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ON THE DIFFERENCES OF SIMULTANEOUS RATIONAL INTERPOLANTS

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1. Introduction

Let D be a region (open convex set) in C and $\{f_i\}$, $i = 1, \ldots, d$, analytic functions in D. Let $m > 0$ and let a system of d integers $\{\delta_i\}$, $i = 1, \ldots, d$, be given such that $\sum_{i=1}^{d} \delta_i = m$. Consider a triangular table $\alpha = (\alpha_{k,n})$, $k = 1, \ldots, n+1$, whose limit points are all in D.

DEFINITION 1. We say that the system of rational functions $\{r_{n,i} = p_{n,i}/q_n\}$, $i = 1, \ldots, d$, is an (n, m) *simultaneous rational interpolant of* $\{f_i\}$, $i = 1, \ldots, d$, *associated to* $\{\delta_i\}$, $i = 1, \ldots, d$, *in the n-th row of* α if for all sufficiently large n

$$\deg(p_{n,i}) \leqslant n - \delta_i, \qquad \deg(q_n) \leqslant m,$$

and $r_{n,i}(\alpha_{k,n}) = f_i(\alpha_{k,n})$, $k = 1, 2, \ldots, n+1$, $i = 1, 2, \ldots, d$, in the sense of Hermite.

We wish to remark that in the case of $d = 1$ the definition coincides with that of generalized multipoint Padé approximants. The problem of constructing such a system in the classical case of interpolation at the origin is considered in [1] where an explicit solution is given in terms of determinants.

When $d = 1$, the study of differences of interpolants of a fixed function has its origin in a paper of Walsh (see [11]) in which he considers a particular case of differences of polynomial interpolants and proves an overconvergence result; that is, that the difference of these polynomial interpolants converges in a larger region than that of analyticity of the fixed analytic function which they interpolate. Various extensions of Walsh's result can be found in [2], [7] and [10]. We are particularly interested in the extension of Saff, Sharma and Varga [10]. Before formulating it, we need to introduce some concepts and notation.

Let E be a compact subset of C whose complement Ω (with respect to the extended complex plane) is connected and regular in the sense that Ω has Green's function $G(z)$ with pole at infinity (without loss of generality we may assume that $0 \in E$). Let Γ_σ, $\sigma > 1$, be the level curve $\Gamma_\sigma = \{z \in C\colon G(z) = \log \sigma\}$ and E_σ the region limited by Γ_σ. Consider two triangular tables $\alpha = (\alpha_{k,n})$, $k = 1, \ldots, n+1$, $\alpha' = (\alpha'_{k,n})$, $k = 1, \ldots, n+1$, $n \in N$, whose limit points are in E, and define

$$w_n(z) = \prod_{k=1}^{n+1} (z - \alpha_{k,n}), \qquad w_{-1}(z) \equiv 1,$$

$$w'_n(z) = \prod_{k=1}^{n+1} (z - \alpha'_{k,n}), \qquad w'_{-1}(z) \equiv 1.$$

Let f be an analytic function in E which is meromorphic with exactly m poles in E_R where $R > 1$ is fixed. The problem solved in [11] was to find conditions that would yield the overconvergence of differences of multipoint Padé approximants $r_{n,m}$ and $r'_{n,m}$ that interpolate f in the $(n+m)$th row of α and α' respectively. First, a natural condition of extremality was imposed on the table α, namely

$$\lim |w_n(z)|^{1/n} = \Delta \exp(G(z)) \tag{1.1}$$

where Δ is the transfinite diameter (or capacity) of E, and the limit is uniform on each compact subset of Ω.

Also, a condition of "nearness" of the tables α and α' is needed.

