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ON STANLEY-REISNER RINGS

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0. Introduction

Let $R = \bigoplus_{i \geqslant 0} R_i$ be a graded algebra over $R_0 = k$ (a field), generated as an algebra by R_1 and let $\dim_k R_1 = n$. The algebra $R = \tilde{R}/I$, where $\tilde{R} = k[X_1, X_2, \ldots, X_n]$, is said to have a t-linear resolution if all elements in a minimal set of generators for I are of degree t and I has a minimal $\tilde{R}$-resolution with all maps homogeneous of degree one. This is equivalent to the fact that $\mathrm{Tor}_{i,j}^{\tilde{R}}(R, k) = 0$ for $j \neq i+t-1$ and for all $i > 0$. We recall that R is 2-linear if and only if it is both a Golod and a Koszul algebra, see [Ba-Fr, Theorem 7]. Let Δ be an abstract simplicial complex on the vertices $X_1, X_2, \ldots, X_n$. The Stanley–Reisner ring of Δ is $k[\Delta] = \tilde{R}/I$, where I is generated by $\{X_{i_1} X_{i_2} \ldots X_{i_k};\ i_1 < i_2 < \ldots < i_k,\ \{i_1, i_2, \ldots, i_k\} \notin \Delta\}$. This paper contains a characterization of those simplicial complexes Δ whose Stanley–Reisner rings have 2-linear resolutions. For Stanley–Reisner rings with relations only in degree two the complex Δ is in a precise way determined by its one-skeleton and thus the characterization is given graph-theoretically. This is the content of the first section and in the second section we restrict to the Cohen–Macaulay case. The third section deals with some numerical invariants of Stanley–Reisner rings, in particular of those with 2-linear resolutions. In the fourth section we give an easy algebraic proof of a known characterization of the depth of a Stanley–Reisner ring.

This paper is in final form and no version of it will be submitted for publication elsewhere.

1. Stanley–Reisner rings with 2-linear resolutions

We denote the complete graph on r vertices by K_r. The f-vector $(f_0, \ldots, f_k)$ associated to a graph is a vector whose i'th coordinate f_{i-1} is the number of sub-K_i's in G. (The reason for this indexing is that in the f-vector of a simplicial complex f_i denotes the number of faces of dimension i. We want the f-vector of a simplicial complex to be the same as the f-vector of the graph which is its 1-skeleton.) The following inductive definition of d-trees is customary:

(i) K_{d+1} is a d-tree.

(ii) If G is a d-tree and v a new vertex, and v is adjoined to G via a sub-K_d of G (so $\{v\} \cup K_d$ is complete) then $\{v\} \cup G$ is a d-tree.

Thus a 0-tree is a set of points and a 1-tree is a usual tree. The f-vector of a d-tree is

$$\left(\binom{d+1}{1}, \binom{d+1}{2}, \ldots, \binom{d+1}{d+1}\right) + n\left(\binom{d}{0}, \binom{d}{1}, \ldots, \binom{d}{d}\right).$$

We define a generalized d-tree inductively as follows:

(i) K_{d+1} is a generalized d-tree.

(ii) If G is a generalized d-tree and we attach a K_i to G in a K_j, where $0 \leqslant j < i \leqslant d+1$ (here K_0 should be interpreted as the empty set), then the new graph is a generalized d-tree.

Call a graph chordal (triangulated, rigid circuit graph, ...) if each circuit of length > 3 has a chord. Dirac shows [Di, Theorems 1 and 2] that the generalized d-trees are exactly the chordal graphs.

Let G be a graph and let $\Delta(G)$ be the simplicial complex associated to G in the following way: $\Delta(G)$ has the same vertices and 1-simplexes as G and furthermore $\Delta(G)$ contains every simplex $(X_{i_0}, X_{i_1}, \ldots, X_{i_k})$ for which the complete graph on $\{X_{i_0}, X_{i_1}, \ldots, X_{i_k}\}$ is a subgraph of G. It is easy to see that if $k[\Delta] = \tilde{R}/I$, then I is generated by elements of degree two if and only if $\Delta = \Delta(G(\Delta))$, where $G(\Delta)$ is the 1-skeleton of Δ.

There is a topological characterization of Stanley–Reisner rings with 2-linear resolutions, see [Fr, Theorem 9]. Namely, $k[\Delta]$ has a 2-linear resolution if and only if $\tilde{H}_i(\Delta') = 0$ for all $i > 0$ and all full subcomplexes Δ' of Δ. For such a ring $k[\Delta]$ we furthermore have that $k[\Delta]$ is Cohen–Macaulay if and only if also $\tilde{H}_0(\Delta') = 0$ for all full subcomplexes Δ' of Δ with at most d points removed from Δ, where $d = \dim \Delta - 1$, [Fr, Lemma 7].

THEOREM 1. *If G is chordal, then $k[\Delta(G)]$ has a 2-linear resolution. Conversely, if $k[\Delta]$ has a 2-linear resolution, then $\Delta = \Delta(G(\Delta))$ and $G(\Delta)$ is chordal.*

Proof. Let G be chordal. We have to show that $H_i(\Delta') = 0$ for all $i > 0$ and all full subcomplexes Δ' of $\Delta(G)$. We do this inductively according to the definition of a generalized d-tree, the induction start being trivial. Let H be chordal and $G = H \cup_{K_j} K_i$. Take a full subcomplex Δ' of $\Delta(G)$ and let $M = G(\Delta')$. Let $M \cap H = A$, $M \cap K_j = B$ and $M \cap K_i = C$. Then $\Delta' = \Delta(M) = \Delta(A) \cup_{\Delta(B)} \Delta(C)$. The Mayer–Vietoris sequence

$$\ldots \to H_i(\Delta(A)) \oplus H_i(\Delta(C)) \to H_i(\Delta(M)) \to H_{i-1}(\Delta(B)) \to \ldots$$

yields the result. Also for the converse we use the topological characterization. Suppose $k[\Delta]$ has a 2-linear resolution. Then all relations are of degree two, hence $\Delta = \Delta(G(\Delta))$. Now take a circuit $(x_0, x_1, \ldots, x_k = x_0)$, $k > 3$. Since $H_1(\Delta') = 0$ for all full subcomplexes Δ', in particular for the full subcomplex generated by $\{x_1, \ldots, x_k\}$, we see that this circuit must have a chord.

