1)\sim},$$

$$u = x' \cdot x'^{\sigma} \cdot (x' \cdot x^{-1} u^{n'})^{-\sigma} \in \mathcal{O}^{(h+h_1+\dots+k-1)\sim},$$

since $x'x^{-1}u^{n'} \in \mathcal{O}^{(h+h_1+\ldots+k-1)\sim}$ (in fact, this is equivalent to say $x^{-1}u^{n'} \in \mathcal{O}^{(h+h_1+\ldots+k-1)\sim}(x')$, which is true since $m' \not\equiv 0$ and $(x^{-1}u^{n'})^{m'} \doteq y' \in \mathcal{O}^{(h+h_1+\ldots+k-1)\sim}(x')$, so the conclusion follows from the Hensel lemma applied to the polynomial $T^{m'}-y'$ over the complete local ring $\mathcal{O}^{(h+h_1+\ldots+k-1)\sim}(x')$). Thus, the other containtement is a consequence from the fact that the ring $\mathcal{O}^{(h+h_1+\ldots+k-1)\sim}$ is saturated with respect to x (this follows from the above expression for x); so $\tilde{\mathcal{O}}_x^* = \mathcal{O}^{(h+h_1+\ldots+k-1)\sim}$.

Now, if $n' \not\equiv 0 \pmod p$, one has

$$x' = w\cdot(yw^{-m'})^{\tau} \in \tilde{\mathcal{O}}_y^{**},$$

$$y' = x'\cdot(y\cdot w^{-m'})^{n'} \in \tilde{\mathcal{O}}_y^{**},$$

so $\mathcal{O}^{(h+h_1+\ldots+k-1)\sim} \subset \mathcal{O}_y^{**}$. Conversely, one has

$$y = x'^{m'}(x'y')^{-\sigma}\cdot x'^{\sigma} \in \mathcal{O}^{(h+h_1+\ldots+k-1)\sim},$$

$$w = x'\cdot x'^{\tau}\cdot(x'\cdot y\cdot w^{-m'})^{-\tau} \in \mathcal{O}^{(h+h_1+\ldots+k-1)\sim}$$

since, as above, $x'\cdot y\cdot w^{-m'} \in \mathcal{O}^{(h+h_1+\ldots+k-1)\sim}$. □

COROLLARY 4. *The ring $\tilde{\mathcal{O}}_x^*$ depends only on the ring $\mathcal{O}$.*

COROLLARY 5. *With notations as above, one has*

$$\tilde{\mathcal{O}}^*(n) = \tilde{\mathcal{O}}(m) \quad \textit{and} \quad \tilde{\mathcal{O}}^{**}(m) = \tilde{\mathcal{O}}(m).$$

Proof. Let e denote the greatest common divisor of n and m. Then one has

$$\tilde{\mathcal{O}}^*(n) = \tilde{\mathcal{O}}_n^*(e) = k[[x', x'y']]^\sim(e) = k[[x', y'-\alpha]]_{x'}^\sim = \tilde{\mathcal{O}}(m),$$

the above equalities being a consequence of Corollaries 2 and 3 and Proposition 2. In the same way, one has

$$\tilde{\mathcal{O}}^{**}(m) = \tilde{\mathcal{O}}_y^{**}(e) = k[[x', x'y']]^\sim(e) = k[[x', y'-\alpha]]_{x'}^\sim = \tilde{\mathcal{O}}(m). \quad \square$$

COROLLARY 6. *If $\{\beta_0, \ldots, \beta_g\}$ denote the set of characteristic exponents of $\mathcal{O}$ and $\{\beta_0^*, \ldots, \beta_{g^*}^*\}$* (resp. $\{\beta_0^{**}, \ldots, \beta_{g^{**}}^{**}\}$) *the set of characteristic exponents of $\mathcal{O}^*$* (resp. $\mathcal{O}^{**}$) *then one has*

If $n \mid m$, then

$$g^* = g, \quad \beta_0^* = e, \quad \beta_\nu^* = \beta_\nu - m + \beta_0, \quad \nu \geqslant 1;$$

$$g^{**} = g, \quad \beta_0^{**} = e, \quad \beta_\nu^* = \beta_\nu, \quad \nu \geqslant 1.$$

If $n \nmid m$, then

$$g^* = g-1, \quad \beta_0^* = e, \quad \beta_\nu^* = \beta_{\nu+1} - m + \beta_0, \quad \nu \geqslant 1.$$

$$g^{**} = g-1, \quad \beta_0^* = e, \quad \beta_\nu^{**} = \beta_{\nu+1}, \quad \nu \geqslant 1.$$