Since the polynomials $w_j(z)$ and $w'_j(z)$ are monic polynomials of degree $j+1$, for each n there exist $n+1$ unique constants $\mu_j(n)$, $0 \leqslant j \leqslant n$, such that

$$w'_n(z) = w_n(z) + \sum_{j=0}^{n} \mu_j(n)\, w_{j-1}(z). \tag{1.2}$$

We suppose that there exists $-\infty \leqslant \theta < 1$ such that

$$\limsup \Big\{\sum_{j=0}^{n} |\mu_j| (\Delta R)^j\Big\}^{1/n} \leqslant \Delta R^{\theta} \quad (< \Delta R). \tag{1.3}$$

It was proved that the extremality of α and the condition (1.3) yield the extremality of α', i.e.

$$\lim |w'_n(z)|^{1/n} = \Delta \exp(G(z)). \tag{1.4}$$

For this case the following result was proved:

THEOREM A. *Under the above conditions we have for each* γ, $R \leqslant \gamma < \infty$,

$$\limsup \{\max |r_{n,m}(z) - r'_{n,m}(z)|,\ z \in H\}^{1/n} \leqslant \gamma / R^{2-\theta},$$

where H *is an arbitrary compact subset of* $\bar{E}_r \setminus \{$*poles of* $f\}$.

Essential to the proof is the generalization of Montessus de Ballore's Theorem given by Saff in [9].

When studying the case when $d > 1$ it is natural to ask whether it is possible to obtain a similar result for the difference of simultaneous rational interpolants. In the following section we will prove the following theorem.

THEOREM. *Let E, E_R $(R > 1)$ be as before. Consider a system of functions $\{f_i\}$, $i = 1, \ldots, d$, analytic in E and meromorphic in E_R such that f_i has exactly δ_i poles in E_R where $\{\delta_i\}$, $i = 1, \ldots, d$, is a system of non-negative integers such that $\sum_{i=1}^{d} \delta_i = m > 0$. Suppose additionally that if $i \neq j$ then f_i and f_j have no common poles. Take two triangular tables α and α' such that α satisfies* (1.1) *and* (1.3) *holds. If $\{r_{n,i} = p_{n,i}/q_n\}$, $i = 1, \ldots, d$, and $\{r'_{n,i} = p'_{n,i}/q'_n\}$, $i = 1, \ldots, d$, are systems of (n, m) simultaneous rational interpolants of $\{f_i\}$, $i = 1, \ldots, d$, with respect to $\{\delta_i\}$, $i = 1, \ldots, d$, and the tables α and α' respectively, then there exist a_i such that for each γ, $R \leqslant \gamma < \infty$,*

$$\limsup \{\max |r_{n,i}(z) - r'_{n,i}(z)|,\ z \in H\}^{1/n} \leqslant \gamma / R^{2-a_i}, \qquad i = 1, \ldots, d, \tag{1.5}$$

where H is an arbitrary closed subset of $\bar{E}_\gamma \setminus \bigcup_{j=1}^{m} \{z_j\}$, $\bigcup_{j=1}^{m} \{z_j\}$ is the set of poles of the system of functions in E_R,

$$a_i = \begin{cases} \max\{\log_R r_i, \theta\} & \text{if } \delta_i \neq m, \\ \theta & \text{if } \delta_i = m, \end{cases}$$

θ is that of condition (1.3) *and r_i is the smallest number for which $\{$poles of f_j, $j \neq i\} \subset E_{r'}$, for each $r' > r_i$.*

2. Proof of the theorem

Before proving the theorem it is necessary to present some ideas given by Graves-Morris and Saff in [4].

For $d > 1$, in general, it is not possible to guarantee the uniqueness of the corresponding approximant. In the context of convergence results this may be ensured through the following:

DEFINITION 2. Let each of the functions $f_1, \ldots, f_d$ be meromorphic in the open set U and let non-negative integers $\delta_1, \ldots, \delta_d$ be given for which $\sum_{i=1}^{d} \delta_i > 0$. Then the functions $f_i(z)$ are said to be *polewise independent with respect to the numbers δ_i in U* if there do not exist polynomials $p_1(z), \ldots, p_d(z)$, at least one of which is non-null, satisfying:

(i) $\deg\{p_i(z)\} \leqslant \delta_i - 1$ if $\delta_i \geqslant 1$.

(ii) $p_i(z) = 0$ if $\delta_i = 0$.

(iii) $\theta(z) = \sum_{i=1}^{d} p_i(z) f_i(z)$ is analytic throughout U.

We note that the conditions imposed on the system of functions in the theorem that we are about to prove are a particular case of polewise independence.