The following corollary is immediate and characterizes these complexes topologically.

Corollary 1. *The following three conditions on a simplicial complex Δ are equivalent*:

(1) $H_i(\Delta') = 0$ *for all full subcomplexes Δ' of Δ and all $i \geqslant 1$.*
(2) $H_1(\Delta') = 0$ *for all full subcomplexes Δ' of Δ.*
(3) *The 1-skeleton $G(\Delta)$ of Δ is chordal and $\Delta = \Delta(G(\Delta))$.*

A purely algebraic corollary is:

Corollary 2. *If $k[\Delta_1]$ and $k[\Delta_2]$ both have 2-linear resolutions, then their fibre product $k[\Delta_1] \prod k[\Delta_2]$ also has a 2-linear resolution.*

Proof. The fibre product $k[\Delta_1] \prod k[\Delta_2]$ is the Stanley–Reisner ring of the disjoint union $\Delta = \Delta_1 \cup \Delta_2$ of Δ_1 and Δ_2. Then

$$\Delta(G(\Delta)) = \Delta(G(\Delta_1) \cup G(\Delta_2)) = \Delta(G(\Delta_1)) \cup \Delta(G(\Delta_2)) = \Delta_1 \cup \Delta_2 = \Delta$$

and $G(\Delta_1 \cup \Delta_2) = G(\Delta_1) \cup G(\Delta_2)$ is chordal.

2. The Cohen–Macaulay case

In the next section we will give an alternative way to characterize the rings with 2-linear resolutions that are Cohen–Macaulay. We choose however here an algebraic way to get further numerical information.

Let $R = \bigoplus_{i \geqslant 0} R_i$ be a graded algebra over $R_0 = k$ (a field). The Hilbert series of R is

$$R(Z) = \sum_{i \geqslant 0} (\dim_k R_i) Z^i.$$

Lemma 1. *Suppose R is a graded Cohen–Macaulay ring of Krull dimension d and with $\dim_k R_1 = n + d$. Then R has a 2-linear resolution if and only if $R(Z) = (1 + nZ)/(1 - Z)^d$.*

Proof. Extend, if necessary, k to an infinite field. This does not affect the Hilbert series and the property of having a 2-linear resolution. Thus we can suppose that there is a regular sequence $y_1, y_2, \ldots, y_d$ of elements of degree one in R. The exact sequence

$$\text{(E)} \qquad 0 \to R/(y_1, \ldots, y_i) \xrightarrow{y_{i+1}} R/(y_1, \ldots, y_i) \to R/(y_1, \ldots, y_{i+1}) \to 0$$

gives $R/(y_1, \ldots, y_{i+1})(Z) = (1-Z)R/(y_1, \ldots, y_i)(Z)$ which gives $R/(y_1, \ldots, y_d)(Z) = (1-Z)^d R(Z)$. Now $R/(y_1, y_2, \ldots, y_d)$ has a 2-linear resolution if and only if R has a 2-linear resolution, see e.g. [Fr-La, Section 4]. Thus it is enough to show that an artinian graded ring S has a 2-linear resolution if and only if $S(Z) = 1 + nZ$ where $\dim_k S_1 = n$, i.e. if and only if $S \simeq k[X_1, X_2, \ldots, X_n]/(X_1, X_2, \ldots, X_n)^2$. Let

$$\text{(R)} \qquad 0 \to \sum_{i=1}^{b_n} \tilde{R}[-m_{n,i}] \to \ldots \to \sum_{i=1}^{b_1} \tilde{R}[-m_{1,i}] \to \tilde{R} \to S \to 0$$

be a minimal graded resolution of a graded artinian ring S over $\tilde{R} = k[X_1, X_2, \ldots, X_n]$ where $m_{k,1} \leqslant m_{k,2} \leqslant \ldots \leqslant m_{k,b_k}$ for $k = 1, 2, \ldots, n$, and where $\tilde{R}[-k]_i = \tilde{R}_{i-k}$. The resolution is 2-linear if and only if $m_{k,i} = k+1$ for $k = 1, 2, \ldots, n$ and all i. Since the resolution is minimal we have $2 \leqslant m_{1,1} < m_{2,1} < \ldots < m_{n,1}$. Now $\mathrm{Hom}_{\tilde{R}}(\mathbf{R}, \tilde{R})$ is a minimal $\tilde{R}$-resolution of $\mathrm{Ext}^n_{\tilde{R}}(S, \tilde{R})$, since S is artinian, which gives that $m_{1,b_1} < m_{2,b_2} < \ldots < m_{n,b_n}$. Hence, we see that the resolution is 2-linear if and only if $m_{n,b_n} = n+1$ or if and only if $m_{n,1} = m_{n,2} = \ldots = m_{n,b_n} = n+1$. Another way to formulate this is to say that $\mathrm{Tor}^{\tilde{R}}_{n,i}(S, k) = 0$ for $i \neq n+1$ (tensor $(\mathbf{R})$ with k and take homology). The usual Koszul complex is an $\tilde{R}$-resolution of k. Tensoring the Koszul complex with S and taking homology yields $\mathrm{Tor}^{\tilde{R}}_n(S, k) \simeq \mathrm{Soc}\, S[-n]$ so we see that S has a 2-linear resolution if and only if the socle of S is in degree one only. This is true if and only if $S \simeq k[X_1, X_2, \ldots, X_n]/(X_1, X_2, \ldots, X_n)^2$. This concludes the proof of the lemma.