THEOREM 2 (see [4], Theorem 4). *If $\mathcal{O}$ is the local ring a strange (of type* I) *branch with characteristic exponents* $\{\beta_0, \ldots, \beta_g\}$ *then one has either* $\beta_0 \not\equiv 0 \pmod{p}$ *and* $\beta_1 \equiv \beta_2 \equiv \ldots \equiv \beta_g \equiv 0 \pmod{p}$, *or* $\beta_0 \equiv \ldots \equiv \beta_i \equiv 0 \pmod{p}$ *and* $\beta_{i+1} \equiv \ldots \equiv \beta_g \equiv c \not\equiv 0 \pmod{p}$ *for some* i, $0 \leqslant i \leqslant g-1$, *and some* c, $1 \leqslant c \leqslant p-1$.

Proof. First note that of $\mathcal{O}$ is strange of type I then $\mathcal{O}^*$ (resp. $\mathcal{O}^{**}$) is again strange of type I, when it is defined. Thus, to prove the theorem we will use induction on the number of quadratic transformations needed to desingularize $\mathcal{O}$. If $\mathcal{O}$ is regular the statement is evident, since $g = 0$ and $\beta_0 = 1 \not\equiv 0 \pmod{p}$.

In the inductive step we shall distinguish several cases: If $\beta_0 \not\equiv 0 \pmod{p}$ then $n' \not\equiv 0$, $e \not\equiv 0$, $m = m_0 \equiv 0 \pmod{p}$, so $\beta_0^{**} \not\equiv 0 \pmod{p}$. By the induction hypothesis one has $\beta_\nu^{**} \equiv 0 \pmod{p}$, which implies $\beta_1 \equiv \ldots \equiv \beta_g \equiv 0 \pmod{p}$, in the two possible cases $n \mid m$ or $n \nmid m$ in Corollary 6.

If $\beta_0 \equiv 0$, $\beta_1 \not\equiv 0 \pmod{p}$, we have $m_0 = k\beta_0$, with $k \not\equiv 0 \pmod{p}$. If $k > 1$, we have $m = m_0$, so from Corollary 6, the characteristic exponents of $\mathcal{O}^*$ are $\beta_0^* = \beta_0$ and $\beta_\nu^* = \beta_\nu - (k-1)\beta_0$ with $g^* = g$. By the induction hypothesis, since $\beta_0^* \equiv 0 \pmod{p}$ and $\beta_1^* \not\equiv 0 \pmod{p}$, it follows that $\beta_1^* \equiv \ldots \equiv \beta_g^* \equiv 0 \pmod{p}$ and hence $\beta_1 \equiv \ldots \equiv \beta_g \not\equiv 0 \pmod{p}$. If $k = 1$, then $m = \beta_1$, so $e \not\equiv 0 \pmod{p}$. Since $n \nmid m$, the characteristic exponents of $\mathcal{O}^*$ are $\beta_0^* = e$, $\beta_\nu^* = \beta_{\nu+1} - \beta_1 + \beta_0$ with $g^* = g-1$. By the induction hypothesis one has $\beta_1^* \equiv \ldots \equiv \beta_{g-1}^* \equiv 0 \pmod{p}$. Thus, $\beta_1 \equiv \ldots \equiv \beta_g \not\equiv 0 \pmod{p}$.

Finally, if $\beta_0 \equiv \beta_1 \equiv 0 \pmod{p}$, then, always, one has $n \equiv m \equiv 0 \pmod{p}$ so according to Corollary 6, by applying the induction hypothesis to $\mathcal{O}^*$ or $\mathcal{O}^{**}$ one concludes $\beta_0 \equiv \ldots \equiv \beta_i \equiv 0 \pmod{p}$, and $\beta_{i+1} \equiv \ldots \equiv \beta_g \not\equiv 0 \pmod{p}$ for some $i \geqslant 1$. □

Remark 2. In [4], it is also considered the case of type II stranges branchs, i.e., branches corresponding to projective plane curves centered at a point $\mathcal{O}$ throughout which every tangent line to the projective curve passes. It is clear that by making one quadratic transformation, a type II strange branch one obtains a type I strange branch, so characteristic exponents of type II branches can be studied from the above result. In [4] it is also indicated how the arithmetical conditions on $\beta_0, \ldots, \beta_g$ in Theorem 2 are not sufficient in order that to guarantee the existence of a strange branch having $\beta_0, \ldots, \beta_g$ as characteristic exponents. However, it is clear that the technique in the above proof based in the formulae in Corollary 6, provides an algorithm which says us if a set $\beta_0, \ldots, \beta_g$ corresponds or not to a strange branch. This algorithm will be analyzed in a forthcoming paper.