THEOREM B (see [4]). *Let E and E_R $(R > 1)$ be as before and let $\{f_i\}$, $i = 1, \ldots, d$, be a system of analytic functions in E_R except for m possible poles (not necessarily distinct) at $z_j \in E_R \setminus E$, $1 \leqslant j \leqslant m$ (if z_j is repeated exactly p times then each f_i is allowed to have a pole at z_j of order at most p). Let $\{\delta_i\}$, $i = 1, \ldots, d$, be a set of non-negative integers such that $\sum_{i=1}^{d} \delta_i = m > 0$ and the $\{f_i\}$, $i = 1, \ldots, d$, are polewise independent with respect to $\{\delta_i\}$, $i = 1, \ldots, d$, in E_R. If α is a triangular table that satisfies* (1.1) *and all its limit points are in E, then there exists a unique system $\{r_{n,i} = p_{n,i}/q_n\}$, $i = 1, \ldots, d$, of simultaneous rational interpolants associated to $\{\delta_i\}$, $i = 1, \ldots, d$, in the n-th row of α such that*

$$\limsup \|q_n - q\|_A^{1/n} \leqslant r/R, \tag{2.1}$$

$$\limsup \|f_i - r_{n,i}\|_H^{1/n} \leqslant \sigma/R, \quad i = 1, \ldots, d, \tag{2.2}$$

where A and H are any compact subsets of C and of $E_R \setminus \bigcup_{j=1}^{m} \{z_j\}$ respectively; r and σ are the smallest numbers satisfying $\bigcup_{j=1}^{m} \{z_j\} \subset E_{r'}$ for each $r' > r$ and $H \subset E_{\sigma'}$ for each $\sigma' > \sigma$ respectively.

In the proof of the theorem an important role is played by the fact that

If u_k is a pole of order m_k for some f_i then for each j, $1 \leqslant j \leqslant m_k$,

$$\limsup |q_n^{(j)}(u_k)|^{1/n} \leqslant \sigma_k/R \quad \text{if } u_k \in \Gamma_{\sigma_k}, \tag{2.3}$$

which is a consequence of the polewise independence of the system (see [4]).

Proof of the theorem. By Theorem B, the polynomials of the denominator $q_n(z)$ and $q'_n(z)$ satisfy

$$\lim_n q_n(z) = \lim_n q'_n(z) = \prod_{j=1}^{m} (z - z_j) = q(z) \tag{2.4}$$

uniformly on every compact subset of the plane.

Let ${}_iq(z) = \prod_{j=\delta_i+1}^{m} (z - u_j)$, where u_j, $j = \delta_i + 1, \ldots, m$, are the poles of f_k, $k \neq i$ if $\delta_i \neq m$ (${}_iq(z) \equiv 1$ if $\delta_i = m$) and $q_n(z) = q_{n,\delta_i}(z)\, q_{n,\delta'_i}(z)$, $q'_n(z) = q'_{n,\delta_i}(z)\, q'_{n,\delta'_i}(z)$, where

$$\limsup \|q_{n,\delta'_i} - {}_iq\|_A^{1/n} \leqslant r_i/R, \qquad \limsup \|q'_{n,\delta'_i} - {}_iq\|_A^{1/n} \leqslant r_i/R \tag{2.5}$$

and A is any compact subset of C (see (2.1)) if $\delta_i \neq m$ (if $\delta_i = m$, $q_{n,\delta'_i}(z) = q'_{n,\delta'_i}(z) \equiv 1$).

Now, define $J_{n,i}(z) = q_{n,\delta_i}(z) f_i(z)\, q'_{n,\delta_i}(z)$. Thus for each $i = 1, \ldots, d$,

$q'_{n,\delta}(z)\, p_{n,i}(z)$ is the unique polynomial in the set P_n of polynomials of degree at most n which interpolates $q_{n,\delta_i'}(z) J_{n,i}(z)$ at $\alpha_{1,n}, \alpha_{2,n}, \ldots, \alpha_{n+1,n}$ in the Hermite sense. Similarly, $q_{n,\delta_i}(z)\, p'_{n,i}(z)$ is the unique polynomial in P_n which interpolates $q'_{n,\delta_i'}(z) J_{n,i}(z)$ at $\alpha'_{1,n}, \alpha'_{2,n}, \ldots, \alpha'_{n+1,n}$. Since $J_{n,i}(z)$, $i = 1, 2, \ldots, d$, are analytic on E, there exists a constant $s > 1$ such that all $J_{n,i}(z)$, $i = 1, 2, \ldots, d$, are analytic on and inside the level curve Γ_s. Then, for each n sufficiently large, Hermite's formula gives