We are now ready to characterize Cohen–Macaulay Stanley–Reisner rings with 2-linear resolutions.

THEOREM 2. *Let Δ be a simplicial complex. Then the following are equivalent*:

(i) *$k[\Delta]$ is Cohen–Macaulay of Krull dimension $d+1$ and has a 2-linear resolution.*

(ii) *The 1-skeleton $G(\Delta)$ of Δ is a d-tree and $\Delta = \Delta(G(\Delta))$.*

Proof. (ii) $\Rightarrow$ (i): The f-vector of Δ is

$$f = \left(\binom{d+1}{1}, \ldots, \binom{d+1}{d+1}\right) + n\left(\binom{d}{0}, \binom{d}{1}, \ldots, \binom{d}{d}\right)$$

(Here the ith coordinate f_{i-1} denotes the number of faces of dimension $i-1$ in Δ.) Thus, since

$$\text{(H)} \qquad k[\Delta](Z) = 1 + f_0 Z/(1-Z) + \ldots + f_d Z^{d+1}/(1-Z)^{d+1}$$

(see [Fr, Lemma 6]), we have

$$k[\Delta](Z) = \sum_{i=0}^{d+1}\left(\binom{d+1}{i}Z^i/(1-Z)^i\right) + n\sum_{i=0}^{d}\left(\binom{d}{i}Z^{i+1}/(1-Z)^{i+1}\right)$$
$$= (1+Z/(1-Z))^{d+1} + nZ/(1-Z)(1+Z/(1-Z))^d = (1+nZ)/(1-Z)^{d+1}$$

Hence, it only remains to show that $k[\Delta]$ is Cohen–Macaulay. This can certainly be done in lots of ways, we choose a purely algebraic way which also gives information about a set of parameters of degree one. A d-tree has a unique vertex colouring in $d+1$ colours $C_1, C_2, \ldots, C_{d+1}$. Let $y_i = \sum_{X_j \in C_i} x_j$. We will show that $y_1, y_2, \ldots, y_{d+1}$ constitutes a regular sequence in $k[\Delta]$. The exact sequence (E) shows that if we have a sequence $y_1, y_2, \ldots, y_{d+1}$ in $k[\Delta]$, then this sequence is regular if and only if

$$k[\Delta]/(y_1, y_2, \ldots, y_{d+1})(Z) = (1-Z)^{d+1}\, k[\Delta](Z).$$

Hence, it is enough to prove that $k[\Delta]/(y_1, y_2, \ldots, y_{d+1})(Z) = 1+nZ$, i.e. that $T = k[\Delta]/(y_1, y_2, \ldots, y_{d+1}) \simeq k[X_1, X_2, \ldots, X_n]/(X_1, X_2, \ldots, X_n)^2$. We have that $x_i x_j = 0$ in T if the vertices X_i and X_j do not belong to a common K_{d+1} in S. We have that $x_i^2 = -x_i \sum_{X_j \in C} x_j$ in T, where C is the colour class of X_i, hence $x_i^2 = 0$ in T. Finally suppose X_i and X_j belong to a common K_{d+1} in S and that $X_i \in C_1$, $X_j \in C_2$. Then $x_i x_j = -x_i \sum_{X_k \in C_2} x_k$. Here we can suppose that each X_k belongs to a K_{d+1} to which also X_i belongs. Then use the relation for x_i then for all x_k and so on alternatively. In this process we make a walk where the distance from the original K_{d+1} increases by one each time we use a relation. Since the graph is finite we get $x_i x_j = 0$ after a finite number of steps. Since we have found a regular sequence of length $d+1$ in $k[\Delta]$ we can conclude that $k[\Delta]$ is Cohen–Macaulay and the proof of the implication (ii) $\Rightarrow$ (i) is finished.

(i) $\Rightarrow$ (ii): The calculation of the Hilbert series in the first part of the proof made backwards shows that if $k[\Delta]$ has a 2-linear resolution then the 1-skeleton of Δ has the f-vector of a d-tree where $d = \dim \Delta - 1$. It is well known that $\dim k[\Delta] = \dim \Delta + 1$ (this follows e.g. from the above mentioned [Fr, Lemma 6]). Suppose that $\dim k[\Delta] = 1$, i.e. $\dim \Delta = 0$. Then Δ is a set of points, i.e. a 0-tree. Thus we can suppose $\dim k[\Delta] \geqslant 2$. It is well known that if $k[\Delta]$ is Cohen–Macaulay then Δ is connected if $\dim k[\Delta] \geqslant 2$. It is also well known that if $k[\Delta]$ is Cohen–Macaulay then Δ is pure, i.e. all maximal faces of Δ have the same dimension. (These two results can also be found e.g. in [Fr].) But a generalized d-tree which is pure and connected and has the f-vector of a d-tree is a d-tree. (This is easily shown by induction on the number of vertices.)