3. The characteristic zero case

Let $\mathcal{O}$ be the ring of an irreducible algebroid curve over the algebraically closed field k. Take $w \in \mathfrak{m}$, $w \neq 0$, and denote by m the value of w, i.e.,

$m = v(w)$. Assume that $m \not\equiv 0 \pmod p$, p being the characteristic of k. (If $p = 0$, we mean that any m verifies this condition). Finally take an uniformizing t in $\mathcal{O}$ such that $w = t^m$. The results and comments in this section will follow from the following result.

PROPOSITION 3 (see [9] and [8]). *If $\mathcal{O}$ is saturated relative to w and S is the semigroup of values of $\mathcal{O}$, then one has*

$$\mathcal{O} = \{\sum_{\gamma \in S} a_\gamma t^\gamma \mid a_\gamma \in k \ \forall \gamma \in S\},$$

i.e., *$\mathcal{O}$ is the monomial curve corresponding to the semigroup S.*

Proof. Take $\gamma \in S$. We claim $t^\gamma \in \mathcal{O}$. In fact, take $z \in \mathcal{O}$ such that $v(z) = \gamma$ and $z = t^\gamma +$ higher order terms. The statement $t^\gamma \in \mathcal{O}$ is equivalent to $t^\gamma z^{-1} \in \mathcal{O}(z)$. Now $t^\gamma z^{-1}$ is a root of the polynomial

$$Y^m - t^{\gamma m} z^{-m} = Y^m - w^\gamma z^{-m}.$$

This polynomial has coefficients in $\mathcal{O}(z)$ since $z \cdot w^\gamma z^{-m} \in \mathcal{O}$, as $\mathcal{O}$ is saturated relative w. The residual polynomial in $\mathcal{O}/\mathfrak{m}[Y]$ is $Y^m - 1$ and it has all its roots distinct and in k, since $m \not\equiv 0 \pmod p$. So the roots of $Y^m - t^{\gamma m} z^{-m}$ exist and they are in $\mathcal{O}(z)$. In particular $t^\gamma z^{-1} \in \mathcal{O}(z)$ as required.

Since $\mathcal{O}$ is complete, $\mathcal{O}$ contains all the elements of type $\sum a_\gamma t^\gamma$, so $\mathcal{O}$ contains the monomial curve given by S. Since both $\mathcal{O}$ and the monomial curve have the same semigroup, they must be equal. This completes the proof of the proposition. □

PROPOSITION 4 (see [8]). *If S is a subsemigroup of $\mathbf{Z}_+$ which verifies property* (A_m) *with* $m \in S$ (see the Introduction) *and if $\mathbf{Z}_+ - S$ is finite, then the monomial ring*

$$\mathcal{O}_S = \{\sum_{\gamma \in S} a_\gamma t^\gamma \mid a_\gamma \in k \ \forall \gamma \in S\}$$

is saturated with respect to $w = t^m$.

Proof. Take $z, z_1, \ldots, z_r, w_1, \ldots, w_s$ nonzero elements in $\mathcal{O}_S$ and $l \in \mathbf{Z}$ such that $z \cdot z_i^{-1}$, $z \cdot w_j^{-1}$, $z_1 \ldots z_r (w_1 \ldots w_s)^{-1} \cdot w^l \in \mathcal{O}_S$. Set $z = t^\gamma u$, $z_i = t^{\gamma_i} u_i$, $w_j = t^{\gamma'_j} u'_j$, u, u_i, u'_j being units in the monomial rings $\mathcal{O}_{S(\gamma)}$, $\mathcal{O}_{S(\gamma_i)}$, $\mathcal{O}_{S(\gamma'_j)}$, where for each $\delta \in S$, the set $S(\delta)$ is defined by

$$S(\delta) = \{\varrho \in S \mid \varrho = \delta' - \delta, \text{ for some } \delta' \geqslant \delta,\ \delta' \in S\}.$$

Note that from property (A_m) it follows that $S(\delta)$ is a semigroup of a finite complement in $\mathbf{Z}_+$.