$$q'_n(z)\, p_{n,i}(z) = \frac{1}{2\pi i} \int_{\Gamma_s} \frac{\{w_n(t) - w_n(z)\}\, q'_{n,\delta_i'}(z)\, q_{n,\delta_i'}(t)\, J_{n,i}(t)}{w_n(t)(t-z)}\, dt,$$

$$q_n(z)\, p'_{n,i}(z) = \frac{1}{2\pi i} \int_{\Gamma_s} \frac{\{w'_n(t) - w'_n(z)\}\, q_{n,\delta_i'}(z)\, q'_{n,\delta_i'}(t)\, J_{n,i}(t)}{w'_n(t)(t-z)}\, dt.$$

Subtracting, we have

$$q'_n(z)\, p_{n,i}(z) - q_n(z)\, p'_{n,i}(z) = \frac{1}{2\pi i} \int_{\Gamma_s} \frac{A_{n,i}(t, z)\, J_{n,i}(t)}{w_n(t)\, w'_n(t)(t-z)}\, dt, \qquad z \in C, \tag{2.6}$$

where

$$\begin{aligned} A_{n,i}(t, z) = {} & w_n(t)\, w'_n(t)\, \{q_{n,\delta_i'}(t)\, q'_{n,\delta_i'}(z) - q'_{n,\delta_i'}(t)\, q_{n,\delta_i'}(z)\} \\ & + w_n(t)\, w'_n(z)\, q'_{n,\delta_i'}(t)\, q_{n,\delta_i'}(z) \\ & - w'_n(t)\, w_n(z)\, q_{n,\delta_i'}(t)\, q'_{n,\delta_i'}(z). \end{aligned}$$

Next, let $\{u_j^*\}$, $j = 1, 2, \ldots, s_i$, $s_i \leqslant \delta_i$, denote the distinct poles of f_i in $E_R - E$. Let M be any constant such that $\max\{1, R^\theta\} < M < R$ and all the poles of $f_i(z)$ lie inside Γ_M. By $c_j = \{t\colon |t - u_j^*| = \varepsilon_i\}$, $1 \leqslant j \leqslant s_i$, we denote small circles that are mutually exterior and satisfy $c_j \subset E_M \setminus E$ for each $j = 1, 2, \ldots, s_i$. By setting $c_{s_i+1} = \Gamma_M$, Cauchy's theorem applied to the integral (2.6) gives, for all n sufficiently large

$$q'_n(z)\, p_{n,i}(z) - q_n(z)\, p'_{n,i}(z) = \sum_{j=1}^{s_i+1} I_{j,i}^{(n)}(z), \qquad i = 1, 2, \ldots, d, \tag{2.7}$$

where

$$I_{j,i}^{(n)}(z) = \frac{1}{2\pi i} \int_{c_j} \frac{A_{n,i}(t, z)\, J_{n,i}(t)}{w_n(t)\, w'_n(t)(t-z)}\, dt$$

and c_{s_i+1} is taken positively oriented, while the remaining contours c_j, $1 \leqslant j \leqslant s_i$, are all negatively oriented.

Using (1.2) we express $I_{j,i}^{(n)}(z)$ as

$$I_{j,i}^{(n)}(z) = \frac{1}{2\pi i} \int_{c_j} \frac{\{(w_n(t) - w_n(z))\, w_n(t)\, Q(t, z) + F(t, z)\}\, J_{n,i}(t)}{w_n(t)\, w'_n(t)(t-z)}\, dt \tag{2.8}$$

where

$$Q(t, z) = q_{n,\delta_i'}(t)\, q'_{n,\delta_i'}(z) - q'_{n,\delta_i'}(t)\, q_{n,\delta_i'}(z), \tag{2.8a}$$

$$F(t, z) = \sum_{j=0}^{n-1} \mu_j(n) \Big\{ w_{j-1}(t) \{ w_n(t) Q(t, z) - w_n(z) q_{n,\delta_i'}(t) q'_{n,\delta_i'}(z) \} + w_{j-1}(z) w_n(t) q'_{n,\delta_i'}(t) q_{n,\delta_i'}(z) \Big\}. \tag{2.8b}$$