COROLLARY. *Let G be a graph. Then G is a d-tree if and only if the following two conditions hold*:

(1) $\tilde{H}_i(\Delta') = 0$ *for all subcomplexes* Δ' *of* $\Delta(G)$ *and for all* $i > 0$.

(2) $\tilde{H}_0(\Delta') = 0$ *for all full subcomplexes* Δ' *of* $\Delta(G)$ *with at most* $d-1$ *points removed from* $\Delta(G)$.

3. Numerical invariants

There are lots of nonisomorphic Cohen–Macaulay k-algebras of embedding dimension n and dimension d which have 2-linear resolutions. They have however all the same Hilbert series (in particular multiplicity), Betti numbers (in particular complete intersection defect) and Poincaré series. This is no longer true if we consider the non-Cohen–Macaulay case and we will in this section examine how these invariants can vary for Stanley–Reisner rings with 2-linear resolutions. We will see that the ones with maximal multiplicity or Hilbert series are exactly the Cohen–Macaulay ones. The same is true for minimal Betti numbers, complete intersection defect or Poincaré series. We will also identify the rings on the other extreme (i.e. the ones with minimal Hilbert series a.s.o.).

We start with a topological characterization of the multiplicity of $k[\Delta]$, true for any simplicial complex Δ.

LEMMA 2. *For any Stanley–Reisner ring* $k[\Delta]$ *we have that the multiplicity of* $k[\Delta]$ *equals the number of faces of maximal dimension in* Δ.

Proof. We use the formula (H) for the Hilbert series. The multiplicity for any graded algebra R equals $\lim_{Z\to 1}(1-Z)^d R(Z)$, where $d = \dim R$.

From now on we restrict in this section to Stanley–Reisner rings with 2-linear resolutions. First we consider the depth.

The connectivity $\varkappa(G)$ of a graph G is the number of vertices in a minimal disconnecting set of G.

LEMMA 3. *Suppose* $k[\Delta]$ *has a 2-linear resolution. Then the depth of* $k[\Delta]$ *equals* $1+\varkappa(G(\Delta))$, *where* $G(\Delta)$ *is the 1-skeleton of* Δ.

Proof. We have $g = \operatorname{depth} k[\Delta] = \max\{j;\ \operatorname{Tor}^R_{n-j}(k[\Delta], k) \neq 0\}$. Since $k[\Delta]$ has a 2-linear resolution we have that $\operatorname{Tor}^R_{i,j}(k[\Delta], k) = 0$ for $j \neq i+1$. From [Fr, lemma 4] it follows that these two conditions are equivalent to that $\tilde{H}_i(\Delta') = 0$ for all $i > 0$ and for all full subcomplexes Δ' and $\tilde{H}_0(\Delta') = 0$ for all full subcomplexes Δ' with at most $g-2$ points deleted from Δ but $\tilde{H}_0(\Delta') \neq 0$ for some Δ' with $g-1$ points deleted. The condition on $\tilde{H}_0$ is equivalent to that all subgraphs G' of G with at most $g-2$ points deleted of the 1-skeleton are connected, but not all with $g-1$ points deleted. This is of course equivalent to $\varkappa(G(\Delta)) = g-1$.

Here follows the alternative proof of Theorem 2, the characterization of 2-linear Stanley–Reisner rings which are Cohen–Macaulay.

COROLLARY. *If G is chordal, then $k[\Delta]$ is Cohen–Macaulay of dimension $d+1$ if and only if G is a d-tree.*

Proof. We use the definition of generalized d-trees from the beginning of Section 1. If we delete the j points in an attaching K_j we disconnect the graph into two components, but it is not possible to disconnect the graph by deleting fewer points. Thus $\varkappa(G) = \min\{j;\ K_j \text{ attaching graph}\}$. But G is a d-tree if and only if this minimum equals d.

Next we will consider Hilbert series. We let the dimension of a graph G (or the clique number) be the dimension of $k[\Delta(G)]$, i.e. $\dim G = d$ if G contains a sub-K_d but no sub-K_{d+1}. We partially order the graphs so that $G \leqslant H$ if G is a subgraph of H. If we restrict to chordal graphs of dimension d with n vertices we have a unique minimal element in this partial order, namely the graph Γ which is the disjoint union of a K_d and $n-d$ points.

LEMMA 4. *Let G be a chordal graph of dimension d with n vertices. Then, coefficientwise, we have*

$$\left(n, \binom{d}{2}, \binom{d}{3}, \ldots, \binom{d}{d}\right) \leqslant f(G) \leqslant$$

$$\left(\binom{d}{1}, \binom{d}{2}, \ldots, \binom{d}{d}\right) + (n-d)\left(\binom{d-1}{0}, \binom{d-1}{1}, \ldots, \binom{d-1}{d-1}\right).$$

We have equality to the left if and only if G is the disjoint union of a K_d and $(n-d)$ points. We have equality to the right if and only if G is a $(d-1)$-tree.