Now one has

$$z^* = z(z_1 \ldots z_r)(w_1 \ldots w_s)^{-1} w^l = t^\varrho u u_1 \ldots u_r (w_1 \ldots u'_s)^{-1}.$$

Since each u_i (resp. $u_j'^{-1}$) is a sum of terms of type $a_\sigma t^\sigma$ with $\sigma \in S(\gamma_i)$

(resp. $\sigma \in S(\gamma'_i)$), it follows from property (A_m) that z^* is a sum of terms of type $a_\sigma t^\sigma$ with $\sigma \in S$, so $z^* \in \mathcal{O}_S$ and $\mathcal{O}_S$ is saturated relative to w, and the proposition is proved. $\square$

Now, consider a system of generators of $\mathfrak{m}$ of type $w_1, w_2, \ldots, w_N$, where $w_1 = w = t^m$ is as above with $m \not\equiv 0 \pmod p$. This system of generators needs not be minimal and m needs not be the minimum among the $v(w_i)$'s. Then, one has a Puiseux's type parametrization

$$\text{(I)} \qquad \begin{aligned} w_1 &= t^m, \\ w_j &= \sum_{i>0} a_i^{(j)} t^i \qquad (j = 2, \ldots, N). \end{aligned}$$

Consider the set A formed by m and the indices i such that $a_i^{(j)} \neq 0$ for some j. Denote by S' the minimum subsemigroup of $\boldsymbol{Z}_+$ containing A and verifying property (A_m).

PROPOSITION 5 (see [8]). *The saturated ring $\tilde{\mathcal{O}}_w$ is the monomial ring $\mathcal{O}_{S'}$ corresponding to the semigroup S'.*

Proof. $\mathcal{O}_{S'}$ is saturated with respect to $w = t^m$ in view of the above proposition, and, since $w_1, \ldots, w_N \in \mathcal{O}_{S'}$, it is clear that $\tilde{\mathcal{O}}_w \subset \mathcal{O}_{S'}$. Conversely, if S is the semigroup of values of $\tilde{\mathcal{O}}_w$ one has $\tilde{\mathcal{O}}_w = \mathcal{O}_S$ and, since $w_j \in \tilde{\mathcal{O}}_w$, it follows that $m \in S$ and $i \in S$ for every i such that $a_i^{(j)} \neq 0$. Thus, one has $A \subset S$ and, hence, $S' \subset S$ as S verifies (A_m). So $\mathcal{O}_{S'} \subset \mathcal{O}_S = \tilde{\mathcal{O}}_w$ which completes the proof of the proposition. $\square$

For a set of parametric equations as (I), one can define the sets of numbers $\beta_0, \ldots \beta_g$ (Puiseux's characteristic exponents) as follows:

$$\beta_0 = m,$$

$$\beta_{v+1} = \min\{i \in \boldsymbol{Z}_+ \mid a_i^{(j)} \neq 0 \text{ for some } j \text{ and } \text{g.c.d.}(\beta_0, \ldots, \beta_v, i) < \text{g.c.d.}(\beta_0, \ldots, \beta_v)\},$$

$$\text{g.c.d.}(\beta_0, \ldots, \beta_g) = 1.$$

It follows from Proposition 5 that the data $\{\beta_0, \ldots, \beta_g\}$ is equivalent to the data (S', m).

COROLLARY 7 (invariance of characteristics exponents). *Let $\mathcal{O}$ be a plane branch with multiplicity n, $n \not\equiv 0 \pmod p$. Then, the characteristic exponents of $\mathcal{O}$ defined in Section 1 are the same than the characteristic exponents associated with every Puiseux parametrization of type* (I), *with $m = n$, for $\mathcal{O}$.*

This result shows us that the characteristic exponents of Puiseux series only depend on the ring structure of $\mathcal{O}$ (compare with [1]). Next, we will derive the formulae for the inversion of characteristic exponents, also included in [1].

THEOREM 3 (inversion of characteristic exponents) (see [8]). *Let $\mathcal{O}$ be a plane branch and $\{x, y\}$ a system of generators of* $\mathfrak{m}$ *such that $v(x) = n$, $v(y) = m$ with $n \not\equiv 0$, $m \not\equiv 0$ (mod p), and $n < m$. Take uniformizing parameters t, t^* in $\bar{\mathcal{O}}$ such that $x = t^n$, $y = t^{*m}$ and set*

$$\text{(I)}\qquad x = t^n, \quad y = \sum_{i=m}^{\infty} a_i t^i,$$

$$\text{(II)}\qquad x = \sum_{j=n}^{\infty} b_j t^{*j}, \quad y = t^{*m}.$$

If $\{\beta_0, \ldots, \beta_g\}$ (resp. $\{\beta_0^*, \ldots, \beta_{g^*}^*\}$) *are the characteristic exponents of the Puiseux parametrization* (I) (resp. (II)), *then one has*

(i) $g = g^* - 1$, $\beta_0 = \beta_1^*$ *and* $\beta_\nu = \beta_{\nu+1}^* + m - n$, $\nu \geqslant 1$, *if* $n \mid m$;

(ii) $g = g^*$, $\beta_0 = \beta_1^*$ *and* $\beta_\nu = \beta_{\nu+1}^* + m - n$, $\nu \geqslant 1$, *if* $n \nmid m$.