Let $\gamma \geqslant R$. According to (2.5) and (2.8a) we have

$$\limsup \{\max |Q(t, z)|;\ t \in c_j,\ z \in \Gamma_\gamma\}^{1/n} \leqslant r_i/R. \tag{2.9}$$

From the hypotheses (1.1) and (1.3) we obtain

$$\limsup \{\max |F(t, z)|;\ z \in \Gamma_\gamma,\ t \in \Gamma_M\}^{1/n} \leqslant \Delta^2 \gamma R^\theta, \tag{2.10}$$

$$\limsup \{\max |w_n(t) - w_n(z)|\, |w_n(t)|;\ z \in \Gamma_\gamma,\ t \in \Gamma_M\}^{1/n} \leqslant \Delta^2 \gamma R. \tag{2.11}$$

From (1.1) and (1.4) we have

$$\lim \{\min |w_n(t) w_n'(t)|;\ t \in \Gamma_M\}^{1/n} = (\Delta M)^2, \tag{2.12}$$

thus

$$\limsup_n \{\max |I^{(n)}_{s_i+1,i}(z)|;\ z \in \Gamma_\gamma\}^{1/n} \leqslant \frac{\gamma \max(r_i, R^\theta)}{M^2}. \tag{2.13}$$

Next, to estimate the integrals around the poles z_j^* we note that for each $j = 1, 2, \ldots, s_i$, $I_{j,i}(z)$ is just the residue at $t = z_j^*$ of the function

$$\frac{\big\{ w_n(t) \{ w_n(t) - w_n(z) \} Q(t, z) + F(t, z) \big\} J_{n,i}(t)}{w_n(t) w_n'(t) (z - t)}. \tag{2.14}$$

From (2.3) we obtain

$$\limsup |q^{(k)}_{n,\delta_i}(z_j^*)|^{1/n} \leqslant \sigma_j^*/R \tag{2.15}$$

and similarly

$$\limsup |q'^{(k)}_{n,\delta_i}(z_j^*)|^{1/n} \leqslant \sigma_j^*/R \tag{2.16}$$

where $k = 0, 1, \ldots, m_j - 1$, m_j is the multiplicity of z_j^* (of f_i) and $z_j^* \in \Gamma_{\sigma_j^*}$.

Using (1.1), (2.5), (1.3), (2.15), (2.16), (1.4) we obtain the following rates:

$$\limsup \Big\{\max |\partial^k \{ w_n(t) (w_n(t) - w_n(z)) \} / \partial t^k|_{t = z_j^*},\ z \in \Gamma_\gamma \Big\}^{1/n} \leqslant \Delta^2 \sigma_j^* \gamma, \tag{2.17}$$

$$\limsup \Big\{\max |\partial^k \{ Q(t, z) \} / \partial t^k|_{t = z_j^*},\ z \in \Gamma_\gamma \Big\}^{1/n} \leqslant r_i/R, \tag{2.18}$$

$$\limsup \Big\{\max |\partial^k \{ F(t, z) \} / \partial t^k|_{t = z_j^*},\ z \in \Gamma_\gamma \Big\}^{1/n} \leqslant \Delta^2 \gamma R^\theta \quad (\gamma \geqslant R), \tag{2.19}$$

$$\limsup |d^k \{ w_n(t) w_n'(t) \}^{-1} / dt^k|^{1/n} \leqslant 1/(\Delta \sigma_j^*)^2 \quad \text{at } t = z_j^*, \tag{2.20}$$

$$\limsup |d^k \{ J_{n,i}(t) (t - z_j^*)^m \} / dt^k|^{1/n} \leqslant (\sigma_j^*/R)^2 \quad \text{at } t = z_j^*. \tag{2.21}$$

Using the combination of (2.21), (2.20), (2.19), (2.18), and (2.17) for estimating the residue at $t = z_j^*$ of the function in (2.14), we obtain, for each $j = 1, 2, \ldots, s_i$,

$$\limsup \{\max |I_{j,i}^{(n)}(z)|, z \in \Gamma_\gamma\}^{1/n} \leqslant \gamma \max(r_i, R^\theta)/R^2 \quad (\gamma \geqslant R). \tag{2.22}$$