Proof. The left inequality follows from the fact that $f(\Gamma) = (n, \binom{d}{2}, \binom{d}{3}, \ldots, \binom{d}{d})$. For the right inequality start with a K_d and build up the graph inductively. If a K_i is attached to the graph in a K_j $(0 \leqslant j < i \leqslant d)$ we increase the f-vector with

$$\left(\binom{i}{1}, \binom{i}{2}, \ldots, \binom{i}{i}, 0, 0, \ldots, 0\right) - \left(\binom{j}{1}, \binom{j}{2}, \ldots, \binom{j}{j}, 0, 0, \ldots, 0\right).$$

In the process of building a $(d-1)$-tree we increase the f-vector each time with

$$\left(\binom{d}{1}, \binom{d}{2}, \ldots, \binom{d}{d}\right) - \left(\binom{d-1}{1}, \binom{d-1}{2}, \ldots, \binom{d-1}{d-1}\right)$$

$$= \left(\binom{d-1}{0}, \binom{d-1}{2}, \ldots, \binom{d-1}{d-1}\right).$$

Now it is enough to prove that $\binom{i}{k} - \binom{j}{k} \leqslant (i-j)\binom{d-1}{k-1}$ if $1 \leqslant k \leqslant j$ and $\binom{i}{k} \leqslant (i-j)\binom{d-1}{k-1}$ if $j < k \leqslant i$. First suppose that $k \leqslant j$. Let k, j and d be fixed and let $f(i) = i\binom{d-1}{k-1} - \binom{i}{k}$ for $i \geqslant k$. We show that f is increasing. Now $f(m+1) - f(m) = \binom{d-1}{k-1} - \binom{m}{k-1} \geqslant 0$. Hence it is enough to prove the inequality for $i=j+1$. But for $i=j+1$ the inequality is trivial. Now suppose $k>j$. The same argument as above shows that it is enough to show the inequality for $i=k$. But $(k-j)\binom{d-1}{k-1}\geq 1$. By examining the inequalities we also see that there is equality if and only if $j+1=k=i=d$.

THEOREM 3. *Let $k[\Delta]$ be a Stanley–Reisner ring of embedding dimension n and dimension d and with a 2-linear resolution. Then, coefficientwise, we have*

$$1/(1-Z)^d+(n-d)Z/(1-Z)\leq k[\Delta](Z)\leq \bigl(1+(n-d)Z\bigr)/(1-Z)^d.$$

We have equality to the left if and only if Δ is the disjoint union of a $(d-1)$-simplex and $(n-d)$ points. We have equality to the right if and only if the 1-skeleton of Δ is a $(d-1)$-tree and if and only if $k[\Delta]$ is Cohen–Macaulay.

Proof. Formula (**H**) shows that if $f(\Delta_1)\leq f(\Delta_2)$ (coordinatewise) then $k[\Delta_1](Z)\leq k[\Delta_2](Z)$. Lemma 4 gives the result together with the characterization of d-trees in Theorem 2 or the corollary to Lemma 3.

COROLLARY. *Let $k[\Delta]$ be a Stanley–Reisner ring of embedding dimension n, dimension d and with a 2-linear resolution. Then the multiplicity of $k[\Delta]$ is at most $n-d+1$ with equality if and only if $k[\Delta]$ is Cohen–Macaulay.*

Proof. We use Lemma 2. Then Lemma 4 shows that the maximal number of maximal faces is $n-d+1$.

We have noted that a generalized d-tree (a chordal graph) can be constructed in the following way: Start with a K_{d+1} and attach a finite number N of times a K_{i_n}, say G_{i_n}, to the constructed graph G in a K_{j_n}, say G_{j_n}, $0\leq j_n<i_n\leq d+1$ for each $n=1,2,\ldots,N$. Denote the vertices in $G_{i_n}-G_{j_n}$ by $v_1,v_2,\ldots,v_{i_n-j_n}$ and choose a K_d in G, say G_d, of which G_{j_n} is a subgraph. Adjoin all edges form v_1 to G_d. Then adjoin all edges from v_2 to a sub-K_{d-1}, say G_{d-1}, of G_d, all edges from v_3 to a sub-K_{d-2} of G_{d-1} and so on. This construction shows that we can always get a d-tree from a generalized d-tree by adjoining edges.

Now we make the statement about Hilbert series more precise.

LEMMA 5. *Let G be a chordal graph with n vertices and of dimension d. Suppose that the connectivity of G is exactly $g-1$. Then, coefficientwise,*

$$(n-d)\left(\binom{g-1}{0},\binom{g-1}{1},\ldots,\binom{g-1}{g-1},0,\ldots,0\right)$$

$$\leq f(G)-\left(\binom{d}{1},\binom{d}{2},\ldots,\binom{d}{d}\right)\leq(n-d-1)\left(\binom{d-1}{0},\binom{d-1}{1},\ldots,\binom{d-1}{d-1}\right)$$

$$+\left(\binom{g-1}{0},\binom{g-1}{1},\ldots,\binom{g-1}{g-1},0,\ldots,0\right).$$

We have equality to the left if and only if G is a K_d with a finite number of $(g-1)$-trees, each one attached to it in a K_{g-1}. We have equality to the right if and only if G is a $(d-1)$-tree with a K_g attached to it in a K_{g-1}.

Proof. A chordal graph as above is built up from a K_d by attaching a K_i to the constructed graph in a K_j where $j < i < d$. Since $\varkappa(G) = g-1$ we have $j \geq g-1$. Each time we attach a K_i in such a way we increase the f-vector by $(\binom{i}{1}, \binom{i}{2}, \ldots, \binom{i}{i}, 0, \ldots, 0) - (\binom{j}{1}, \binom{j}{2}, \ldots, \binom{j}{j}, 0, \ldots, 0)$. Suppose that we each time attach a K_g in a K_{g-1}. Then we increase the f-vector by $(\binom{g-1}{0}, \binom{g-1}{1}, \ldots, \binom{g-1}{g-1}, 0, \ldots, 0)$. Hence, in order to prove the left inequality, we have to prove that