Proof. Proposition 5 shows us the semigroups of $\tilde{\mathcal{O}}_y$ and $\tilde{\mathcal{O}} = \tilde{\mathcal{O}}_x$ in terms of the Puiseux characteristic exponents of (I) and (II), and from Corollary 2 one has $\tilde{\mathcal{O}}_y = k + x\tilde{\mathcal{O}}(m)$, so by identifying the semigroups of both sides in this equalities one obtains the formulae (i) and (ii). □

Remark 3. As said in the Introduction, the results in this section are taken from [8]. There, the equality $\tilde{\mathcal{O}}_y = k + x\tilde{\mathcal{O}}(m)$ is directly proved for the case of multiplicities prime to the characteristic. In this way, the actual proof in [8] is even shorter than our proof of Proposition 2.

4. The case of several branches

In this section we will assume that $\operatorname{Spec} \mathcal{O} \to \operatorname{Spec} k$ is a reduced algebroid curve over the algebraically closed field k. Then one has $\bar{\mathcal{O}} = \bar{\mathcal{O}}_1 \times \ldots \times \bar{\mathcal{O}}_r$ (r the number of branches) and $\bar{\mathcal{O}}_i = k[[t_i]]$ for each i. If v_i: $\bar{\mathcal{O}} \to \boldsymbol{Z}_+ \cup \{\infty\}$, is the function given by $v_i(z_1, \ldots, z_r) = \operatorname{ord}_{t_i} z_i$; to the subring $\tilde{\mathcal{O}}$ of $\bar{\mathcal{O}}$ one can associate its semigroup of values S given by

$$S = \{(v_1(z), \ldots, v_r(z)) \in \boldsymbol{Z}_+^r \mid z \in \tilde{\mathcal{O}} \text{ and } z \text{ is not a zero divisor}\}.$$

The set S is in fact an additive subsemigroup of $\boldsymbol{Z}_+^r$, and as for $r = 1$ its structure can be determined in a simple way as indicate below.

For each $\gamma \in S$ and $z \in \tilde{\mathcal{O}}$ such that $v(z) = (v_1(z), \ldots, v_r(z)) = \gamma$ set $\tilde{\mathcal{O}}(z) = \{w \in \tilde{\mathcal{O}} \mid w \cdot z \in \tilde{\mathcal{O}}\}$. As in Section 1 one has that $\tilde{\mathcal{O}}(z)$ is a complete semilocal ring which only depends on γ. The maximal ideals of $\tilde{\mathcal{O}}(z) = \tilde{\mathcal{O}}(\gamma)$ correspond bijectively to the cosets of the equivalence relation on $\{1, 2, \ldots, r\}$ given by

$$i \sim j \quad \Leftrightarrow \quad \text{If } \delta \geqslant \gamma \text{ and } \delta_i = \gamma_i \text{ then } \delta_j = \gamma_j.$$

(Here $\delta = (\delta_1, \ldots, \delta_r)$, $\gamma = (\gamma_1, \ldots, \gamma_r)$ and $\delta \geqslant \gamma$ means $\delta_i \geqslant \gamma_i$ $\forall i$).

On the other hand, since $\tilde{\mathcal{O}}$ is local, there exists an element $\mu \in S$ such that $\mu \neq \mathbf{0} = (0, \ldots, 0)$ and $\mu \leqslant \gamma$ $\forall \gamma \in S - \{\mathbf{0}\}$. Let L denote the set of elements in S which lie on the line passing through $\mathbf{0}$ and μ. For each $\gamma \in S$ set $S(\gamma) = \{\delta \in S \mid \delta \geqslant \gamma\}$ and $S_0(\gamma) = \{\delta - \gamma \mid \delta \in S(\gamma)\}$. Since $\tilde{\mathcal{O}}$ has a conductor, i.e., $(t_r^{\delta_1}) \times \ldots \times (t_r^{\delta_r}) \subset \tilde{\mathcal{O}}$ for some values of $\delta_1, \ldots, \delta_r$, it follows that the set $I_0 = \{\gamma \in S \mid S \subset L \cup S(\gamma)\}$ is finite and, of course, it is totally ordered by $\leqslant$. Finally, set $\gamma^0 = \max I_0$. The following result sums up the arithmetical properties of S, and indicates the recursive possibilities for studying S.