Thus, from (2.22) and (2.13) it follows that for $\gamma \geqslant R$

$$\limsup \{\max |q_n'(z) p_{n,i}(z) - q_n(z) p_{n,i}'(z)|, z \in \Gamma_\gamma\}^{1/n} \leqslant \gamma \max(r_i, R^\theta)/M^2 \tag{2.23}$$

and so, letting M approach R and applying the maximum principle, we obtain

$$\limsup \{\max |q_n'(z) p_{n,i}(z) - q_n(z) p_{n,i}'(z)|, z \in E_\gamma\}^{1/n} \leqslant \gamma / R^{2-a_i} \tag{2.24}$$

where $a_i = \max\{\log_R r_i, \theta\}$ if $\delta_i \neq m$.

Using the same method we can obtain (2.24) for $a_i = \theta$ if $\delta_i = m$. Now we immediately obtain (1.5) from (2.4) and (2.24).

NOTE. If $d = 1$ then $\delta_1 = m$ and we got the result of [10].

COROLLARY. *Let $\{f_i\}$, $i = 1, \ldots, d$, be a system of analytic functions in $[|z| \leqslant 1]$ such that for each $i = 1, \ldots, d$, f_i has exactly δ_i poles in $[|z| < R]$ $(R > 1)$ where $\{\delta_i\}$, $i = 1, \ldots, d$, is a set of non-negative integers such that $\sum_{i=1}^d \delta_i = m > 0$. Then if $\{r_{n,i} = p_{n,i}/q_n\}$, $i = 1, \ldots, d$, and $\{r_{n,i}' = p_{n,i}'/q_n'\}$, $i = 1, \ldots, d$, denote the (n, m) simultaneous rational interpolants of $\{f_i\}$, $i = 1, \ldots, d$, with respect to $\{\delta_i\}$, $i = 1, \ldots, d$, at the origin and at the $(n+1)$-roots of unity then for each γ, $R \leqslant \gamma < \infty$,*

$$\limsup \{\max |r_{n,i}(z) - r_{n,i}'(z)|, z \in H\}^{1/n} \leqslant \gamma / R^{2-a_i} \tag{2.25}$$

where H is an arbitrary closed subset of $[|z| \leqslant \gamma] \setminus \bigcup_{j=1}^m \{z_j\}$, $\bigcup_{j=1}^m \{z_j\}$ is the set of poles of the system in $[|z| < R]$ and

$$a_i = \begin{cases} \log_R(\max\{|u|, u \text{ is pole of } f_j, j \neq i\}) & \text{if } \delta_i \neq m, \\ \theta & \text{if } \delta_i = m. \end{cases}$$

Proof. This is a special case of the theorem. Let $E = [|z| \leqslant 1]$, so that E has capacity $\Delta = 1$. The associated Green function is then simply $G(z) = \log|z|$, and the level curves Γ_σ are the circles $[|z| = \sigma]$. Now, $w_n(z) = z^{n+1}$ and $w_n'(z) = z^{n+1} - 1$. Trivially $w_n(z)$ satisfies (1.1) and the inequallity of (1.3) is valid for every $R > 1$, with $\theta = 0$. Thus the theorem gives (2.25). Furthermore, slight modifications in the proof of the theorem show that, for these special interpolation schemes, we can indeed allow some or all of the poles $\{f_i\}$, $i = 1, 2, \ldots, d$, to lie in the punctured disc $0 < |z| < 1$.

As the reader can observe in our result on the difference of simultaneous rational interpolants it has been necessary, due to the method used in the proof, to impose more severe restrictions than those in Graves-Morris and

Saff's Theorem B (no common poles for the functions in the system). It would be interesting to obtain a result similar to ours but, for instance, under the general condition of polewise independence. It would also be interesting to know whether the results obtained are exact, that is, whether in general it is not possible to obtain a larger region of overconvergence for an arbitrary system of analytic functions $\{f_i\}$, $i = 1, \ldots, d$, satisfying the conditions imposed in the theorem.

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