$$\binom{i}{k} - \binom{j}{k} \geq (i-j)\binom{g-1}{k-1}$$

if $1 \leq k \leq j$ and

$$\binom{i}{k} \geq (i-j)\binom{g-1}{k-1}$$

if $j < k \leq i$. Then it is of course enough to prove that

$$\binom{i}{k} - \binom{j}{k} \geq (i-j)\binom{j}{k-1}.$$

Let j and k be fixed and set $f(i) = \binom{i}{k} - i\binom{j}{k-1}$. Then $f(m+1) - f(m) = \binom{m}{k-1} - \binom{j}{k-1} > 0$ so it is enough to prove the inequality for $i = j+1$. But then it is an equality. We also see that we have equality if and only if $j+1 = i = g-1$. In order to prove the right inequality we use the fact that a chordal graph of dimension d has maximal f-vector if and only if it is a $(d-1)$-tree (Lemma 4). A $(d-1)$-tree with $n-1$ vertices on which a K_g is attached in a K_{g-1} can be extended to a $(d-1)$-tree by adjoining $d-g$ edges. The reasoning above shows that this is the minimal number possible of edges one has to adjoin to a chordal graph of connectivity $g-1$ to get a $(d-1)$-tree. Hence the f-vector of chordal graph G of dimension d and of connectivity $g-1$ is maximal if G is a $(d-1)$-tree with a K_g attached in a K_{g-1}.

THEOREM 4. *Let $k[\Delta]$ be a ring with 2-linear resolution of embedding dimension n, dimension d and depth g. Then, coefficientwise, we have*

$$1/(1-Z)^d + (n-d)Z/(1-Z)^g \leq k[\Delta](Z) \leq \big(1+(n-d-1)Z\big)/(1-Z)^d + Z/(1-Z)^g.$$

Proof. This follows from Lemma 5, exactly as Theorem 3 follows from Lemma 4.

The next invariants to be considered are the Betti numbers.

THEOREM 5. *Let $k[\Delta]$ be a Stanley–Reisner ring of embedding dimension n, dimension d and with a 2-linear resolution. Denote the Betti number* $\dim_k \mathrm{Tor}_i^R(k[\Delta], k)$ *by* b_i. *Then*

$$\binom{n-d}{i}(n-d)-\binom{n-d}{i+1} \leqslant b_i \leqslant \binom{n-1}{i}(n-d)-\binom{n-d}{i+1}$$

We have equality to the left if and only if $G(\Delta)$ is a $(d-1)$-tree and if and only if $k[\Delta]$ is Cohen–Macaulay. We have equality to the right if and only if Δ is the disjoint union of a $(d-1)$-simplex and $(n-d)$ points.

Proof. The Betti number b_i is equal to $\sum \dim_k \tilde{H}_0(\Delta')$, where the sum is taken over all full subcomplexes Δ' on $i+1$ points of Δ, [Fr, Lemma 4]. The simplicial complex Γ consisting of the disjoint union of one $(d-1)$-simplex and $n-d$ points is as remarked above a subcomplex of all complexes of dimension d on n points. Fix an embedding of Γ in Δ. Let S be the subset of the vertices of Δ (and of Γ) and let Δ' (and Γ' respectively) be the corresponding full subcomplexes of Δ (and Γ). Then obviously $\dim_k H_0(\Delta') \leqslant \dim_k \tilde{H}_0(\Gamma')$. We have shown that $k[\Gamma]$ has the largest Betti numbers of all Stanley–Reisner rings of embedding dimension n and dimension d with a 2-linear resolution. The Hilbert series of $k[\Gamma]$ is

$$\begin{aligned}1/(1-Z)^d+(n-d)Z/(1-Z) &= (1+(n-d)Z(1-Z)^{d-1})/(1-Z)^d \\ &= ((1-Z)^{n-d}+(n-d)Z(1-Z)^{n-1})/(1-Z)^n.\end{aligned}$$

For a ring with a 2-linear resolution we have for $i>0$ that $b_i=(-1)^i c_{i+1}$, where c_{i+1} is the coefficient of Z^{i+1} in the numerator of the Hilbert series $\sum c_i Z^i/(1-Z)^n$ (e.g. [Fr-La, Section 4]). Thus the inequality to the right is proved. We have seen that each chordal graph can be extended to a $(d-1)$-tree by adjoining edges. Thus each chordal graph has at least as large Betti numbers as some $(d-1)$-tree. But $(d-1)$-trees with n vertices all have the same Hilbert series, hence the same Betti numbers. The Hilbert series of a $(d-1)$-tree on n vertices is

$$(1+(n-d)Z)/(1-Z)^d = ((1-Z)^{n-d}+(n-d)Z(1-Z)^{n-d})/(1-Z)^n$$

so $b_i=(n-d)\binom{n-d}{i}-\binom{n-d}{i+1}$.

The complete intersection defect of a graded algebra R is $\mathrm{cid}(R)=b_1(R)-n+d$.

COROLLARY. *Let $k[\Delta]$ be of embedding dimension n and dimension d and suppose that $k[\Delta]$ has a 2-linear resolution. Then*

$$\binom{n-d}{2} \leqslant \mathrm{cid}(k[\Delta]) \leqslant (n-d)(n+d-3)/2.$$

There is equality to the left if and only if $k[\Delta]$ is Cohen–Macaulay and to the right if and only if Δ is the disjoint union of a $(d-1)$-simplex and $n-d$ points.