THEOREM 4 (see [3], Sect. 3). (a) *The equivalence relation associated to γ^0 has at least two cosets.*

(b) *If $J_1, \ldots, J_l$ are the cosets corresponding to the relation associated to γ^0 then*

$$S = L \cup \{\gamma^0 + S_1 \times \ldots \times S_l\},$$

where $S_0(\gamma^0) = S_1 \times \ldots \times S_l$ and the S_α's are the semigroups of saturated local rings $\tilde{\mathcal{O}}_1, \ldots, \tilde{\mathcal{O}}_r$ such that $\tilde{\mathcal{O}}(\gamma^0) = \tilde{\mathcal{O}}_1 \times \ldots \times \tilde{\mathcal{O}}_r$.

(c) *For each i, $1 \leqslant i \leqslant r$, $pr_i S \subset \mathbf{Z}_+$ is the semigroup of the saturation of the i-th branch of $\mathcal{O}$.*

Proof. (a) Set $\gamma^0 = (\gamma_1, \ldots, \gamma_r)$ and $\mu = (\mu_1, \ldots, \mu_r)$ and order the coordinates in such a way that $\gamma_1/\mu_1 \geqslant \gamma_2/\mu_2 \geqslant \ldots \geqslant \gamma_r/\mu_r$.

Take elements x, y such that $v(x) = \mu$ and $v(y) = \gamma^0$. Then one has $y^* = y \cdot y^{\mu_r} x^{-\gamma_r} \in \tilde{\mathcal{O}}$, since, by construction, one has $y^{\mu_r} x^{-\gamma_r} \in \tilde{\mathcal{O}}$. Now, if $\gamma^0 \in L$, it is clear that the relation has more than one coset since γ^0 is the maximum of I_0. If $\gamma^0 \notin L$ then $\gamma_1/\mu_1 > \gamma_r/\mu_r$, so the index 1 is not equivalent to r since $\delta^* = v(y^*) \geqslant \gamma^0$ and $\delta_1^* > \gamma_1$ but $\delta_r^* = \mu_r$.

(b) The rings $\tilde{\mathcal{O}}_\alpha$ are nothing but the localizations of $\tilde{\mathcal{O}}(\gamma^0)$, at its maximal ideals. It is clear that each $\mathcal{O}$ is again saturated. The result follows from this fact.

(c) For each $J \subset \{1, 2, \ldots, r\}$ denote by ε_J the element in $\{0, 1\}^r$ given by $pr_i \varepsilon_J = 0$ if $i \notin J$ and $pr_i \varepsilon_J = 1$ if $i \in J$. If J is a coset for the relation associated to γ^0 and if we choose elements $x_0, x_1, \ldots, x_m, y \in \tilde{\mathcal{O}}$ such that $v(x_0) \leqslant v(x_1) < \ldots < v(x_m) < v(y)$ are the values in I_0, then one has

$$\tilde{\mathcal{O}} = kx_0 + kx_1 + \ldots + kx_m + y\tilde{\mathcal{O}}_1 \times \ldots \times \tilde{\mathcal{O}}_l,$$

so it is clear that $\tilde{\mathcal{O}}\varepsilon_J = kx_0\varepsilon_J + \ldots + kx_m\varepsilon_J + y\varepsilon_J\tilde{\mathcal{O}}_\alpha$ is a saturated subring of $Z_+^{\#J}$ ($\tilde{\mathcal{O}}_\alpha$ corresponds to the coset J), and in fact $\tilde{\mathcal{O}}\varepsilon_J$ is the saturation of $\mathcal{O}\varepsilon_J$. Making induction on r, it becames clear that $S\varepsilon_{\{i\}} = pr_i S$ is the semigroup of values of the saturation of the ith branch of Spec $\mathcal{O}$. □

Remark 4. In [9] it is determined the arithmetical conditions on a subsemigroup $S \subset Z_+^r$ in order that S be the semigroup of values of the saturation of same algebroid curve, proving the above theorem directly from these conditions. It is also proved, in this paper, that, for such a S, there exists a "monomial algebroid curve" having S as semigroup, and that, conversely, if charac $k = 0$, and $\tilde{\mathcal{O}}$ is a saturated curve over k then $\tilde{\mathcal{O}}$ is isomorphic to a monomial one. This result connects with those given in Section 3 for the case $r = 1$.

In [3] it is also proved that if Spec $\mathcal{O}$ is an algebroid plane curve, then the equisingularity type of Spec $\mathcal{O}$ determines and it is determined by the semigroup of values S of the saturation $\tilde{\mathcal{O}}$. Thus, in particular, intersection multiplicities for two branches can be computed from S.