The last invariant we consider is the Poincaré series

$$P_{k[\Delta]}(Z) = \sum_{i \geq 0} \dim_k \operatorname{Tor}_i^{k[\Delta]}(k, k) Z^i.$$

THEOREM 6. *Let $k[\Delta]$ be of embedding dimension n and dimension d. Suppose that $k[\Delta]$ has a 2-linear resolution. Then, coefficientwise,*

$$(1+Z)^d/(1-(n-d)Z) \leq P_{k[\Delta]}(Z) \leq (1+Z)^d/(1-(n-d)Z(1+Z)^{d-1}).$$

Proof. Since rings with 2-linear resolutions are Golod [Ba-Fr, Theorem 7] we have $P_{k[\Delta]}(Z) = (1+Z)^n/(1-\sum_{i>0} b_i Z^{i+1})$, where n is the embedding dimension and the b_i's are the Betti numbers. Thus the Poincaré series is maximal (minimal resp.) when the Betti numbers are maximal (minimal resp.).

A graded algebra $R = k[X_1, X_2, \ldots, X_n]/(f_1, f_2, \ldots, f_r)$ of dimension d and depth g is called extremal if

$$R(Z) = (1-Z)^{-g} \max\left(\left[(1-Z)^{g-n} \prod_{i=1}^{r} (1-Z^{d_i})\right], (1-Z)^{g-d}\right)$$

where $d_i = \deg f_i$ and where the brackets stand for taking the initial positive part of a power series. This is the minimal possible value of $R(Z)$ given the numerical character $(n, d, g, (d_1, d_2, \ldots, d_r))$, see [Fr1].

THEOREM 7. *Let G be a graph with n vertices consisting of one K_d and a $(g-1)$-tree attached to this K_d in a K_{g-1} $(1 \leq g \leq d)$. Then $k[\Delta(G)]$ is an extremal ring of numerical character $(n, d, g, (2, 2, \ldots, 2))$. (The number of 2's is $\binom{n+1}{2} - \binom{d+1}{2} - (n-d)g$.)*

Proof. The f-vector of $\Delta(G)$ is

$$\left(\binom{d}{1}, \binom{d}{2}, \ldots, \binom{d}{d}\right) + (n-d)\left(\binom{g-1}{0}, \binom{g-1}{1}, \ldots, \binom{g-1}{g-1}, 0, 0, \ldots, 0\right).$$

Hence

$$\begin{aligned} k[\Delta](Z) &= 1 + \sum_{i=1}^{d} \binom{d}{i} Z^i/(1-Z)^i + (n-d) \sum_{i=0}^{g-1} \binom{g-1}{i} Z^{i+1}/(1-Z)^{i+1} \\ &= 1/(1-Z)^d + (n-d)Z/(1-Z)^g = (1-Z)^{-g}((n-d)Z + (1-Z)^{g-d}) \end{aligned}$$

which meets the requirement in the definition of extremality.

We conclude this section with some problems.

PROBLEMS. Stanley–Reisner rings are special in the sense that their depth is always positive (except in the very trivial case of an empty complex). Therefore the bounds in the theorems above can not be expected to hold in the general case of 2-linear algebras. Except for this trivial modification we would rather believe that they do hold in general. We formulate this as problems. Are the following true for arbitrary graded 2-linear algebras R of embedding dimension n, dimension d and depth g?

(a) $1/(1-Z)^d-(n-d)Z \leqslant R(Z) \leqslant (1+(n-d)Z)/(1-Z)^d$,
(b) $e(R) \leqslant n-d+1$ ($e(R)$ denotes the multiplicity),
(c) $1/(1-Z)^d+(n-d)Z/(1-Z)^g \leqslant R(Z) \leqslant (1+(n-d-1)Z)/(1-Z)^d + Z/(1-Z)^g$,
(d) $\binom{n}{i}(n-d)-\binom{n-d}{i+1} \geqslant b_i(R) \geqslant \binom{n-d}{i}(n-d)-\binom{n-d}{i+1}$,
(e) $(1+Z)^d/(1-(n-d)Z(1+Z)^d) \geqslant P_R(Z) \geqslant (1+Z)^d/(1-(n-d)Z)$,
(f) Is there equality to the right in (a), (b), (d) and (e) if and only if R is Cohen–Macaulay?

What we know is the following. If R is Cohen–Macaulay we do have equality to the right in (a)–(e). We also know that the left inequality is correct in (a) and (c) and that the statement in (d) is equivalent to the statement in (e). Furthermore, if we can prove that equality to the right is equivalent to R being Cohen–Macaulay in one of (a), (d) or (e), then equality is equivalent to R being Cohen–Macaulay also for the other two.

4. Depth of Stanley–Reisner rings

We now turn to the question of the depth of an arbitrary Stanley–Reisner ring $k[\Delta]$. It has been shown in [Sm] that depth $k[\Delta] = 1+\max\{i$; the i-skeleton Δ^i of Δ has a Cohen–Macaulay Stanley–Reisner ring$\}$. He used a topological characterization of Cohen–Macaulay-complexes and his proof has since been simplified in [Wa] using ideas from [Mu]. We will give a purely algebraic proof of this theorem.

THEOREM 8. *Let Δ be a simplicial complex. Then* depth $k[\Delta] = 1+\max\{i;\ k[\Delta^i]$ *is Cohen–Macaulay*$\}$.