5. Nonalgebraically closed base fields

In this section we will assume that ψ: Spec $\mathcal{O} \to$ Spec k is an irreducible algebroid branch over k, where k is not assumed to be algebraically closed. The complete local k-algebra $\mathcal{O}$ is excelent, so $\bar{\mathcal{O}}$ is again a complete discrete valuation ring with normalized valuation denoted also by v. If k' denotes the residue field of $\bar{\mathcal{O}}$, then k' is a finite extension of $\mathcal{O}/\mathfrak{m} \approx k$, but although $\bar{\mathcal{O}}$ has coefficient fields, which are isomorphic to k', it is not true in general that one has a commutative diagram

$$\begin{array}{ccc} k & \xrightarrow{i} & k' \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu'} \\ \mathcal{O} & \xrightarrow{i^*} & \bar{\mathcal{O}} \end{array}$$

for some ring homomorphism μ', where i, i^* are the natural inclusions and μ the homomorphism corresponding to ψ. However, if k'/k is separable there exist a unique μ' making the diagram commutative. In fact, $F = \{z \in \bar{\mathcal{O}} \mid z$ is integral over $\operatorname{Im}\mu\}$ is a subfield of $\bar{\mathcal{O}}$, so it is sufficient to see that the (injective) composing map $F \to \bar{\mathcal{O}} \to \bar{\mathcal{O}}/\bar{\mathfrak{m}} = k'$ is onto. Take $\alpha \in k'$, and let $p(Y) \in k[Y]$ be its irreducible polynomial over k. Denote by $p^\mu(Y) \in \mathcal{O}[Y]$ the polynomial obtaining by changing the coefficients of $p(Y)$ by its images by μ. Since $\bar{\mathcal{O}}$ is henselian and the residual polynomial of $p^\mu(Y)$ is $p(Y) = (Y-\alpha)q(Y)$ and $q(Y) \in k'[Y]$, $q(\alpha) \neq 0$, it follows that $p^\mu(Y)$ has a root $z \in \bar{\mathcal{O}}$ whose residue (module $\bar{\mathfrak{m}}$) is α. It is clear that $z \in F$ and that it is mapped to α. Now μ' is the inverse of the above isomorphism $F \to k'$, and it is clear that μ' is unique.

We will assume in the sequel that k'/k is separable. So if t is a uniformizing for $\bar{\mathcal{O}}$ and $\{x_1, \ldots, x_N\}$ a system of generators for $\mathfrak{m} \subset \mathcal{O}$, one has the identifications $\bar{\mathcal{O}} = k'[[t]]$ and $\mathcal{O} = k[[x_1, \ldots, x_N]] \subset k'[[t]]$ where x_i is identified to its image $x_i = x_i(t)$ in $k'[[t]]$. The saturation $\tilde{\mathcal{O}}$ of $\mathcal{O}$ is, in view of Lemma 2, an irreducible algebroid curve over the same k (see also [3], Prop. 1.5).

To a saturated ring $\tilde{\mathcal{O}}$ as above one can associate its semigroup of values $S = \{v(z) \mid z \in \tilde{\mathcal{O}},\ z \neq 0\}$ and to each $\gamma \in S$ one can associate the residue field $k(\gamma)$ if the local ring $\tilde{\mathcal{O}}(\gamma)$. Denote by $d(\gamma)$ the degree $[k(\gamma):k]$. Since $\tilde{\mathcal{O}}(\gamma)$ is again complete (Lemma 2) and $k(\gamma)/k$ is separable since $k(\gamma) \subset k'$, a similar argument than above shows that there exists a unique ring homomorphism $k(\gamma) \to \tilde{\mathcal{O}}(\gamma)$ making commutative the diagram

$$\begin{tikzcd}
k \arrow[r] \arrow[d] & k(\gamma) \arrow[r] \arrow[d] & k' \arrow[d] \\
\mathcal{O} \arrow[r] & \tilde{\mathcal{O}}(\gamma) \arrow[r] & \bar{\mathcal{O}}
\end{tikzcd}$$

Note, finally, that if $\gamma \leqslant \gamma'$, then $k(\gamma) \subset k(\gamma')$, so $d(\gamma) \mid d(\gamma')$.

Now, assume that $\mathcal{O}$ is an irreducible algebroid plane curve over k and let $\{x, y\}$ be a system of generators of the maximal ideal $\mathfrak{m}$ of $\mathcal{O}$. Then $\mathcal{O} = k[[x(t), y(t)]]$, $\bar{\mathcal{O}} = k'[[t]]$ and the coefficients of the power series $x(t)$, $y(t) \in k'[[t]]$ generate k' over k. Let $f(X, Y) \in k[[X, Y]]$ be a generator of the principal ideal $\mathrm{Ker}(P)$, where $P\colon k[[X, Y]] \to k'[[t]]$ is the homomorphism given by $P(X) = x(t)$, $P(Y) = y(t)$. In the same way, let $g(X, Y) \in k'[[X, Y]]$ be a generator of $\ker(P')$ where $P' = k'[[X, Y]] \to k'[[t]]$ is also given by $P'(X) = x'(t)$, $P'(Y) = y'(t)$.

Since k'/k is a separable extension, take the minimum Galois extension k'' of k containing k' and set $H = \mathrm{Hom}_k(k', k'')$, $G = \mathrm{Gal}(k''/k)$. For each $\sigma \in H$ (resp. $\varphi \in G$) and each power series s with coefficients in k' (resp. k'') denote by s^σ (resp. s^φ) the corresponding power series obtained by replacing the coefficients by its images by σ (resp. φ). Then, since $g \mid f$, one has $g^\sigma \mid f^\sigma = f$ $\forall \sigma \in H$ and the g^σ are pairwise different irreducible factors, since they correspond to different parametric equations $\{x^\sigma(t), y^\sigma(t)\}$. Thus, $g^* = \prod_{\sigma \in H} g^\sigma$ divides f and, since $g^{*\varphi} = g^*$ $\forall \varphi \in G$, one has $g^* \in k[[X, Y]]$, and hence $f = u \cdot g^*$, u a unit in $k[[X, Y]]$.

It follows that the algebroid curve $\mathcal{O} \hat{\otimes}_k \bar{k} = \bar{k}[[X, Y]]/(f)$ ($\bar{k}$ an algebraic closure of k) has $\# H = h$ irreducible components which correspond to the g^σ and to the parametrization $\{x^\sigma(t), y^\sigma(t)\}$. The group G acts transitively on the components, so in particular, all the components have the same set of characteristic exponents. Using the same techniques than in Section 1, it is possible to proof a similar expression for $\tilde{\mathcal{O}}$ than in [3], Theorem 1, and deduce the following

THEOREM 5. (a) *The semigroup of values S of $\tilde{\mathcal{O}}$ is the semigroup of values of the saturation of each irreducible component of $\mathcal{O}\hat{\otimes}_k \bar{k}$.*

(b) *The semigroup of values $\bar{S} \subset \mathbf{Z}^h_+$ of $(\mathcal{O}\hat{\otimes}_k \bar{k})^\sim$ determines S and it is determined by S and by the map $S \to [1, h]$ given by $\gamma \mapsto d(\gamma)$.*

(c) *The action of G on S induced by the action of G on the components determines and it is determined by the mapping sending each γ to the field $k(\gamma)$.*

Proof. As said before one can deduces a formula for $\tilde{\mathcal{O}}$ similar to that of [3], Theorem 1. With notations as in [3] for a Hamburger–Noether expansion associated to $\{x(t), y(t)\}$, this formula looks as follows

$$\tilde{\mathcal{O}} = k + k(\beta_0)x_0 + k(2\beta_0)x_0^2 + \ldots + k(\beta_1)y_0 + k(\beta_1 + e_1)y_0 x_1 + \ldots$$

From this formula (a) directly follows, and also (b) and (c) since the Hamburger–Noether expansions and the saturations of the branches can be described using H and G, and hence the intersection multiplicities. We omit details for the sake of simplicity in the exposition of this paper. □

Remark 4. 1. An analogous result to Theorem 5 is obtained in the Dissertation by A. Granja [5], using the semigroup of values of $\mathcal{O}\hat{\otimes}_k \bar{k}$ instead that of $(\mathcal{O}\hat{\otimes}_k \bar{k})^\sim$.

2. If $\operatorname{Spec}\mathcal{O} \to \operatorname{Spec} k$ is reducible but reduced, then the situation is similar to Section 4, since we have as main invariants the semigroup S of $\tilde{\mathcal{O}}$ in $\mathbf{Z}^r_+$ (r the number of irreducible components over k) and a ring $k(\gamma) = k_1(\gamma) \times \ldots \times k_r(\gamma) \subset k'_1 \times \ldots \times k'_r$ and multidegree $d(\gamma) = (d_1(\gamma), \ldots, d_r(\gamma))$ for each $\gamma \in S$.

3. Note that even if k'/k is inseparable, the above data S and $k(\gamma)$ are well defined.

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