Proof. Let $k[\Delta] = k[X_1, X_2, \ldots, X_n]/I$ be of dimension d. The depth of $k[\Delta]$ equals $\max\{j;\ \mathrm{Tor}_i^R(k[\Delta], k) = 0$ for $i > n-j\}$. Let $\sigma = (X_{i_1}, X_{i_2}, \ldots, X_{i_d})$ be a face of maximal dimension in Δ. Removing σ (but none of its faces) from Δ gives a complex with Stanley–Reisner ring

$$k[X_1, X_2, \ldots, X_n]/(I+(X_{i_1}X_{i_2}\ldots X_{i_d})) = k[\Delta-\sigma].$$

Consider the exact sequence

$$0 \to (x_{i_1}x_{i_2}\ldots x_{i_d}) \to k[\Delta] \to k[\Delta-\sigma] \to 0$$

of $k[X_1, X_2, \ldots, X_n]$-modules. It is easy to see that as such the kernel $K = (x_{i_1}x_{i_2}\ldots x_{i_d})$ is isomorphic to $k[Y_1, Y_2, \ldots, Y_d]$. The long exact sequence

$$\ldots \to \mathrm{Tor}_i^R(K, k) \to \mathrm{Tor}_i^R(k[\Delta], k) \to \mathrm{Tor}_i^R(k[\Delta-\sigma], k) \to \mathrm{Tor}_{i-1}^R(K, k) \to \ldots$$

and the fact that $\mathrm{Tor}_i^R(K, k) = 0$ if $i > n-d$ (depth $K = d$) and $\mathrm{Tor}_{n-d}^R(K, k) \neq 0$ gives

$$\mathrm{Tor}_i^R(k[\Delta], k) \simeq \mathrm{Tor}_i^R(k[\Delta-\sigma], k) \qquad \text{if } i > n-d+1$$

and

$$0 \to \mathrm{Tor}_{n-d+1}^R(k[\Delta], k) \to \mathrm{Tor}_{n-d+1}^R(k[\Delta-\sigma], k)$$

is exact. Thus, if $k[\Delta]$ is not Cohen–Macaulay, we have $\operatorname{depth} k[\Delta] = \operatorname{depth} k[\Delta-\sigma]$. Continuing to "factor out" maximal simplexes we get $\operatorname{depth} k[\Delta] = \operatorname{depth} k[\Delta^{d-1}]$. If $k[\Delta^{d-1}]$ is Cohen–Macaulay we are finished, otherwise $\operatorname{depth} k[\Delta^{d-1}] = \operatorname{depth} k[\Delta^{d-2}]$ and so on.

References

[Ba-Fr] J. Backelin and R. Fröberg, *Koszul algebras, Veronese subrings and rings with linear resolutions*, Rev. Roumaine Math. Pures Appl. 30 (2) (1985), 85–97.

[Di] G. A. Dirac, *On rigid circuit graphs*, Abh. Math. Sem. Univ. Hamburg 38 (961), 71–76.

[Fr] R. Fröberg, *Rings with monomial relations having linear resolutions*, J. Pure Appl. Algebra 38 (1985), 235–241.

[Fr1] —, *An inequality for Hilbert series of graded algebras*, Math. Scand. 56 (1985), 117–144.

[Fr-La] R. Fröberg and D. Laksov, *Compressed algebras*, Lecture Notes in Math. Springer, Berlin 1092 (1984), 121–151.

[Mu] J. R. Munkrees, *Topological results in combinatorics*, Michigan Math. J. 31 (1984), 235–241.

[Sm] D. E. Smith, *On the Cohen–Macaulay property in commutative algebra and simplicial topology*, Dissertation Univ. of Oregon, 1986.

[Wa] J. W. Walker, Letter to A. Björner, March 1987.

Addendum

At the conference talk W. Vogel asked for the characterization of simplicial complexes with Stanley–Reisner rings with 2-linear resolutions and which are Buchsbaum. Here is a delayed answer.

P. Schenzel has characterized Buchsbaum Stanley–Reisner rings as thôse belonging to simplexes with Cohen–Macaulay links of vertices [Sc, Theorem 3.2]. A Stanley–Reisner ring $k[\Delta]$ is Cohen–Macaulay if and only if it is pure and has Cohen–Macaulay links of vertices and furthermore $\tilde{H}_i(\Delta) = 0$ for $i < \dim \Delta$ [Re, Theorem 1]. We will use these facts to prove the following theorem.

THEOREM. *Let Δ be a simplicial complex. The following are equivalent.*

(i) *$k[\Delta]$ is Buchsbaum and has a 2-linear resolution.*

(ii) *For each connected component Δ_i of Δ we have that $G(\Delta_i)$ is a d-tree for some d and that $\Delta(G(\Delta_i)) = \Delta_i$ (i.e. $k[\Delta_i]$ is Cohen–Macaulay and has a 2-linear resolution).*

(iii) *$k[\Delta]$ is the fibre product of rings $k[\Delta_i]$ which are Cohen–Macaulay and have 2-linear resolutions.*

Proof. The Stanley–Reisner ring of the disjoint union of Δ_1 and Δ_2 is the fibre product of $k[\Delta_1]$ and $k[\Delta_2]$, hence (ii) and (iii) are equivalent. That (ii) implies (i) follows from Schenzel's and Reisner's theorems. Assume (i). Schenzel's theorem shows that $k[\Delta]$ is Buchsbaum if and only if $k[\Delta_i]$ is Buchsbaum for each connected component Δ_i of Δ. Hence we consider a connected component Δ_i of Δ. We use the inductive process of building Δ_i described above. The Mayer–Vietoris sequence shows that $\tilde{H}_j(\Delta_i) = 0$ for all j. Thus $k[\Delta_i]$ is Buchsbaum if and only if $k[\Delta_i]$ is Cohen–Macaulay.

References

[Re] G. A. Reisner, *Cohen–Macaulay quotients of polynomial rings*, Adv. in Math. 21 (1976), 30–49.

[Sc] P. Schenzel, *On the number of faces of simplicial complexes and the purity of Frobenius*, Math. Z. 178 (1981), 125–142.